Doctoral Training School *Kardar-Parisi-Zhang equation: new trends in theories and experiments*
April 15-26, 2024 — Ecole de Physique des Houches (France)

Interfaces in disordered systems and directed polymer

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Elisabeth Agoritsas

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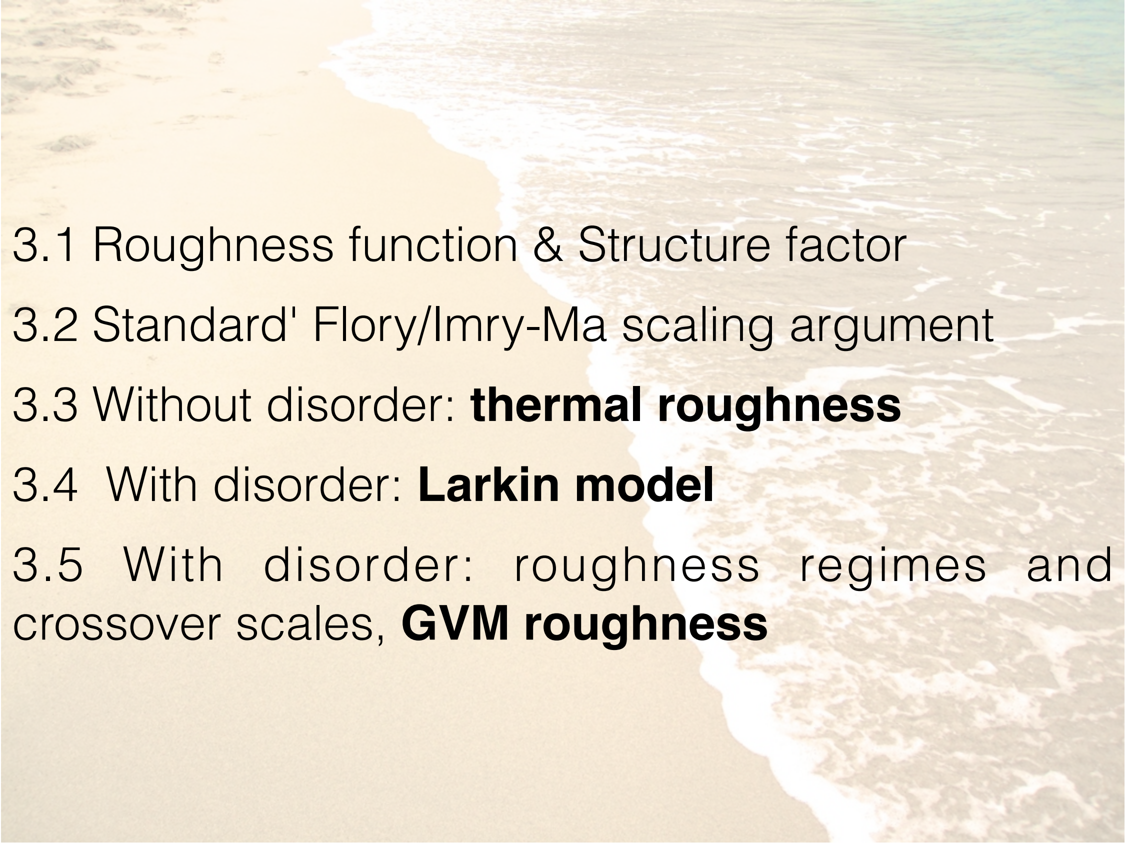
- ➡ 3. Disordered elastic systems: Statics
- 4. Disordered elastic systems: Dynamics
- 5. Concluding remarks



3.1 Roughness function & Structure factor

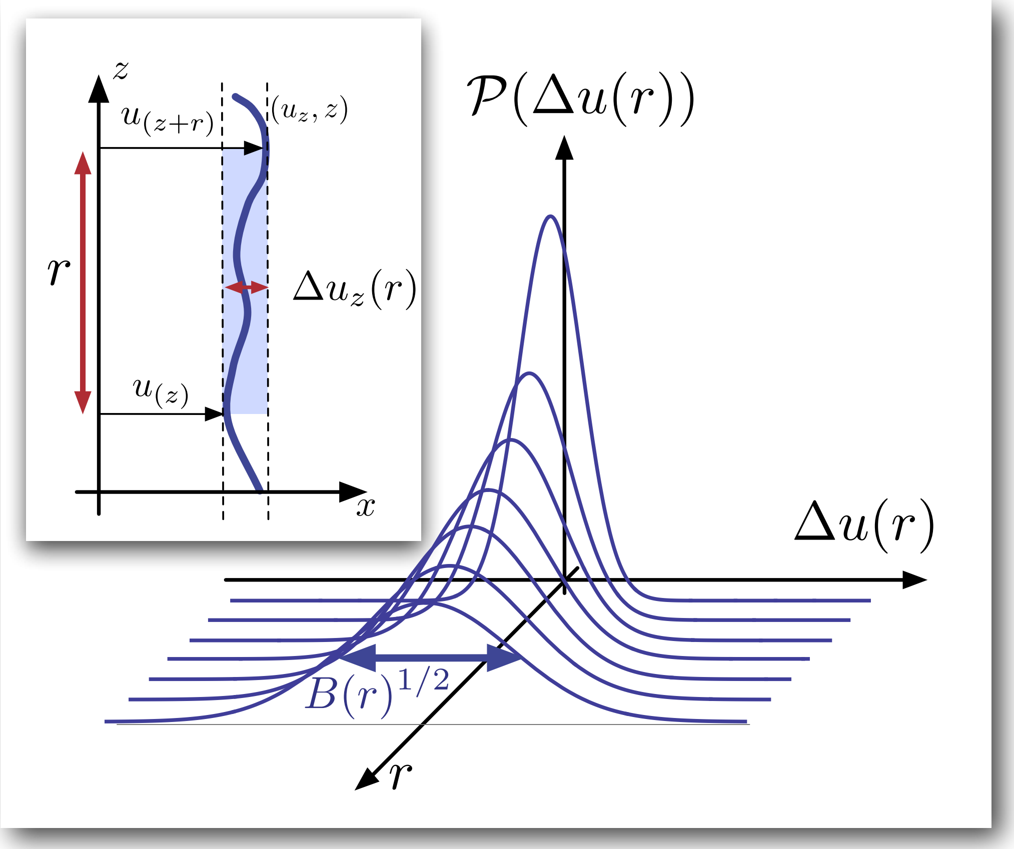
3.2 Standard' Flory/Imry-Ma scaling argument

3.3 Without disorder: **thermal roughness**

3.4 With disorder: **Larkin model**

3.5 With disorder: roughness regimes and crossover scales, **GVM roughness**

Roughness function & Structure factor



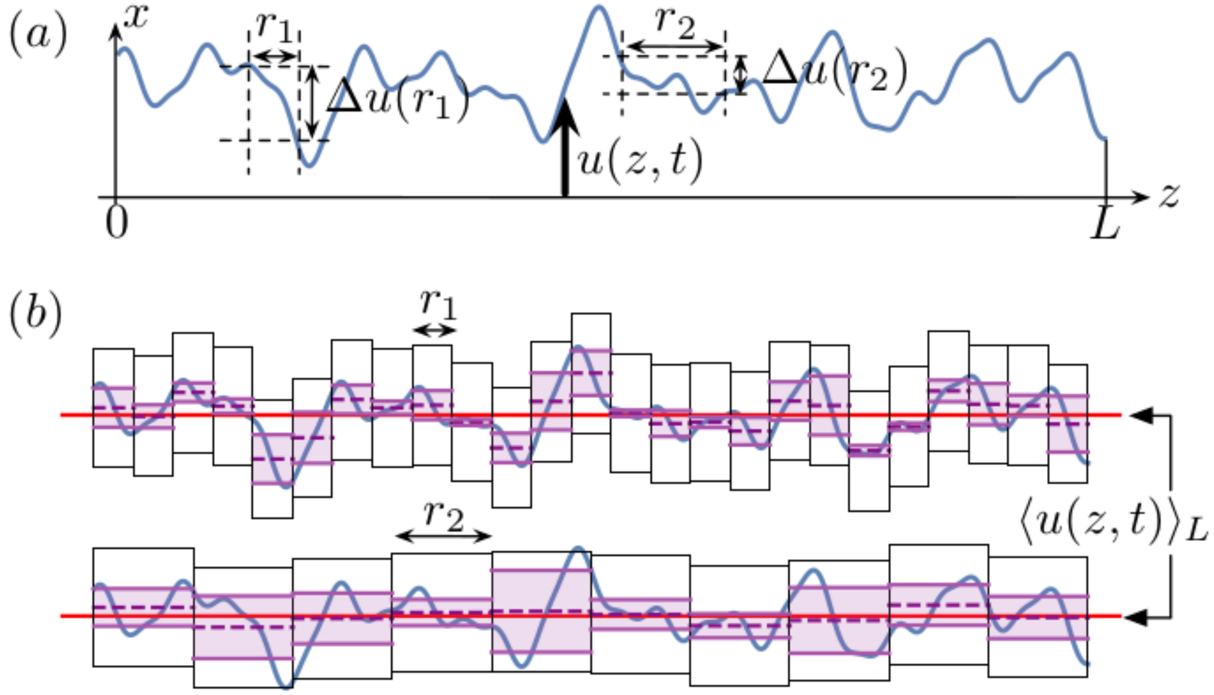
- Our roughness = 'height-height correlation function"

$$B(r) = \overline{\left\langle [u(z+r) - u(z)]^2 \right\rangle} = \left\langle \frac{1}{N_r} \sum_{\text{pairs}} \Delta u_z(r)^2 \right\rangle_{\text{samples}}$$

- Working in Fourier. structure factor

$$S(q) = \overline{\langle \tilde{u}_{-q} \tilde{u}_q \rangle} = \langle |\tilde{u}_q|^2 \rangle$$

$$[\ldots] \quad B(r) = \int \frac{d^d q}{(2\pi)^d} 2\,[1 - \cos(qr)]\, S(q)$$



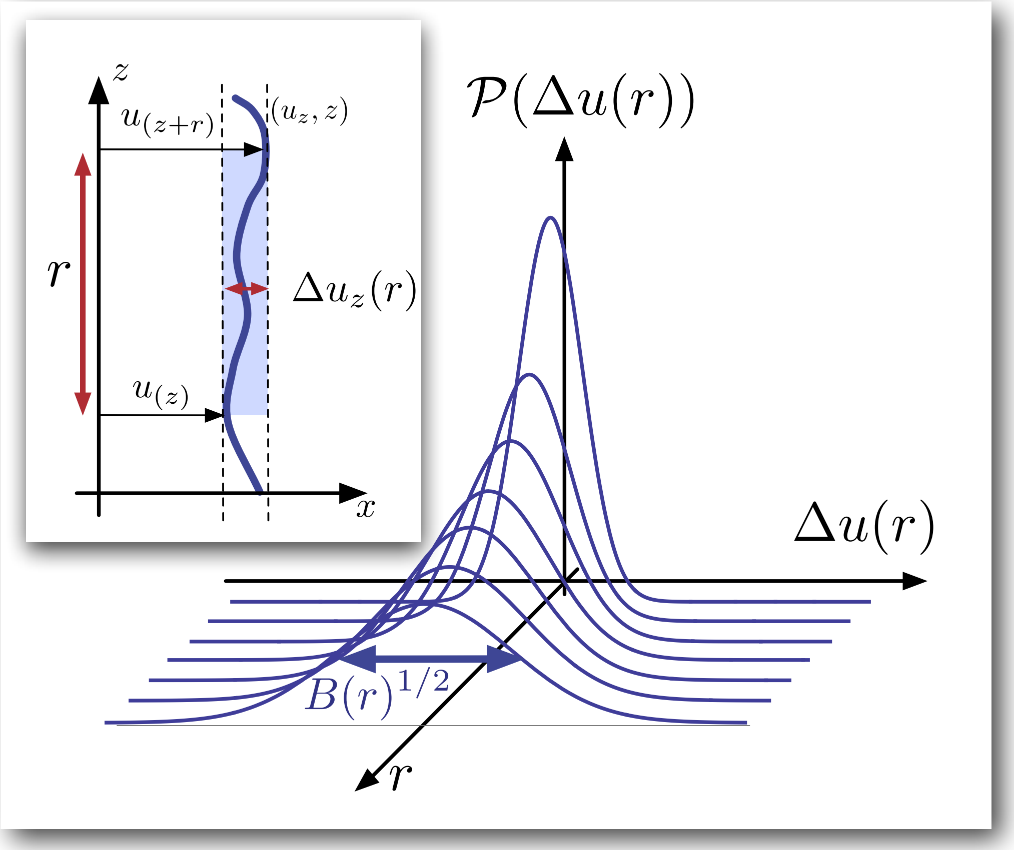
- Alternative definitions/quantities:

local width $$w(r)^2 = \overline{\left\langle [u(z) - \langle u(z) \rangle_r]^2 \right\rangle_r}$$

global width $$W(L)^2 = \overline{\left\langle [u(z) - \langle u(z) \rangle_L]^2 \right\rangle_L}$$

- Our roughness & structure factors are **2-pt spatial correlations** in direct/continuous space, different physical content that the global/local width

Roughness function & Structure factor



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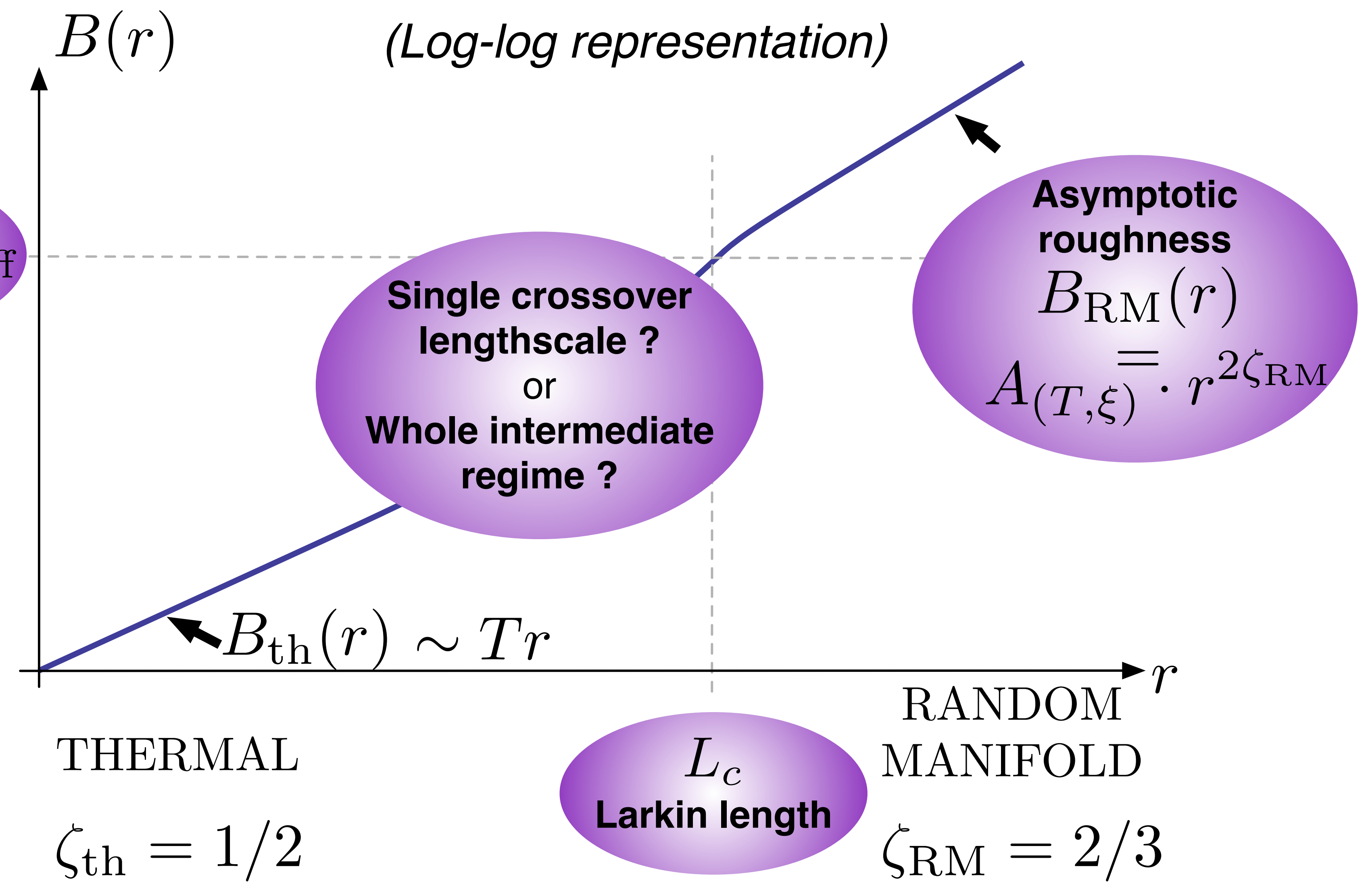
$$[\dots] \quad B(r) = \int \frac{d^d q}{(2\pi)^d} 2 \left[1 - \cos(qr)\right] S(q)$$

- Issues in experimental (and also numerical) studies of roughness:
  - **Finite statistics:** # of samples / N_r= #pairs for a given lengthscale: $N_r \searrow$ when $r \nearrow$
  - **Finite system size:** e.g. in ferroelectric domain walls, L=512 pixels ⇒ relevance of power-law exponents?
  - **Finite-time saturation:** glassy behaviour, *a priori* not fully relaxed systems (experimentally/numerically)
  - **Beware of strong impurities:** might break locally the elastic description ⇒ non-Gaussian artefacts

A.-L. Barabàsi & H. E. Stanley, *Fractal Concepts in Surface Growth,* Cambridge University Press, 1995.
Cf. Preprint: J. Guyonnet, E. Agoritsas, P. Paruch, S. Bustingorry, arXiv:1904.11726 [cond-mat.dis-nn].
S. Bustingorry, J. Guyonnet, P. Paruch, E. Agoritsas, *J. Phys. Condens. Matter* 33, 345001 (2021).

Static roughness regimes & characteristic crossover scales



Effective width ξ_{eff}^2

Single crossover lengthscale ?
or
Whole intermediate regime ?

Asymptotic roughness
$B_{\mathrm{RM}}(r) = A_{(T,\xi)} \cdot r^{2\zeta_{\mathrm{RM}}}$

L_c
Larkin length

Temperature-induced crossover

$$\boxed{T_c = (\xi c D)^{1/3}}$$

$$\begin{cases} T \ll T_c : & f \sim T/T_c \\ T \gg T_c : & f \lesssim 1 \end{cases}$$

Universal scaling

KPZ scaling for the asymptotic roughness

$$B(r) \overset{(r\to\infty)}{\sim} \left(\frac{D}{cT/f}\right)^{2/3} r^{4/3}$$

Non-universal features

Roughness amplitude: $A_{(T,\xi)} \sim \left(\frac{D}{cT/f}\right)^{2/3}$

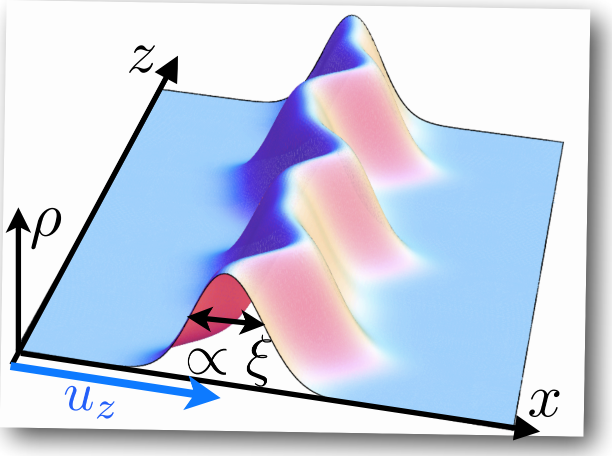
Larkin length: $L_c = \frac{(T/f)^5}{cD^2}$

Interlude: 'Standard' Flory/Imry-Ma scaling argument

- Short-range elasticity & Elastic limit / Quenched random-bond weak disorder

$$\boxed{\mathcal{H}_{\mathrm{DES}} = \mathcal{H}_{\mathrm{el}} + \mathcal{H}_{\mathrm{dis}}}$$

$$\mathcal{H}[u,\widetilde{V}] = \int_{\mathbb{R}} dz \cdot \left[\frac{c}{2}(\nabla_z u(z))^2 + \int_{\mathbb{R}} dx \cdot \rho_\xi(x-u(z))\,\widetilde{V}(z,x)\right]$$



- Dimensional analysis / power counting:
  - $\mathcal{H}_{\mathrm{el}}[u] = \frac{c}{2}\int d^dz \cdot (\nabla u(z))^2 \sim L^d \cdot \left(\frac{u}{L}\right)^2 = L^{d-2}u^2;$

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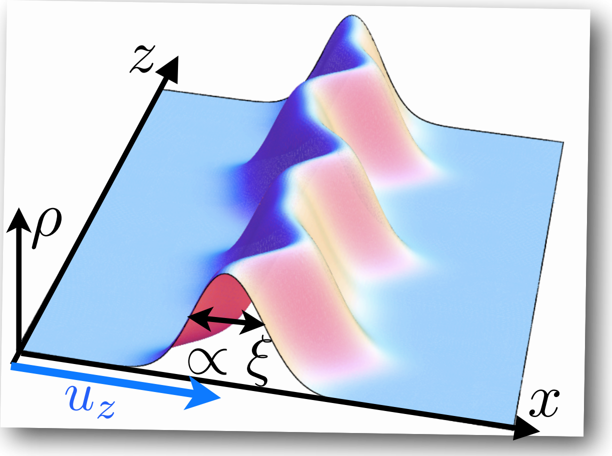

- Dimensional analysis / power counting:
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  - $\overline{V(x,z)V(x',z')} = D\cdot\delta^{(m)}(x-x')\delta^{(d)}(z-z') \Longrightarrow V^2 \sim \left(\frac{1}{x}\right)^m\left(\frac{1}{z}\right)^d \sim u^{-m}\cdot L^{-d};$
  - $\rho(x,z) = \frac{1}{(2\pi\xi^2)^{m/2}}e^{-\frac{(x-u(z))^2}{2\xi^2}} \Longrightarrow \rho \sim \xi^{-m} \sim u^{-m};$
  - $\mathcal{H}_{\mathrm{dis}}\left[u,V\right] = \int d^mx\,d^dz\cdot V(x,z)\rho(x,z) \sim u^m\cdot L^d\cdot u^{-m/2}\cdot L^{-d/2}\cdot u^{-m} = L^{d/2}u^{-m/2};$
  - imposing $\mathcal{H}_{\mathrm{el}} \sim \mathcal{H}_{\mathrm{dis}}$ on the thermal and disorder energetic contributions at the lengthscale L, we thus have:

$$L^{d-2}u^2 \sim L^{d/2}u^{-m/2} \Longleftrightarrow u^{\frac{4+m}{2}} \sim L^{\frac{4-d}{2}} \Longleftrightarrow \boxed{u(L) \sim L^{\frac{4-d}{4+m}} \equiv L^{\zeta_F}}$$

  - and eventually $B(r) \equiv \overline{\left\langle [u(r)-u(0)]^2\right\rangle} \sim u(r)^2 \sim r^{2\zeta_F}$ with

$$\zeta_F = \frac{4-d}{4+m}$$

$$d = m = 1$$

$$\Rightarrow \zeta_F = 3/5$$

Interlude: Fourier transforms with discrete vs continuous modes

- If finite length L and periodic boundary conditions, **Fourier transform with <u>discrete</u> modes** $q \in 2\pi\mathbb{Z}^*/L$

$$\tilde{u}(q) = \int_0^L dz\, e^{iqz}\, u(z)\,, \qquad u(z) = \underbrace{\tilde{u}(q=0)}_{= L\bar{u}} + \sum_{q \in \frac{2\pi\mathbb{Z}^*}{L}} e^{iqz}\, \tilde{u}(q)$$

- Should be physically equivalent to working with continuous Fourier modes with a finite infra-red cutoff

$$q \in [q_{\min}, \Lambda]$$
$$q_{\min} \sim 1/L$$
$$\Lambda \to \infty$$

$$\boxed{u(z) \approx \tilde{u}(q=0) + 2\int_{q_{\min}}^{\infty} \frac{dq}{2\pi}\, e^{iqz}\, \tilde{u}(q)}$$

- NB. Fourier transform of the **Dirac delta**:

$$\sum_{\omega \in 2\pi\mathbb{Z}/t_{\mathrm{f}}} e^{i\omega t} = \delta(t) = \frac{1}{t_{\mathrm{f}}}\delta(t/t_{\mathrm{f}}) = \frac{1}{t_{\mathrm{f}}} \sum_{\hat{\omega} \in 2\pi\mathbb{Z}} e^{i\hat{\omega} t/t_{\mathrm{f}}}$$

- Fourier formulation useful if **elastic energy per mode is quadratic in the displacement field**

$$\mathcal{H}_{\mathrm{el}}[u(z), L] = \frac{1}{2} \sum_{q \in \frac{2\pi\mathbb{Z}^*}{L}} c(q)\, \tilde{u}(-q)\tilde{u}(q) \approx \int_{q_{\min}}^{\infty} \frac{dq}{2\pi} c(q)\tilde{u}(-q)\tilde{u}(q)$$

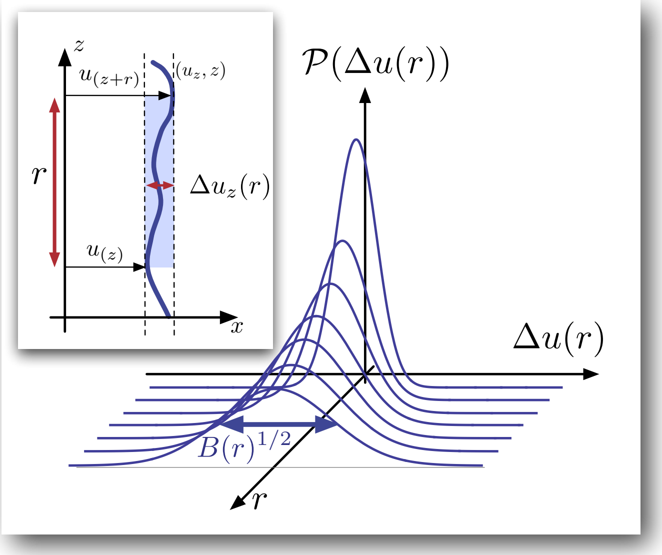
Roughness function & Structure factor — Derivation

- Working in Fourier: structure factor $S(q) = \overline{\langle \tilde{u}_{-q} \tilde{u}_q \rangle} = \langle |\tilde{u}_q|^2 \rangle$

$$[\dots] \quad B(r) = \int \frac{d^d q}{(2\pi)^d} 2 \left[1 - \cos(qr)\right] S(q)$$

- Assuming translation invariance (recovered after averaging over disorder): $\overline{\left\langle u^*_{\tilde{q}} u_q \right\rangle} \propto \delta(q - \tilde{q}) \cdot \overline{\langle u^*_q u_q \rangle}$

$$\begin{aligned}
B(z_1, z_2) \quad &\equiv \overline{\left\langle \left[u(z_1) - u(z_2)\right]^2 \right\rangle} = \overline{\left\langle \left[\int \frac{d^d q}{(2\pi)^d} \left(e^{iqz_1} - e^{iqz_2}\right) u(q) \right]^2 \right\rangle} \\
&= \int \frac{d^d q}{(2\pi)^d} \frac{d^d \tilde{q}}{(2\pi)^d} \left(e^{iqz_1} - e^{iqz_2}\right) \left(e^{i\tilde{q}z_1} - e^{i\tilde{q}z_2}\right) \overline{\langle u_q u_{\tilde{q}} \rangle} \\
&\overset{[\tilde{q} \mapsto -\tilde{q}]}{=} \int \frac{d^d q}{(2\pi)^d} \frac{d^d \tilde{q}}{(2\pi)^d} \left(e^{iqz_1} - e^{iqz_2}\right) \left(e^{-i\tilde{q}z_1} - e^{-i\tilde{q}z_2}\right) \overline{\langle u_q u_{-\tilde{q}} \rangle} \\
&\overset{[u_{-\tilde{q}} = u^*_{\tilde{q}}]}{=} \int \frac{d^d q}{(2\pi)^d} \frac{d^d \tilde{q}}{(2\pi)^d} \left(e^{i(q-\tilde{q})z_1} + e^{i(q-\tilde{q})z_2} - e^{i(qz_1 - \tilde{q}z_2)} - e^{-i(\tilde{q}z_1 - qz_2)}\right) \overline{\left\langle u_q u^*_{\tilde{q}} \right\rangle} \\
&\overset{(2.18)}{=} \int \frac{d^d q}{(2\pi)^d} \cdot \frac{1}{(2\pi)^d} \left(2 - 2\cos(q(z_1 - z_2)\right)) \cdot \overline{\left\langle u^*_q u_q \right\rangle} \\
&= \int \frac{d^d q}{(2\pi)^d} \cdot 2\left(1 - \cos(q(z_1 - z_2)\right)) \cdot \frac{1}{(2\pi)^d} \overline{\left\langle u^*_q u_q \right\rangle}
\end{aligned}$$



⚠ Beware of the normalisation: e.g. as a guide $S_{\text{thermal}}(q) = \dfrac{T}{cq^2}$ ⚠

Roughness function & Structure factor — Derivation

- Structure factor for discrete versus continuous Fourier modes:

$$S(\omega) = \sum_{\omega'\in 2\pi\mathbb{Z}/t_\mathrm{f}} \overline{\langle \tilde{y}(-\omega')\tilde{y}(\omega)\rangle}\,, \qquad S(q) = 2\int_{q_\mathrm{min}}^{\infty} \text{đ}q'\, \overline{\langle \tilde{y}(-q')\tilde{y}(q)\rangle} \overset{(q_\mathrm{min}\to\infty)}{\approx} \int_{\mathbb{R}} \text{đ}q'\, \overline{\langle \tilde{y}(-q')\tilde{y}(q)\rangle}\,.$$

⚠️ Beware of the zero modes that should not be included ⚠️

- Assuming translation invariance (recovered after averaging over disorder):

$$\begin{aligned}
B(\tau; c, D, T, \xi, t_\mathrm{f}) &= \overline{\left\langle \left(y(t+\tau) - y(t)\right)^2 \right\rangle}_{\{c,D,T,\xi,t_\mathrm{f}\}} \\
&= \overline{\left\langle \left[\sum_{\omega_1\in 2\pi\mathbb{Z}^*/t_\mathrm{f}} \left(e^{i\omega_1\tau} - 1\right)\tilde{y}(\omega_1)\right] \left[\sum_{\omega_2\in 2\pi\mathbb{Z}^*/t_\mathrm{f}} \left(e^{i\omega_2\tau} - 1\right)\tilde{y}(\omega_2)\right] \right\rangle} \\
&= \sum_{\omega_1,\omega_2\in 2\pi\mathbb{Z}^*/t_\mathrm{f}} \left(e^{i\omega_1\tau} - 1\right)\left(e^{i\omega_2\tau} - 1\right) \underbrace{\overline{\langle \tilde{y}(\omega_1)\tilde{y}(\omega_2)\rangle}}_{\overset{*}{=}\delta_{\omega_1+\omega_2} S(\omega_1)} \\
&= \sum_{\omega\in 2\pi\mathbb{Z}^*/t_\mathrm{f}} 2\left[1 - \cos(\omega\tau)\right] S(\omega) = \frac{1}{t_\mathrm{f}} \sum_{\hat{\omega}\in 2\pi\mathbb{Z}^*} 2\left[1 - \cos(\hat{\omega}\tau/t_\mathrm{f})\right] S(\hat{\omega}/t_\mathrm{f}) \\
&\approx 2\int_{q_\mathrm{min}}^{\infty} \text{đ}q\, 2\left[1 - \cos(q\tau)\right] S(q)\,,
\end{aligned}$$

Thermal average (at equilibrium)

A sum over all its possible configurations $\{\mathbf{s}\}$ is thus discrete, as denoted thereafter by $\sum_{\{\mathbf{s}\}}$. If the energy of the system is described by a Hamiltonian $\mathcal{H}$, the partition function Z and the *thermal average* of an observable $\mathcal{O}$ are defined by:

$$\langle\mathcal{O}\rangle_{\mathcal{H}} = \sum_{\{\mathbf{s}\}} \mathcal{O}\left(\{\mathbf{s}\}\right) \cdot \frac{e^{-\beta\mathcal{H}(\{\mathbf{s}\})}}{Z} \qquad Z \equiv \sum_{\{\mathbf{s}\}} e^{-\beta\mathcal{H}(\{\mathbf{s}\})}$$

We are interested in physical samples where the disorder is *quenched*, i.e. the inhomogeneities of the medium are fixed. Since a thermal average aims to compute the expected value of an observable for a given physical sample, it should be computed at fixed disorder V. The thermal average of an observable, on our interface and at quenched disorder, is thus:

$$\langle\mathcal{O}\rangle_V \equiv \frac{1}{Z}\int \mathcal{D}u \cdot \mathcal{O}\left[u\right] \cdot e^{-\beta\mathcal{H}[u,V]} = \frac{\int \mathcal{D}u \cdot \mathcal{O}\left[u\right] \cdot e^{-\beta\mathcal{H}[u,V]}}{\int \mathcal{D}u \cdot e^{-\beta\mathcal{H}[u,V]}}$$

- Generic change of variables: $\sum_{\{\mathbf{s}\}} \xrightarrow{(a\to 0)} \int \mathcal{D}u \equiv \prod_i \int du_i \equiv J \prod_{q>0} \int\int du_q^* du_q$

Statistical average over the disorder

In a discrete representation of the interface, the random potential of each site i thus takes a value $V_i \in \mathbb{R}$ with a probability $\mathcal{P}(V_i) \propto e^{-V_i^2/2D}$, i.e. of variance D and of mean value $\overline{V_i} = 0$. If the disorder is uncorrelated between two sites, the probability of realization of a given set $\{V_1, \cdots, V_n\}$ is simply given by the product $\mathcal{P}(V_1) \cdot (\cdots) \cdot \mathcal{P}(V_n)$. Note that physically the random potential cannot be infinite, and so $V_i \in [-\Lambda, \Lambda]$ would be more adequate; but the regulating term $e^{-V_i^2/2D}$ conveniently replaces the cutoff $\pm\Lambda$.

The *average over disorder* of an observable $\mathcal{O}$ is defined as the average over all the possible values of $\mathcal{O}[V]$, weighted by the respective probability of realization of each configuration of disorder V. With the appropriate normalization, it is given by:

$$\begin{aligned}\overline{\mathcal{O}} &= \frac{\int_{-\infty}^{+\infty} dV_1 \cdot e^{-V_1^2/2D} \cdot (\cdots) \cdot \int_{-\infty}^{+\infty} dV_n \cdot e^{-V_n^2/2D} \quad \cdot \quad \mathcal{O}[V_1, \cdots, V_n]}{\int_{-\infty}^{+\infty} dV_1 \cdot e^{-V_1^2/2D} \cdot (\cdots) \cdot \int_{-\infty}^{+\infty} dV_n \cdot e^{-V_n^2/2D}} \\ &= \frac{\left(\prod_i \int_{\mathbb{R}} dV_i\right) \cdot e^{-\frac{D^{-1}}{2}\sum_j V_j^2} \cdot \mathcal{O}[\{V_i\}]}{\left(\prod_i \int_{\mathbb{R}} dV_i\right) \cdot e^{-\frac{D^{-1}}{2}\sum_j V_j^2}}\end{aligned}$$

In a continuous representation of the interface, we have $V_i \mapsto V(x,z)$ and $\prod_i \int dV_i \mapsto \int \mathcal{D}V$, and we can define:

$$\overline{\mathcal{O}} \equiv \frac{1}{C}\int \mathcal{D}V \cdot \mathcal{O}[V] \cdot e^{-\frac{D^{-1}}{2}\int dx\, dz \cdot V(x,z)^2} = \frac{\int \mathcal{D}V \cdot \mathcal{O}[V] \cdot e^{-\frac{D^{-1}}{2}\int dx\, dz \cdot V(x,z)^2}}{\int \mathcal{D}V \cdot e^{-\frac{D^{-1}}{2}\int dx\, dz \cdot V(x,z)^2}}$$

where D is now called the *strength of disorder*. We can check a posteriori that this definition of $\overline{\mathcal{O}}$ corresponds indeed to a 'white Gaussian disorder' :

$$\left\{\begin{array}{lcl}\overline{V(x,z)} &=& 0 \\ \overline{V(x,z)V(x',z')} &=& D \cdot \delta^{(m)}(x-x')\,\delta^{(d)}(z-z')\end{array}\right.$$

$\boxed{d = m = 1}$

Without disorder: thermal roughness

- **Elastic energy per mode quadratic in the displacement field**

$$\mathcal{H}_{\mathrm{el}}[u(z),L] = \frac{1}{2}\sum_{q\in\frac{2\pi\mathbb{Z}^*}{L}} c(q)\,\tilde{u}(-q)\tilde{u}(q) \approx \int_{q_{\min}}^{\infty}\frac{dq}{2\pi}c(q)\tilde{u}(-q)\tilde{u}(q)$$

- **Thermal structure factor:** $S_{\mathrm{th}}(q) = \langle\tilde{u}(-q)\tilde{u}(q)\rangle = \dfrac{\int\mathcal{D}u\,\tilde{u}(-q)\tilde{u}(q)\,e^{-\mathcal{H}_{\mathrm{el}}/T}}{\int\mathcal{D}u\,e^{-\mathcal{H}_{\mathrm{el}}/T}} = \dfrac{T}{c(q)}$

- **Thermal roughness:**

$$B_{\mathrm{th}}(r,L) = \sum_{q\in\frac{2\pi\mathbb{Z}^*}{L}} 2(1-\cos(qr))S_{\mathrm{th}}(q) \underset{\substack{\big\uparrow\\ c(q)\,=\,cq^2}}{=} \frac{Tr}{c}\left(1-\frac{r}{L}\right)$$

- Alternative with continuous modes & finite infra-red cutoff:

$$B_{\mathrm{th}}(t) \approx 2T\int_{q_{\min}}^{\infty}\mathrm{đ}q\,\frac{2[1-\cos(qt)]}{cq^2} = \frac{T}{c}\left\{t + \frac{2\,[1-\cos(q_{\min}t)]}{\pi q_{\min}} - \frac{2t}{\pi}\int_0^{q_{\min}t}\mathrm{d}x\,\frac{\sin(x)}{x}\right\}.$$

$$\frac{c}{T}B_{\mathrm{th}}(t) \overset{(q_{\min}\to 0)}{\approx} t - \frac{t^2 q_{\min}}{\pi} + \mathcal{O}\left(q_{\min}^3\right)$$

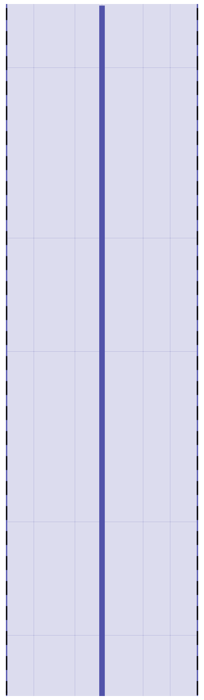
$$\boxed{q_{\min} = \pi/t_{\mathrm{f}}} \qquad B_{\mathrm{th}}(t) \approx \frac{T}{c}\int_{q_{\min}}^{\infty}\mathrm{đ}q\,\frac{2[1-\cos(qt)]}{q^2} \overset{(q_{\min}\to 0)}{\approx} \frac{T}{c}t\left(1-\frac{t}{t_{\mathrm{f}}}\right) \overset{(t_{\mathrm{f}}\to\infty)}{\approx} \frac{Tt}{c}.$$

With disorder (perturbative approach): Larkin model

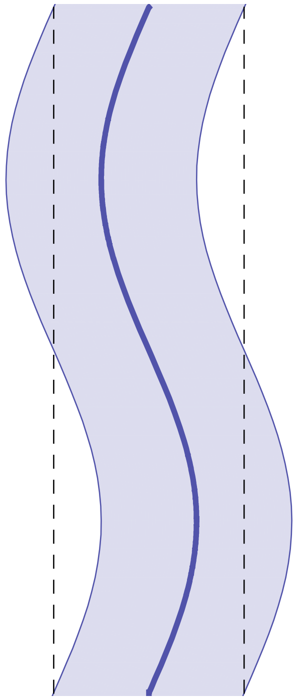
- Hamiltonian for a weakly distorted interface: the Larkin model

  A. I. Larkin, "Effect of inhomogeneities on the structure of the mixed state of superconductors", *Sov. Phys. JETP* 31, 784 (1970).

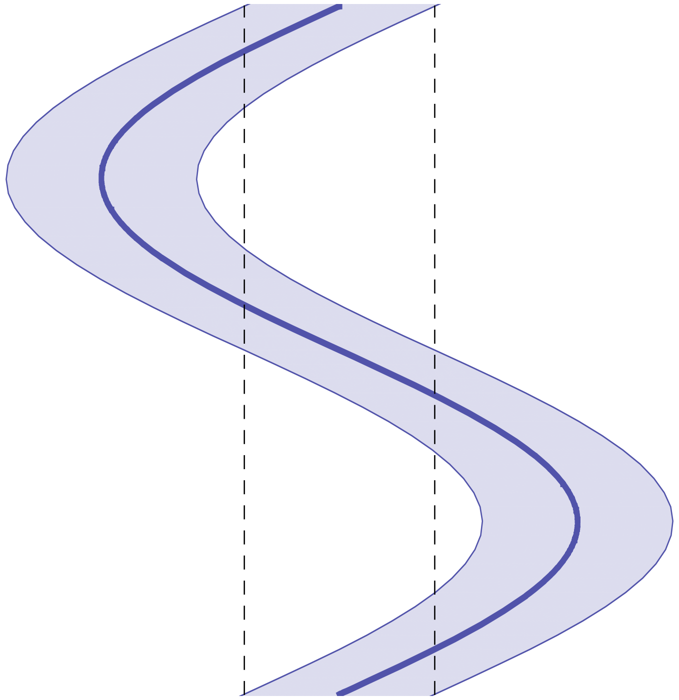
  V. Démery, V. Lecomte, A. Rosso, "The effect of disorder geometry on the critical force in disordered elastic systems", *J. Stat. Mech.* 2014, P03009 (2014)



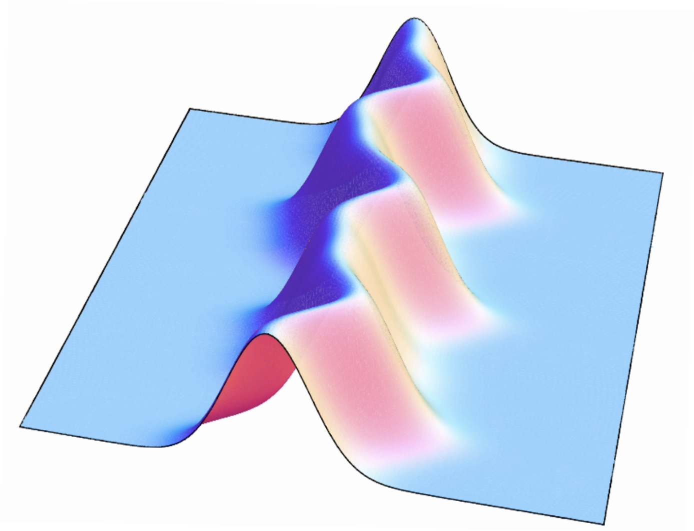
(a) Flat interface



(b) Small distorsions



(c) Large distorsions

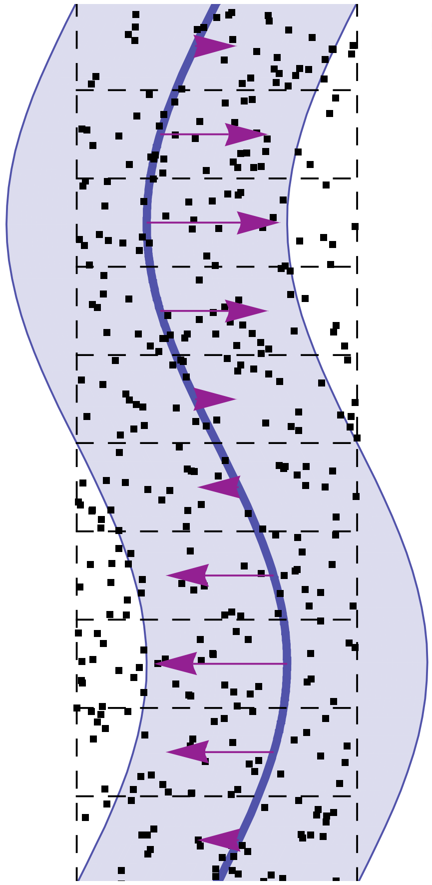


- Expanding the interface density, e.g. a Gaussian function:

$$\rho(x,z) \overset{(2.3)}{=} \frac{1}{\sqrt{2\pi}\xi} e^{-\frac{x^2}{2\xi^2}} e^{\frac{xu(z)}{\xi^2}} e^{-\frac{u(z)^2}{2\xi^2}}$$

$$= \rho(x,z)|_{u(z)=0} \cdot \left(1 + \frac{x}{\xi^2} u(z) + \mathcal{O}\left((u(z)/\xi)^2\right)\right)$$

With disorder (perturbative approach): Larkin model


(a) Small distorsion

- Taylor-expansion of the density valid for small distorsions $|\Delta u(r)| \ll \xi$

$$\mathcal{H}_{\text{dis}}[u,V] = \int d^d z \int d^m x \, \rho_u(x,z) V(x,z)$$

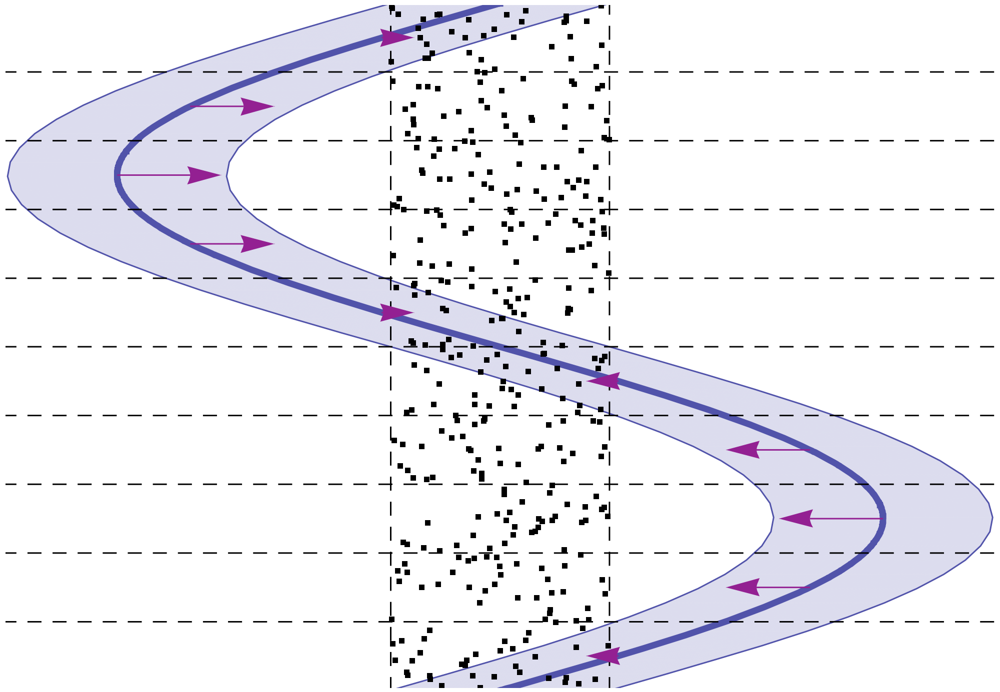
$$\rho_u(x,z) = \rho(x,z)|_{u=0} - \partial_x \rho(x,z)|_{u=0} \, u(z) + \mathcal{O}(u^2)$$

- Keeping only the leading term in $u(z)$, definition of the Larkin Hamiltonian:

$$\Rightarrow \mathcal{H}^L[u,f] = \text{cte} + \int d^d z \, u(z) \, f(z)$$

$$f(z) = \int d^m x \, \rho(x,z)|_{u=0} \underbrace{[-\partial_x V(x,z)]}_{}$$

$$\Rightarrow \mathcal{H}^L[u,f] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \left(c q^2 \tilde{u}_{-q} \tilde{u}_q + f_q \tilde{u}_{-q} + f_{-q} \tilde{u}_q \right)$$


(b) Large distorsions

- Gaussian local force, of zero mean and 2-pt correlation:

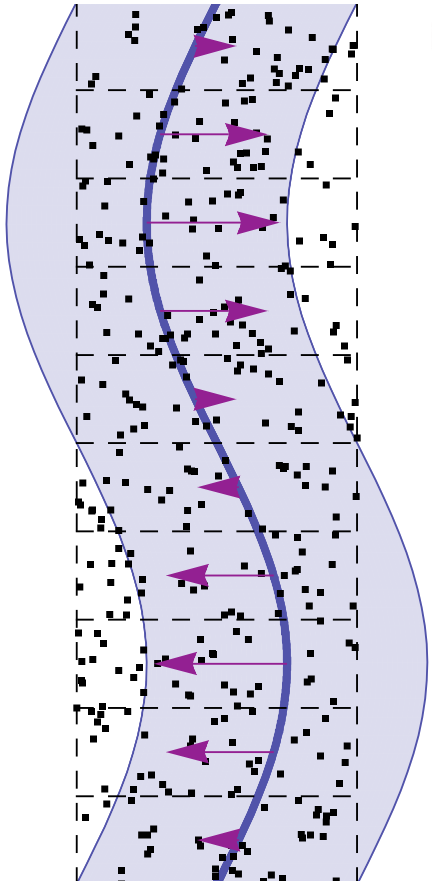
$$\begin{cases} \overline{f(z) f(z')} = \widetilde{D} \cdot \delta^{(d)}(z - z') \\ \widetilde{D} = \frac{D}{4\sqrt{\pi}\xi^3} \end{cases}$$

- Structure factor & Roughness scaling:

$$\boxed{S^L(q) = \frac{T}{cq^2} + \frac{\widetilde{D}}{(cq^2)^2}}$$

$$S(q) \sim \begin{cases} q^{-2} \Rightarrow B(r) \sim r \\ q^{-4} \Rightarrow B(r) \sim r^{4-d} \end{cases}$$

With disorder (perturbative approach): Larkin model



(a) Small distortion

- Taylor-expansion of the density valid for small distorsions $|\Delta u(r)| \ll \xi$

$$\mathcal{H}_{\text{dis}}[u,V] = \int d^d z \int d^m x \, \rho_u(x,z) V(x,z)$$

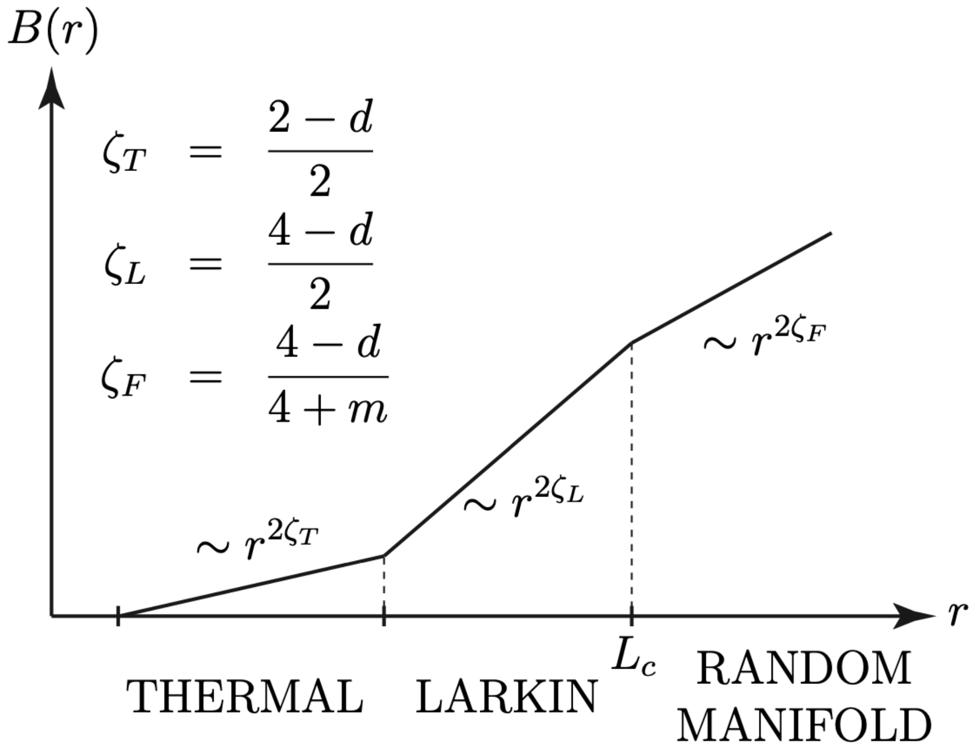
$$\rho_u(x,z) = \rho(x,z)|_{u=0} - \partial_x \rho(x,z)|_{u=0} \, u(z) + \mathcal{O}(u^2)$$

- Keeping only the leading term in $u(z)$, definition of the Larkin Hamiltonian:

$$\Rightarrow \mathcal{H}^L[u,f] = \text{cte} + \int d^d z \, u(z) \, f(z)$$

$$f(z) = \int d^m x \, \rho(x,z)|_{u=0} \underbrace{[-\partial_x V(x,z)]}_{}$$

$$\Rightarrow \mathcal{H}^L[u,f] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \left(c q^2 \tilde{u}_{-q} \tilde{u}_q + f_q \tilde{u}_{-q} + f_{-q} \tilde{u}_q \right)$$



- Gaussian local force, of zero mean and 2-pt correlation:

$$\begin{cases} \overline{f(z) f(z')} = \widetilde{D} \cdot \delta^{(d)}(z - z') \\ \widetilde{D} = \frac{D}{4\sqrt{\pi}\xi^3} \end{cases}$$

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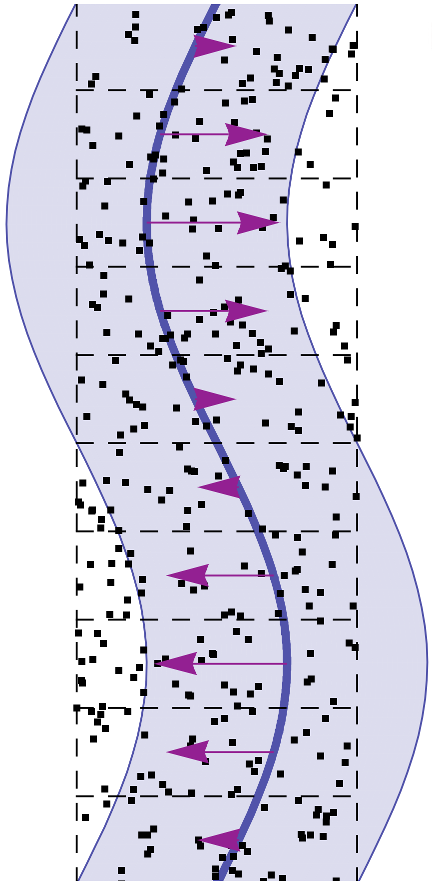
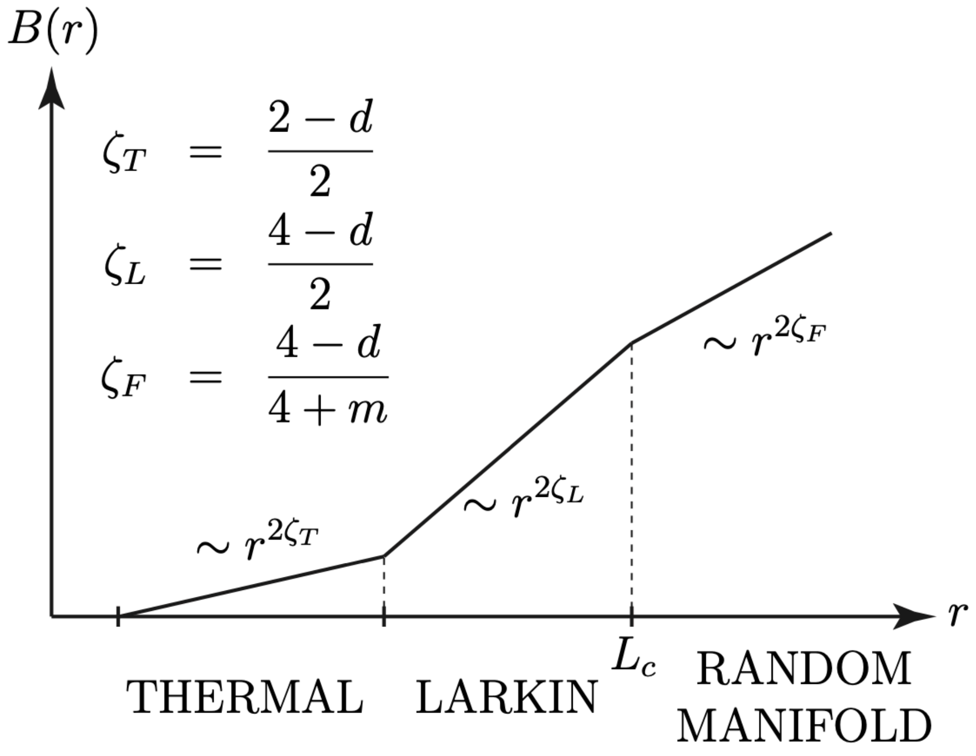


(a) Small distortion

- In Fourier space it can also be put in a quadratic form:

$$\begin{aligned}\mathcal{H}^L[u,f] &= \mathcal{H}_{\text{el}}[u]+\mathcal{H}^L_{\text{dis}}[u,f]\\ &= \frac{c}{2}\int_{\mathcal{D}_z} d^dz\,(\nabla u(z))^2+\int_{\mathcal{D}_z} d^dz\cdot u(z)f(z)\\ &= \frac{1}{2}\int_{(\mathcal{D}_z)_q}\frac{d^dq}{(2\pi)^d}\left(cq^2u_q^*u_q+f_qu_q^*+f_q^*u_q\right)\\ &= \frac{1}{2}\int_{(\mathcal{D}_z)_q}\frac{d^dq}{(2\pi)^d}cq^2\underbrace{\left(u_q^*+\frac{f_q^*}{cq^2}\right)}\underbrace{\left(u_q+\frac{f_q}{cq^2}\right)}-\frac{1}{2}\int_{(\mathcal{D}_z)_q}\frac{d^dq}{(2\pi)^d}\frac{f_q^*f}{cq^2}\end{aligned}$$

$$\overline{\langle u_q^*u_q\rangle}=\underbrace{\overline{\langle u_q^*u_q\rangle_{f\equiv 0}}}_{\overline{\text{cste}}=\text{cste}}+\frac{\overline{f_q^*f_q}}{(cq^2)^2}=\underbrace{\langle u_q^*u_q\rangle_{f\equiv 0}}_{\text{purely thermal}}+\underbrace{\frac{\overline{f_q^*f_q}}{(cq^2)^2}}_{\text{purely disorder}}$$



- Gaussian local force, of zero mean and 2-pt correlation:

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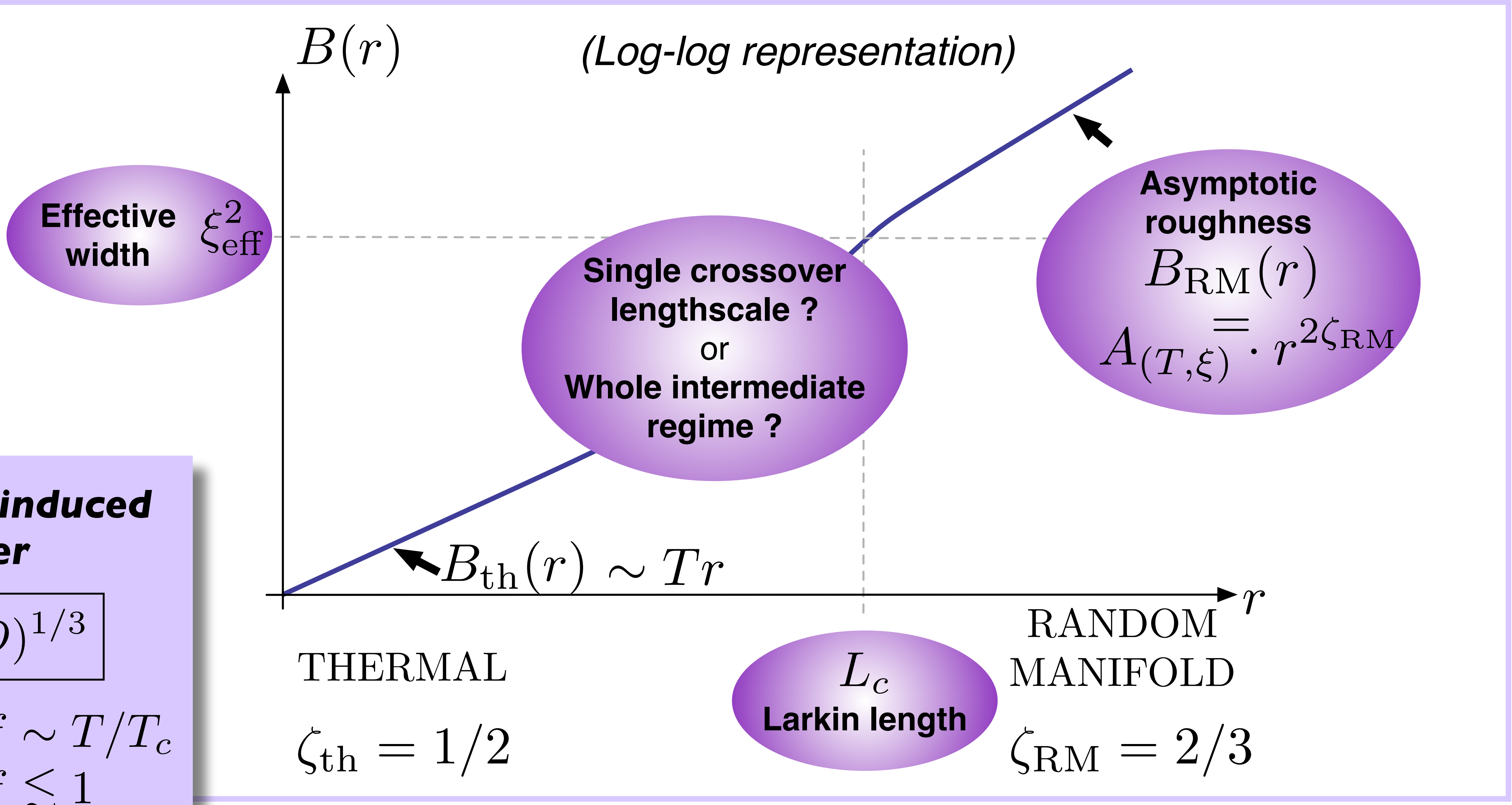
$$S(q)\sim\begin{cases}q^{-2}\Rightarrow B(r)\sim r\\ q^{-4}\Rightarrow B(r)\sim r^{4-d}\end{cases}$$

Static roughness regimes & characteristic crossover scales



Effective width ξ_{eff}^2

Single crossover lengthscale ?
or
Whole intermediate regime ?

Asymptotic roughness
$B_{\mathrm{RM}}(r) = A_{(T,\xi)} \cdot r^{2\zeta_{\mathrm{RM}}}$

L_c
Larkin length

$\zeta_{\mathrm{th}} = 1/2$

$\zeta_{\mathrm{RM}} = 2/3$

Temperature-induced crossover

$$\boxed{T_c = (\xi c D)^{1/3}}$$

$$\begin{cases} T \ll T_c : & f \sim T/T_c \\ T \gg T_c : & f \lesssim 1 \end{cases}$$

Universal scaling

KPZ scaling for the asymptotic roughness

$$B(r) \overset{(r\to\infty)}{\sim} \left(\frac{D}{cT/f}\right)^{2/3} r^{4/3}$$

Non-universal features

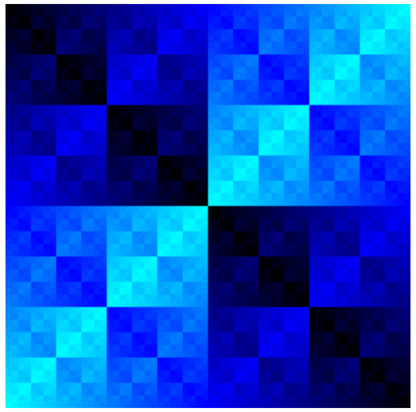
Roughness amplitude: $A_{(T,\xi)} \sim \left(\frac{D}{cT/f}\right)^{2/3}$

Larkin length: $L_c = \frac{(T/f)^5}{cD^2}$

With disorder: Gaussian Variational Method (GVM) roughness computations

- GVM in a nutshell

$$\begin{array}{c}\mathcal{H}\left[u,V\right]\\ \&\text{ Random }V\end{array} \xrightarrow{\text{Replicæ}} \begin{array}{c}\widetilde{\mathcal{H}}\left[u_1,\ldots,u_n\right]\\ \&\ \lim\limits_{n\to 0}\end{array} \overset{\text{GVM}}{\approx} \sum_{\substack{\text{Fourier}\\ \text{modes}}} \vec{u}^T$$



$$\vec{u}$$

- Statistical averages in statics

$$\overline{\langle \mathcal{O} \rangle} = \int \mathcal{D}V\, \bar{\mathcal{P}}[V] \int \mathcal{D}u_1\, \mathcal{O}[u_1]\, \frac{e^{-\frac{1}{T}\mathcal{H}[u_1,V]}}{Z_V}$$

$$\boxed{\frac{1}{Z_V} = \lim_{n\to 0} \frac{Z_V^n}{Z_V} = \lim_{n\to 0} Z_V^{n-1}}$$

- With replicas:

$$\overline{\langle \mathcal{O} \rangle} = \lim_{n\to 0} \int \mathcal{D}u_1(z)(\cdots)u_n(z)\, \mathcal{O}[u_1]\, \exp\left\{-\frac{1}{T}\widetilde{\mathcal{H}}[u_1,\ldots,u_n]\right\}$$

GVM = recipe
to find the best quadratic
replicated Hamiltonian $\mathcal{H}_0$

$$\boxed{e^{-\frac{1}{T}\widetilde{\mathcal{H}}} \overset{\text{(GVM)}}{\approx} e^{-\frac{1}{T}\mathcal{H}_0}}$$

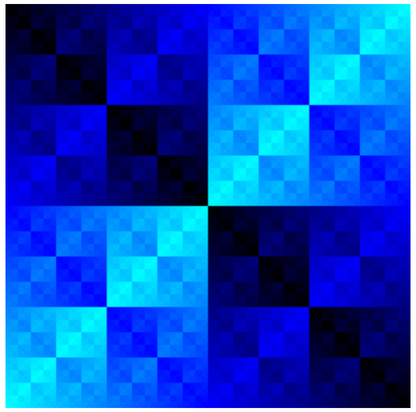
- Our case:

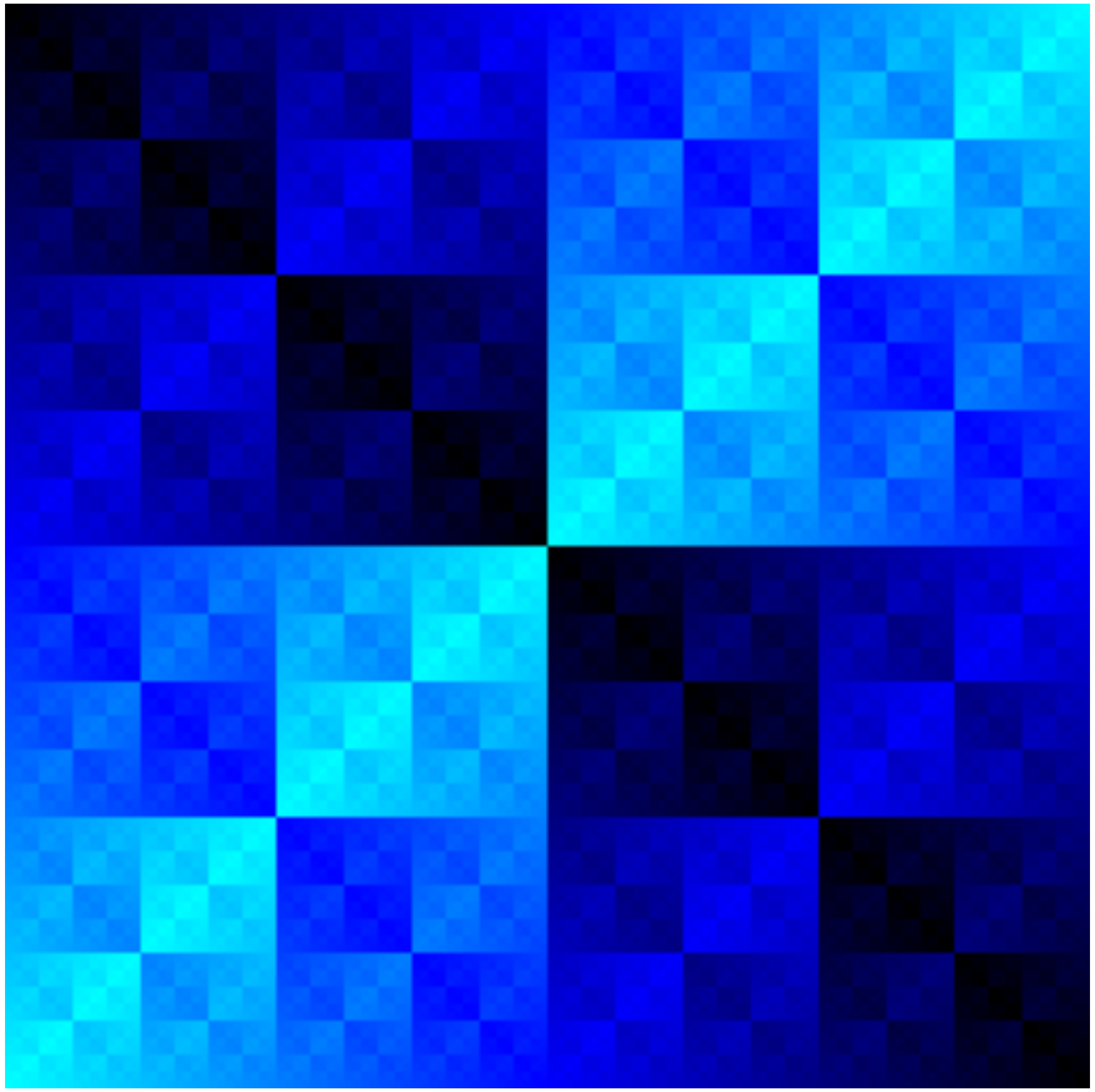
$$\widetilde{\mathcal{H}}[u_1(z),\ldots,u_n(z);L] = \int_0^L dz \frac{c}{2}\sum_{a=1}^{n}(\partial_z u_a(z))^2 - \frac{D}{T}\int_0^L dz \sum_{a,b=1}^{n} R_\xi(u_a(z)-u_b(z))$$

With disorder: Gaussian Variational Method (GVM) roughness computations

- GVM in a nutshell

$$\begin{matrix}\mathcal{H}\left[u,V\right] \\ \text{\& Random } V\end{matrix} \xrightarrow{\text{Replicæ}} \begin{matrix}\widetilde{\mathcal{H}}\left[u_1,\dots,u_n\right] \\ \text{\& } \lim\limits_{n\to 0}\end{matrix} \overset{\text{GVM}}{\approx} \sum_{\substack{\text{Fourier} \\ \text{modes}}} \vec{u}^T \; [\;\cdot\;] \; \vec{u}$$





- **Hierarchical matrices:**

  invariance upon permutation of replica indices
  ⇒ every line/column with reshuffled coefficients

  Example here: color code = coefficient values
  (continuous blend on the first line/column)

- **Algebra of inverting such $n \times n$ matrices in limit $n \to 0$:**

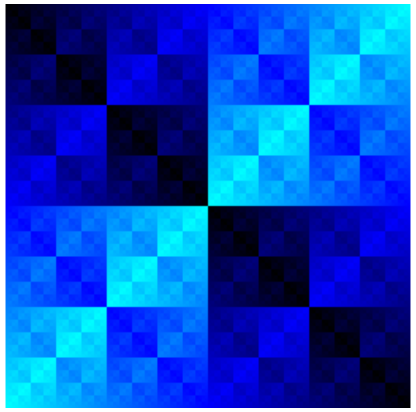
  M. Mézard & G. Parisi, *J. Phys. I* 1, 809 (1991) [Append. II]

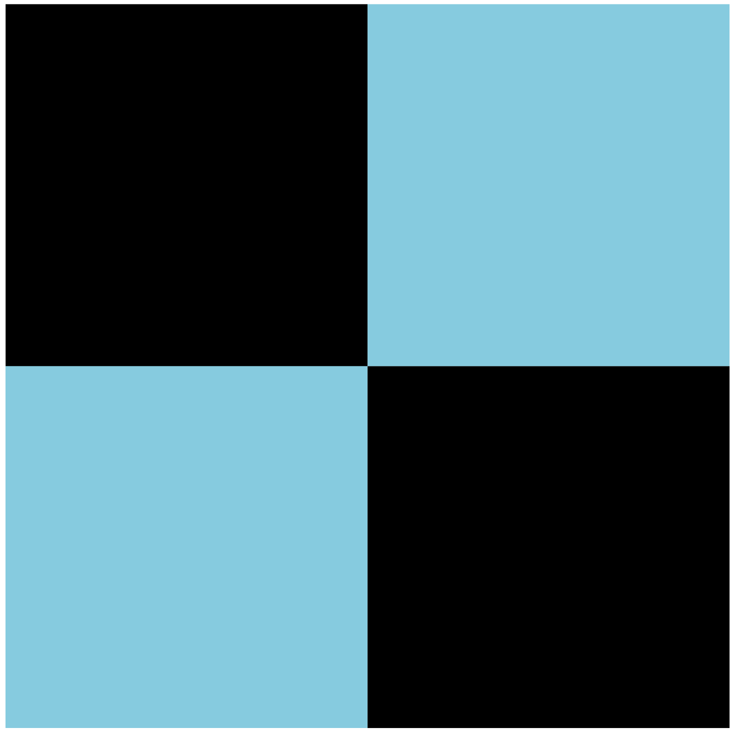
  E. Agoritsas, V. Lecomte, T. Giamarchi, *Phys. Rev. B* 82, 184207 (2010) [Appendix B]

With disorder: Gaussian Variational Method (GVM) roughness computations

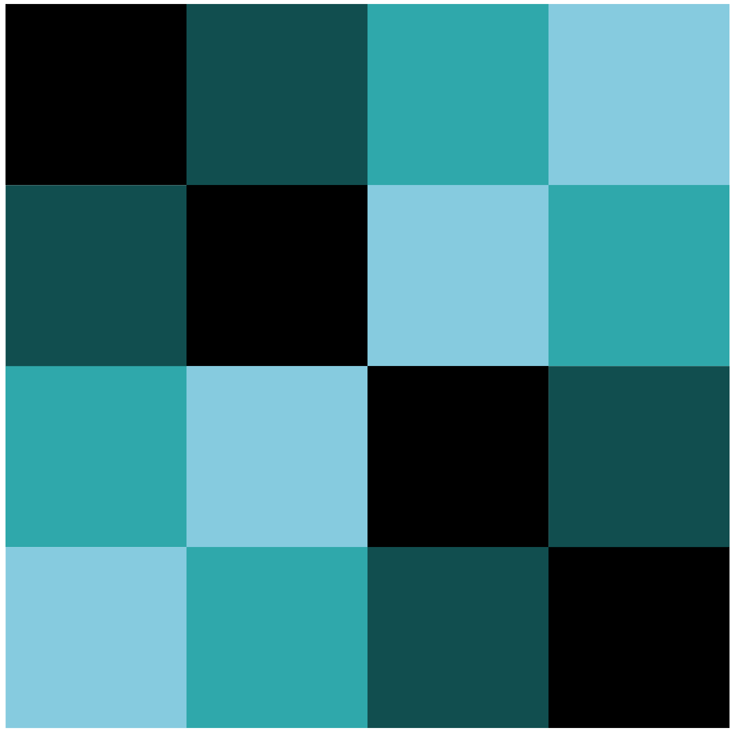
- GVM in a nutshell

$$\begin{array}{c}\mathcal{H}\left[u,V\right]\\ \&\ \text{Random } V\end{array} \overset{\text{Replicæ}}{\longrightarrow} \begin{array}{c}\widetilde{\mathcal{H}}\left[u_1,\ldots,u_n\right]\\ \&\ \lim\limits_{n\to 0}\end{array} \overset{\text{GVM}}{\approx} \sum_{\substack{\text{Fourier}\\ \text{modes}}} \vec{u}^T \;\cdots\; \vec{u}$$

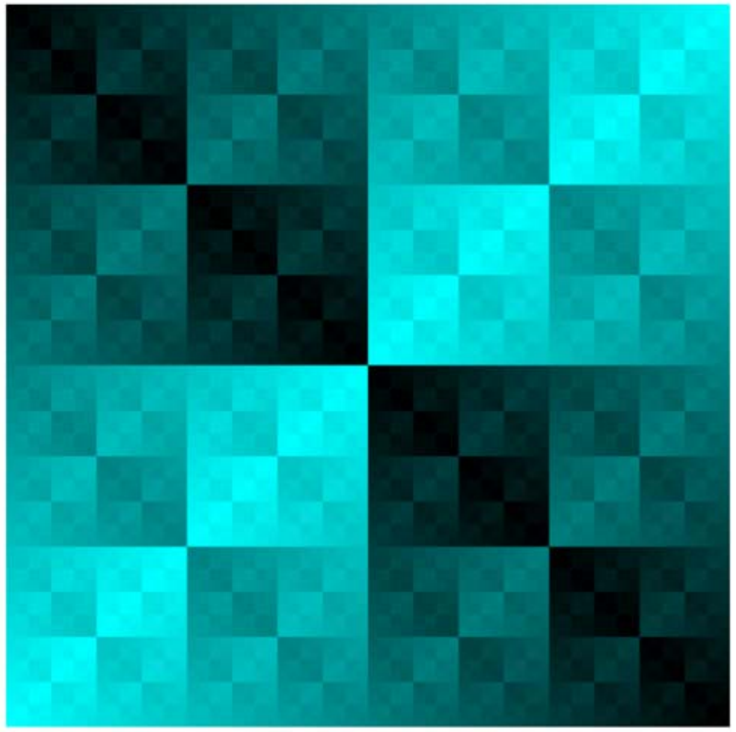




1-Replica-symmetry-breaking



3-RSB



full-RSB

With disorder: Gaussian Variational Method (GVM) roughness computations

- '0x0' inversion formulas for **Replica-Symmetric (RS) Ansatz**

$$\widehat{G}^{-1} = \begin{pmatrix} \widetilde{G}^{-1} & & G^{-1} \\ & \ddots & \\ G^{-1} & & \widetilde{G}^{-1} \end{pmatrix} \Longrightarrow \widehat{G} = \begin{pmatrix} \widetilde{G} & & G \\ & \ddots & \\ G & & \widetilde{G} \end{pmatrix}$$

$$\boxed{\begin{aligned} G_c^{-1} &\equiv \widetilde{G}^{-1} + (n-1)\,G^{-1} \overset{(n\to 0)}{=} \widetilde{G}^{-1} - G^{-1} \\ G_c &\equiv \widetilde{G} + (n-1)\,G \overset{(n\to 0)}{=} \widetilde{G} - G \end{aligned}}$$

$$G_c \cdot G_c^{-1} = 1\,, \quad G = -\frac{G^{-1}}{(G_c^{-1})^2}\,, \quad \widetilde{G} = \frac{\widetilde{G}^{-1} - 2G^{-1}}{(G_c^{-1})^2}$$

- **Illustration on the Larkin model**: quadratic, hence exactly RS in its replicated formulation.

$$\overline{\exp\left(-\beta\sum_i \mathcal{H}^L\left[u_i, f\right]\right)} = \exp\left(-\beta\sum_i \mathcal{H}_{\text{el}}\left[u_i\right]\right)\cdot\overline{\exp\left(-\beta\sum_i \mathcal{H}^L_{\text{dis}}\left[u_i, f\right]\right)}$$

$$\Rightarrow \widetilde{\mathcal{H}}^L\left[\vec{u}\right] = \frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\cdot\vec{u}_q^{*\mathrm{T}}\left(cq^2\mathbb{I}_n - \beta D\begin{pmatrix} 1 & - & 1 \\ | & \diagdown & | \\ 1 & - & 1 \end{pmatrix}\right)\vec{u}_q$$

$$S(q) = \lim_{n\to 0} T\widetilde{G}(q)$$

$$\left\{\begin{array}{rcl} \widetilde{G}^{-1} & = & cq^2 - \beta D \\ G^{-1} & = & -\beta D \\ G_c^{-1} & = & cq^2 \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} G_c & = & \frac{1}{cq^2} \\ G & = & \frac{\beta D}{(cq^2)^2} \\ \widetilde{G} & = & \frac{1}{cq^2} + \frac{\beta D}{(cq^2)^2} \end{array}\right.$$

$$\boxed{\overline{\langle u_q^* u_q \rangle} = (2\pi)^d\cdot\left(\frac{T}{cq^2} + \frac{D}{(cq^2)^2}\right)}$$

With disorder: Gaussian Variational Method (GVM) roughness computations

- '0x0' inversion formulas and GVM roughness for **full-RSB Ansatz** (more general case):

$$\widehat{G}^{-1}(q) = \begin{pmatrix} G_c^{-1} - \tilde{\sigma} & & -\sigma(u) \\ & \ddots & \\ -\sigma(u) & & G_c^{-1} - \tilde{\sigma} \end{pmatrix} \Longrightarrow \widehat{G}(q) = \begin{pmatrix} \widetilde{G}(q) & & G(q,u) \\ & \ddots & \\ G(q,u) & & \widetilde{G}(q) \end{pmatrix}$$

$$[\sigma](v) \equiv v \cdot \sigma(v) - \int_0^v dw \cdot \sigma(w)$$

$$\begin{aligned} G(u) &= \frac{1}{G_c^{-1}} \left(\frac{1}{u} \cdot \frac{[\sigma](u)}{G_c^{-1} + [\sigma](u)} + \int_0^u \frac{dv}{v^2} \frac{[\sigma](v)}{G_c^{-1} + [\sigma](v)} + \frac{\sigma(0)}{G_c^{-1}} \right) \\ \widetilde{G} &= \frac{1}{G_c^{-1}} \left(1 + \int_0^1 \frac{dv}{v^2} \cdot \frac{[\sigma](v)}{G_c^{-1} + [\sigma](v)} + \frac{\sigma(0)}{G_c^{-1}} \right) \\ \widetilde{G} - G(u) &= \frac{1}{u} \cdot \frac{1}{G_c^{-1} + [\sigma](u)} - \int_u^1 \frac{dv}{v^2} \cdot \frac{1}{G_c^{-1} + [\sigma](v)} \\ \widetilde{G} - G(u) &= \frac{1}{G_c^{-1} + [\sigma](1)} + \int_u^1 dv \cdot \frac{\sigma'(v)}{\left(G_c^{-1} + [\sigma](v)\right)^2} \end{aligned}$$

Conventions from: E. Agoritsas, V. Lecomte, & T. Giarmachi, *Phys. Rev. B* 82, 184207 (2010)
Adapted from: M. Mézard & G. Parisi, *J. Phys. I* 1, 809 (1991)

With disorder: Gaussian Variational Method (GVM) roughness computations

- What is the relation between the Flory exponents/crossovers & GVM predictions?
  - GVM doomed to predict the Flory exponent $B_{GVM}(r) \overset{(r\to\infty)}{\sim} r^{2\zeta_F}?$
  - Crossover scales consistent with scaling candidates e.g. $L_c = \frac{(T/f)^5}{cD^2}$
  - GVM interpolates the temperature crossover $\left\{ \begin{array}{l} f^6 \propto (T/T_c)^6(1-f) \\ T_c = (\xi c D)^{1/3} \end{array} \right.$

- Some previous GVM studies on random manifolds:

M. Mézard & G. Parisi, *J. Phys. I* 1, 809 (1991)
M. Mézard & G. Parisi, *J. Phys. I* 2, 2231 (1992)

$\boxed{R(x) \sim x^{(1-\gamma)}}$

J.-P. Bouchaud ,M. Mézard & G. Parisi, *Phys. Rev. E* 52, 3656 (1995)

$\boxed{R_\xi(x) = \xi^{-1} R_1(x/\xi)}$ $\boxed{d > 2}$

T. Giamarchi & P. Le Doussal, *Phys. Rev. B* 52, 1242 (1995) **(periodic systems)**

- Our GVM studies on static 1D interface / 1+1 directed polymer:

$\boxed{R_\xi(x) = \xi^{-1} R_1(x/\xi)}$

E. Agoritsas, V. Lecomte, & T. Giarmachi, *Phys. Rev. B* 82, 184207 (2010)
E. Agoritsas, S. Bustingorry, V. Lecomte, G. Schehr, & T. Giamarchi, *Phys. Rev. E* 86, 031144 (2012)
E. Agoritsas & V. Lecomte, *J. Phys. A* 50, 104001 (2017)

Non-commutable limits: $T \to 0, \xi \to 0, L \to \infty$

With disorder: Gaussian Variational Method (GVM) roughness computations

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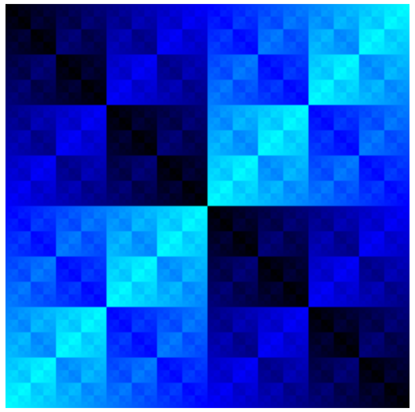
$$\begin{matrix} \mathcal{H}\left[u, V\right] \\ \text{\& Random } V \end{matrix} \overset{\text{Replicæ}}{\to} \begin{matrix} \widetilde{\mathcal{H}}\left[u_1, \ldots, u_n\right] \\ \text{\& } \lim\limits_{n\to 0} \end{matrix} \overset{\text{GVM}}{\approx} \sum_{\substack{\text{Fourier} \\ \text{modes}}} \vec{u}^T$$



$$\vec{u}$$

$$\widetilde{\mathcal{H}}[u_1(z), \ldots, u_n(z); L] = \int_0^L dz \frac{c}{2} \sum_{a=1}^{n} (\partial_z u_a(z))^2 - \frac{D}{T} \int_0^L dz \sum_{a,b=1}^{n} R_\xi(u_a(z) - u_b(z))$$

$$B(r) = \lim_{n\to 0} \int \mathcal{D}u_1(z)(\cdots)\mathcal{D}u_n(z) \left[u_1(r) - u_1(0)\right]^2 e^{-\frac{1}{T}\widetilde{\mathcal{H}}[u_1,\ldots,u_n,L]}$$

$$\boxed{e^{-\frac{1}{T}\widetilde{\mathcal{H}}} \overset{\text{(GVM)}}{\approx} e^{-\frac{1}{T}\mathcal{H}_0}}$$

With disorder: Gaussian Variational Method (GVM) roughness computations

- What is the relation between the Flory exponents/crossovers & GVM predictions?
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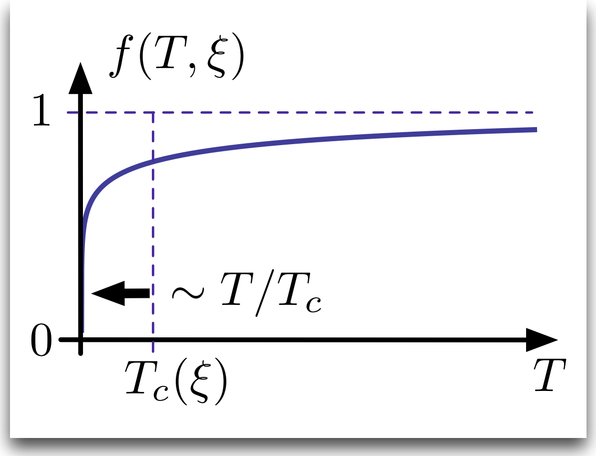
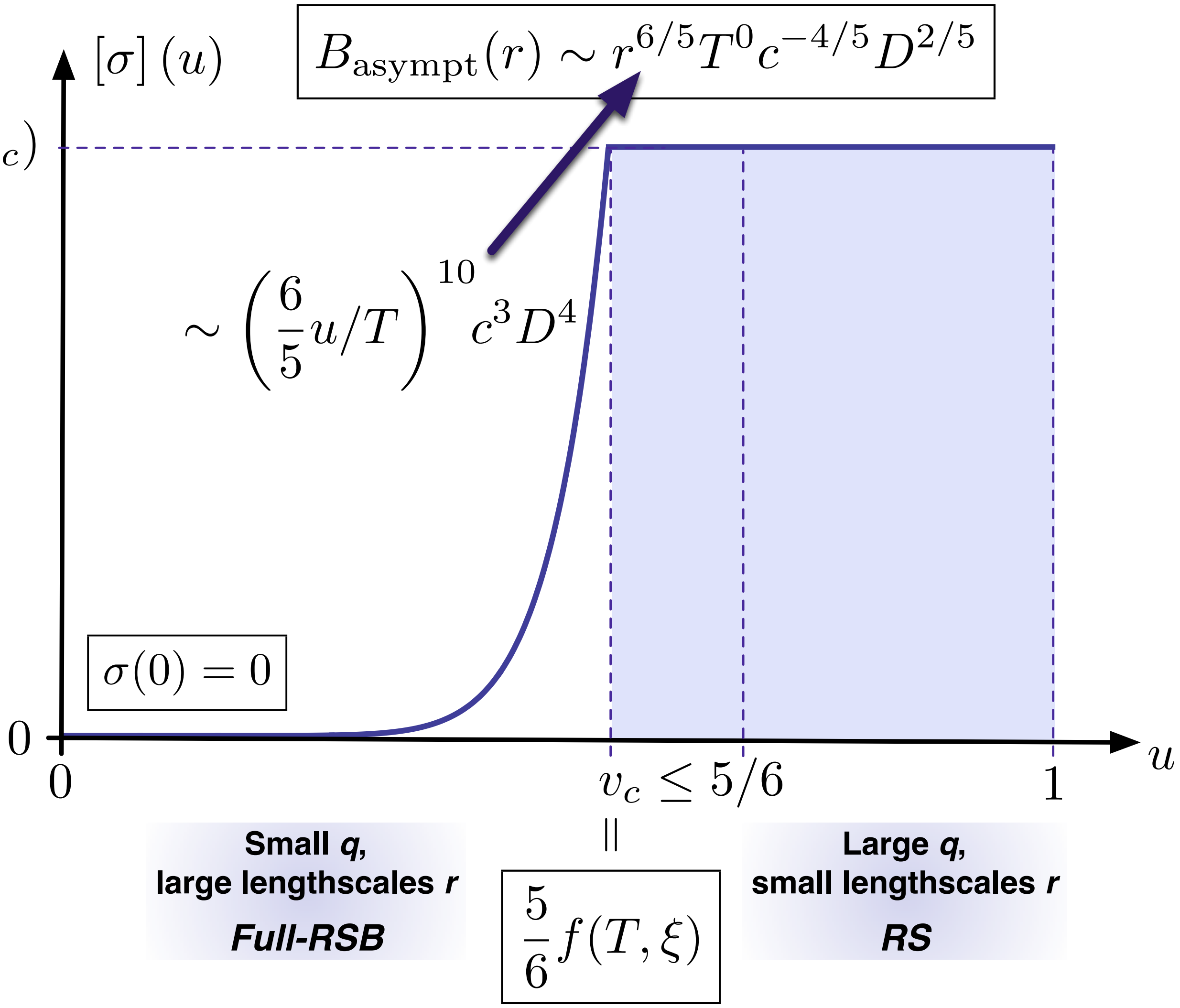
- GVM on the Hamiltonian of a 1D interface

$$B(r) = \int \frac{d^dq}{(2\pi)^d} 2\left[1-\cos(qr)\right] S(q)$$

$$S(q) = \underset{\overset{\uparrow}{B_{\text{th}}(t)}}{\frac{1}{cq^2}} + \underset{\overset{\uparrow}{B_{\text{dis}}(t)}}{\frac{1}{cq^2}} \int_0^1 \frac{du}{u^2}\frac{[\sigma](u)}{cq^2+[\sigma](u)}$$



$\boxed{c/L_c(T,\xi)^2} = [\sigma]\,(v_c)$



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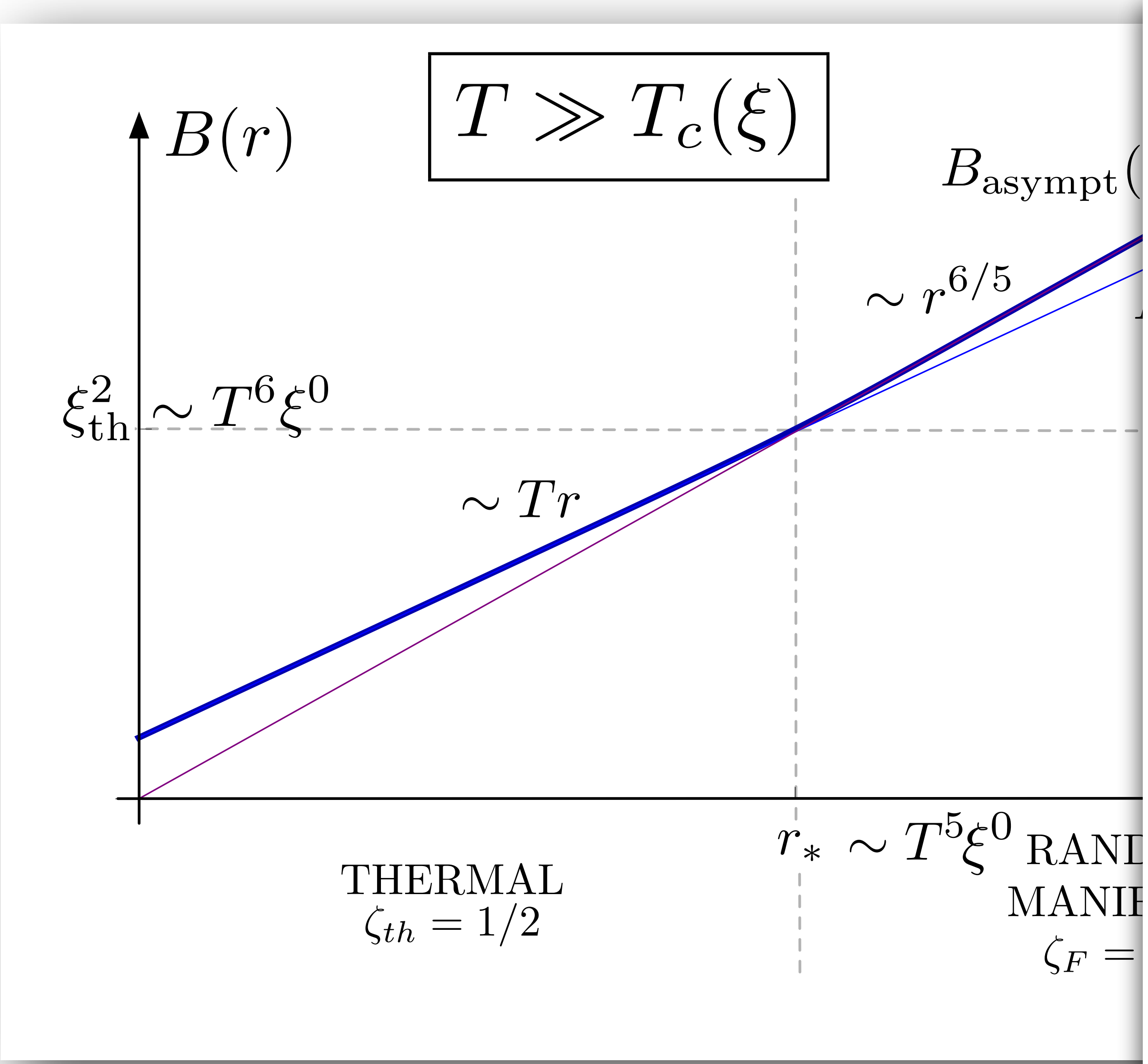

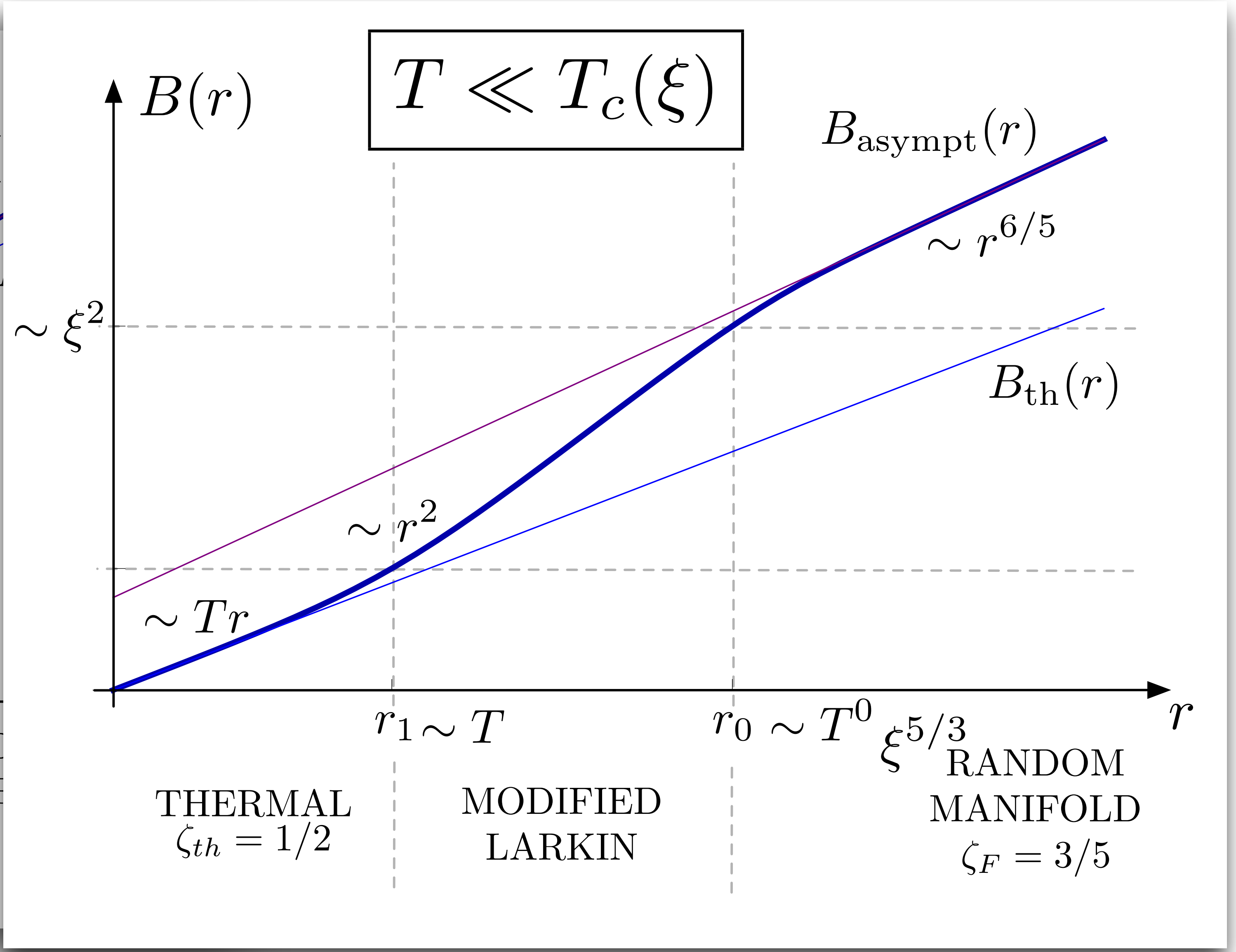

With disorder: Gaussian Variational Method (GVM) roughness computations

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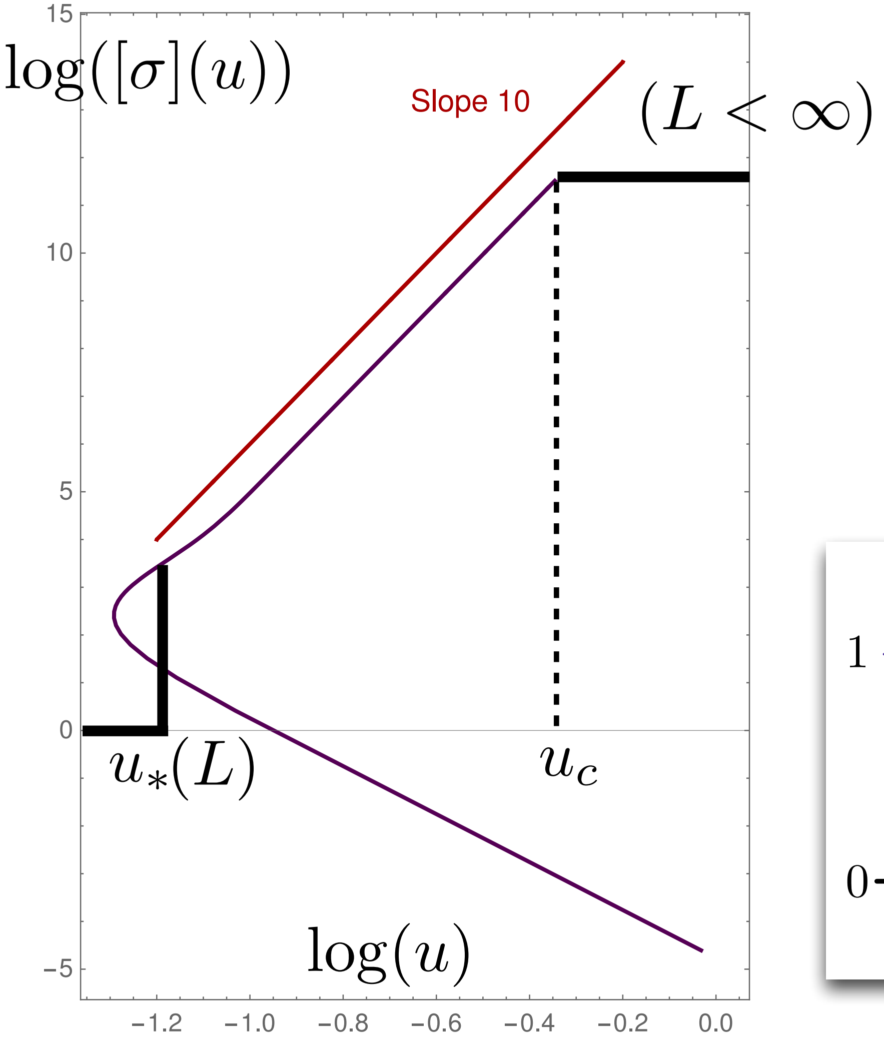
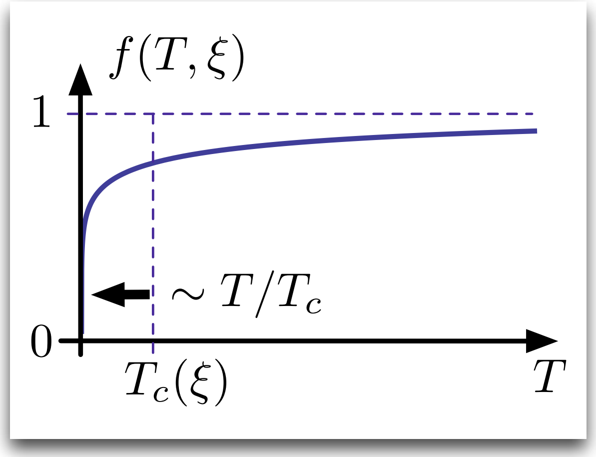
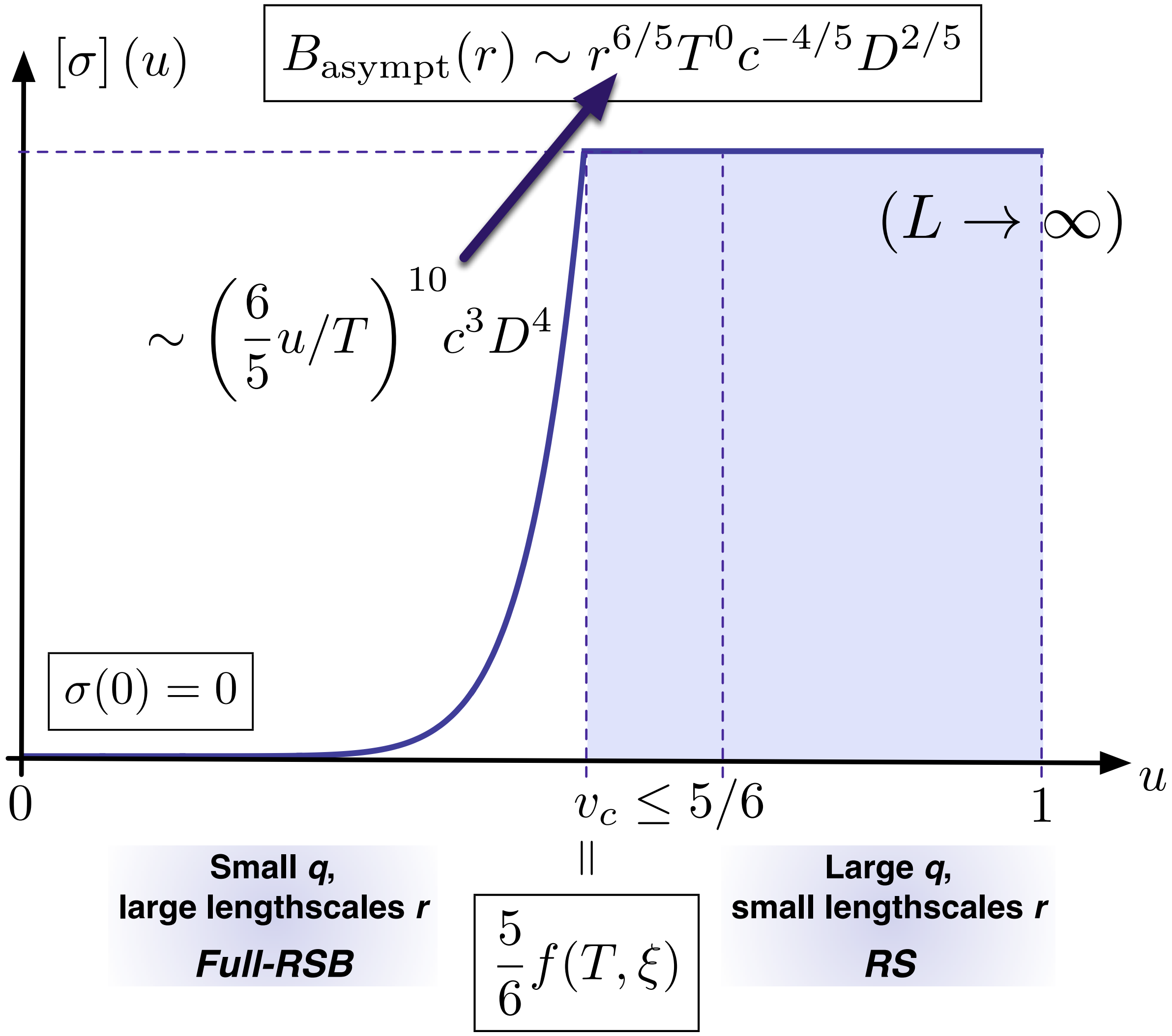
GVM on the Hamiltonian of a 1D interface

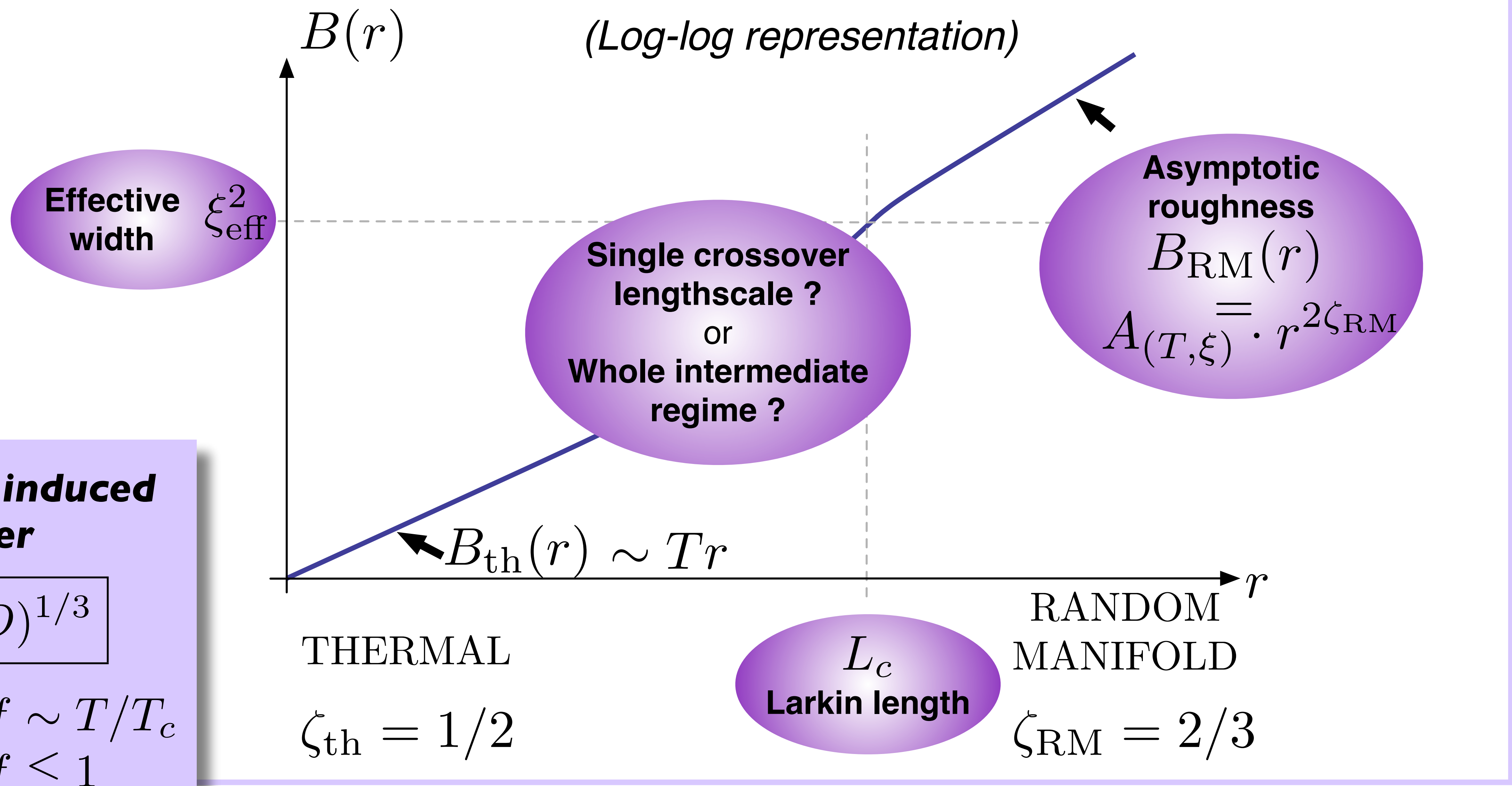




$\boxed{c/L_c(T,\xi)^2} = [\sigma](v_c)$



Roughness regimes & characteristic crossover scales



$B(r)$

(Log-log representation)

Effective width ξ_{eff}^2

Single crossover lengthscale ?
or
Whole intermediate regime ?

Asymptotic roughness
$B_{\text{RM}}(r) = A_{(T,\xi)} \cdot r^{2\zeta_{\text{RM}}}$

$B_{\text{th}}(r) \sim Tr$

r

THERMAL

$\zeta_{\text{th}} = 1/2$

L_c
Larkin length

RANDOM MANIFOLD

$\zeta_{\text{RM}} = 2/3$

Temperature-induced crossover

$$\boxed{T_c = (\xi c D)^{1/3}}$$

$$\begin{cases} T \ll T_c : & f \sim T/T_c \\ T \gg T_c : & f \lesssim 1 \end{cases}$$

- The GVM roughness provides us with
  - a qualitative understanding of the different regimes as a function of the lengthscale
  - the prediction of a temperature-induced crossover, and its associated crossover scales
- Remarkably, the crossover scalings are the same from scaling analysis, GVM computations, perturbative RG computations, and other results obtained from the mapping to the 1+1 DP & KPZ!
- The GVM predicts the standard Flory exponent 3/5 instead of 2/3, but can be corrected at $L < 0$!