Doctoral Training School *Kardar-Parisi-Zhang equation: new trends in theories and experiments*
April 15-26, 2024 — Ecole de Physique des Houches (France)

Interfaces in disordered systems and directed polymer

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Elisabeth Agoritsas

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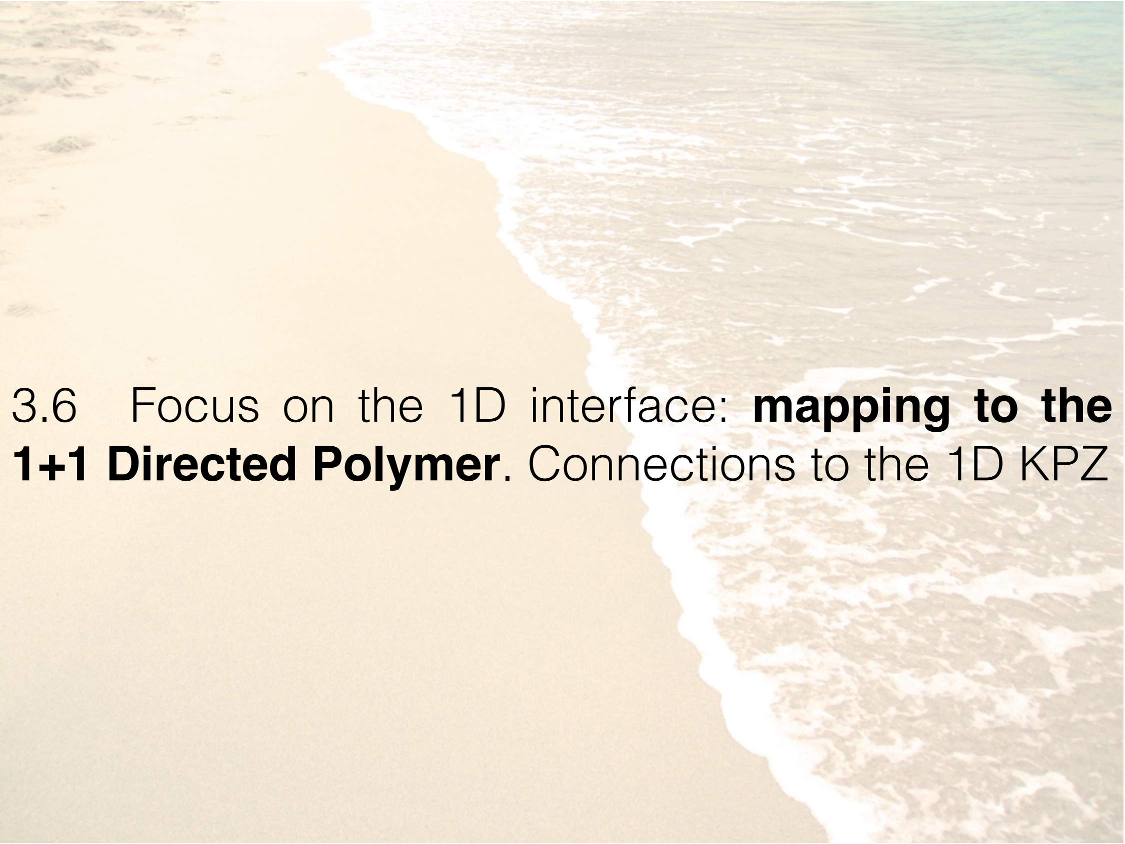
Interfaces in disordered systems and directed polymer

Elisabeth Agoritsas

- 1. Introduction
- 2. Disordered elastic systems: Recipe
- 3. Disordered elastic systems: Statics
- 4. Disordered elastic systems: Dynamics
- 5. Concluding remarks



Related bibliography

Beyond universal scalings in disordered systems:
Case study of the 1D interface in short-range correlated disorder

(i.e.1D KPZ equation **with spatially-correlated noise** & 'sharp-wedge' initial condition)

E. Agoritsas, V. Lecomte, & T. Giamarchi, *Phys. Rev. B* 82, 184207 (2010)
"Temperature-induced crossovers in the static roughness of a one-dimensional interface" **[GVM roughness]**

E. Agoritsas, S. Bustingorry, V. Lecomte, G. Schehr, & T. Giamarchi, *Phys. Rev. E* 86, 031144 (2012)
"Finite-temperature and finite-time scaling of the directed polymer free-energy with respect to its geometrical fluctuations"

E. Agoritsas, V. Lecomte, & T. Giamarchi, *Phys. Rev. E* 87, 042406 (2013)
"Static fluctuations of a thick one-dimensional interface in the 1+1 directed polymer formulation"

E. Agoritsas, V. Lecomte, & T. Giamarchi, *Phys. Rev. E* 87, 062405 (2013)
"Static fluctuations of a thick one-dimensional interface in the 1+1 directed polymer formulation: Numerical study"

E. Agoritsas, PhD thesis, University of Geneva (2013) [http://archive-ouverte.unige.ch/unige:30031]
"Temperature-dependence of a 1D Interface Fluctuations: Role of a Finite Disorder Correlation Length"

E. Agoritsas*, R. García-García*, V. Lecomte, L. Truskinovsky, & D. Vandembroucq, *J. Stat. Phys.* 164, 1394 (2016)
"Driven Interfaces: From Flow to Creep Through Model Reduction"

E. Agoritsas & V. Lecomte, *J. Phys. A* 50, 104001 (2017)
"Power countings versus physical scalings in disordered elastic systems - Case study of the one-dimensional interface"

S. Mathey, E. Agoritsas, T. Kloss, V. Lecomte, & L. Canet, *Phys. Rev. E* 95, 032117 (2017)
"Kardar-Parisi-Zhang equation with short-range correlated noise: Emergent symmetries and nonuniversal observables"

N. Caballero, E. Agoritsas, V. Lecomte, & T. Giamarchi, *Phys. Rev. B* 102, 104204 (2020)
"From bulk descriptions to emergent interfaces: Connecting the Ginzburg-Landau and elastic-line models"

N. Caballero, T. Giamarchi, V. Lecomte, & E. Agoritsas, *Phys. Rev. E* 105, 044138 (2022)
"Microscopic interplay of temperature and disorder of a one-dimensional elastic interface"

Model: 1D interface with finite width / short-range correlated disorder

- Short-range elasticity & Elastic limit / Quenched random-bond weak disorder

$$\boxed{\mathcal{H}_{\mathrm{DES}} = \mathcal{H}_{\mathrm{el}} + \mathcal{H}_{\mathrm{dis}}}$$

Hamiltonian: $$\mathcal{H}[u,\widetilde{V}] = \int_{\mathbb{R}} dz \cdot \left[\underbrace{\frac{c}{2}(\nabla_z u(z))^2}_{\text{Elasticity}} + \underbrace{\int_{\mathbb{R}} dx \cdot \rho_\xi(x-u(z))\,\widetilde{V}(z,x)}_{\text{Effective random potential } V(z,u(z))} \right]$$

- Density $\rho_\xi(x-u(z))$ & random potential $\widetilde{V}(z,x)$

$$\overline{\widetilde{V}(z,x)} = 0$$

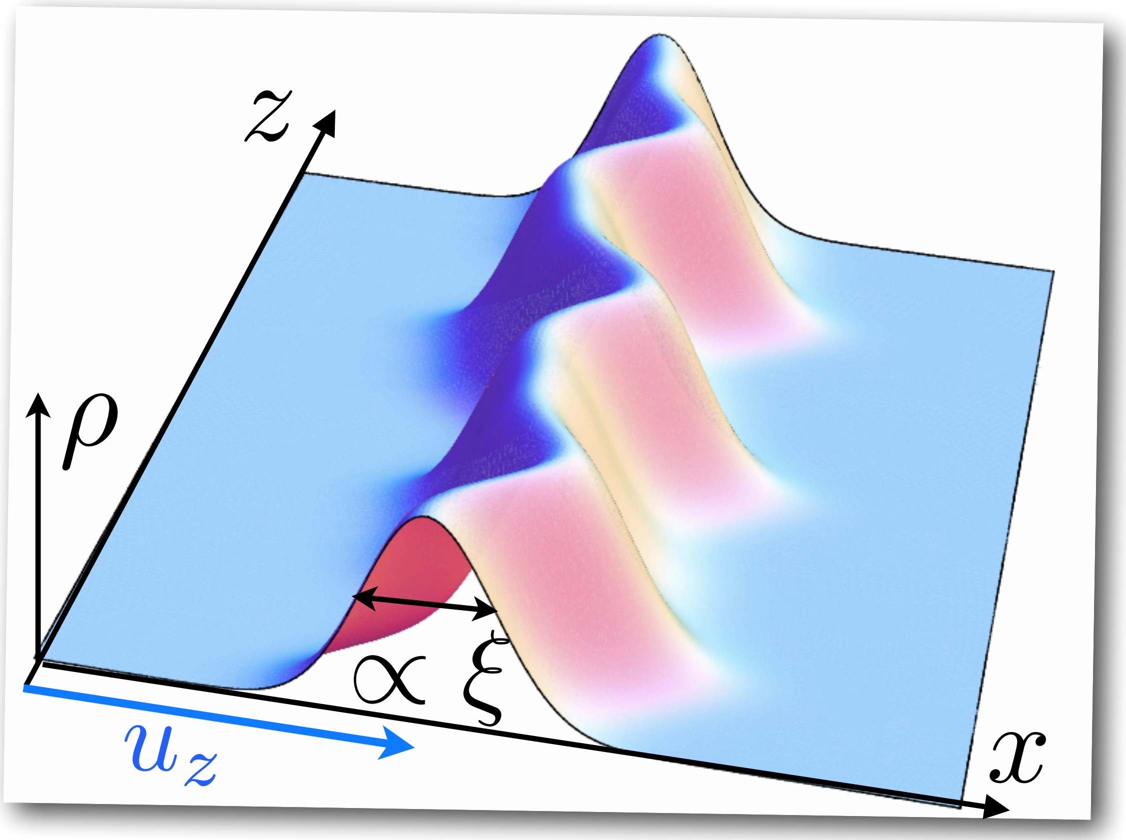
$$\overline{\widetilde{V}(z,x)\widetilde{V}(z',x')} = D\,\delta(z-z')\delta(x-x')$$

- Alternative: ***correlated*** effective potential $V(z,u(z))$

$$\boxed{\overline{V(z,x)V(z',x')} = D\,\delta(z-z')R_\xi(x-x')}$$

- Exponentially decaying with following scaling:

$$\boxed{R_\xi(x) = \xi^{-1}R_1(x/\xi)} \quad \text{e.g.} \quad R_\xi^{\mathrm{G}}(x) = \frac{e^{-x^2/(2\xi^2)}}{\sqrt{2\pi}\xi}$$



Elastic constant c / Width ξ / Disorder strength D / Temperature T

- Overdamped dynamics: 'quenched Edwards-Wilkinson':

$$\gamma\,\partial_t u(z,t) = c\partial_z^2 u(z,t) + F_{\mathrm{dis}}(z,u(z)) + f_{\mathrm{ext}} + \eta_{\mathrm{thermal}}(z,t)$$

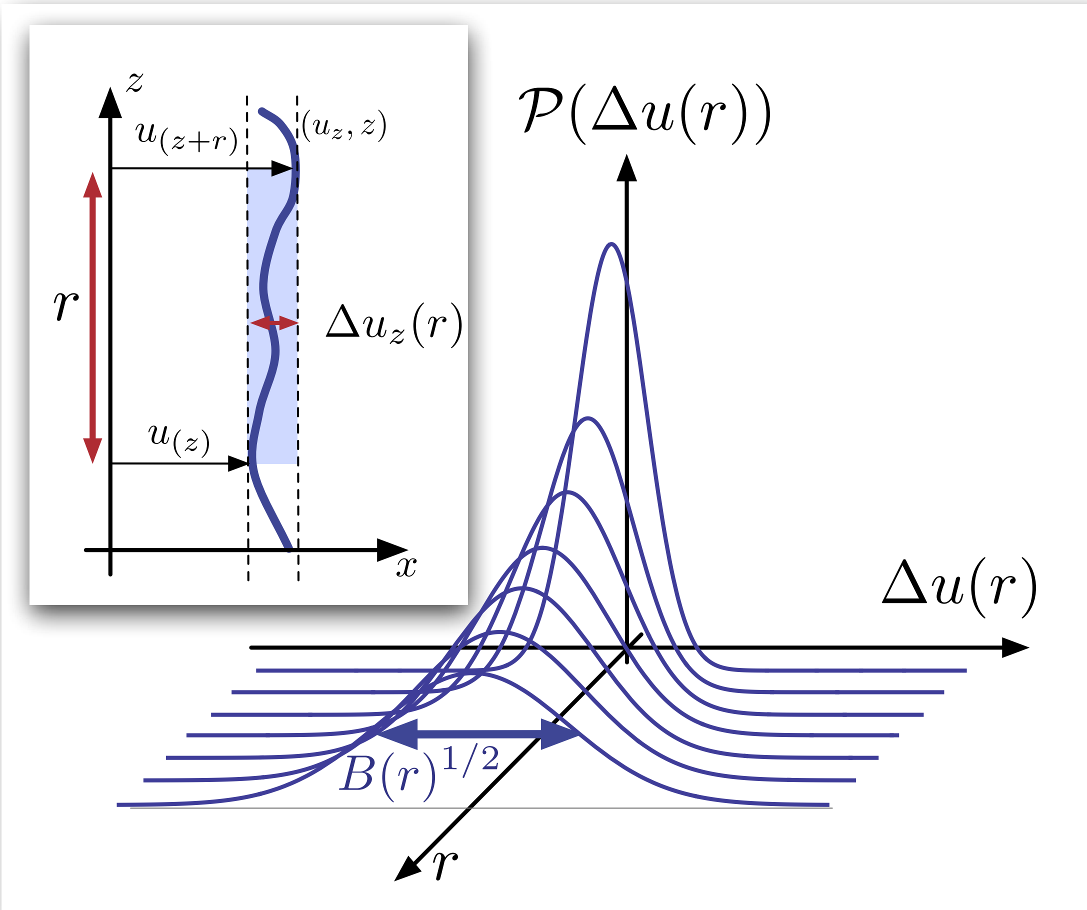
$$\langle \eta_{\mathrm{th}}(z,t)\eta_{\mathrm{th}}(z',t')\rangle = 2\gamma T\delta(z-z')\delta(t-t')$$

$$F_{\mathrm{dis}}(z,x) = -\partial_x V(z,x)$$

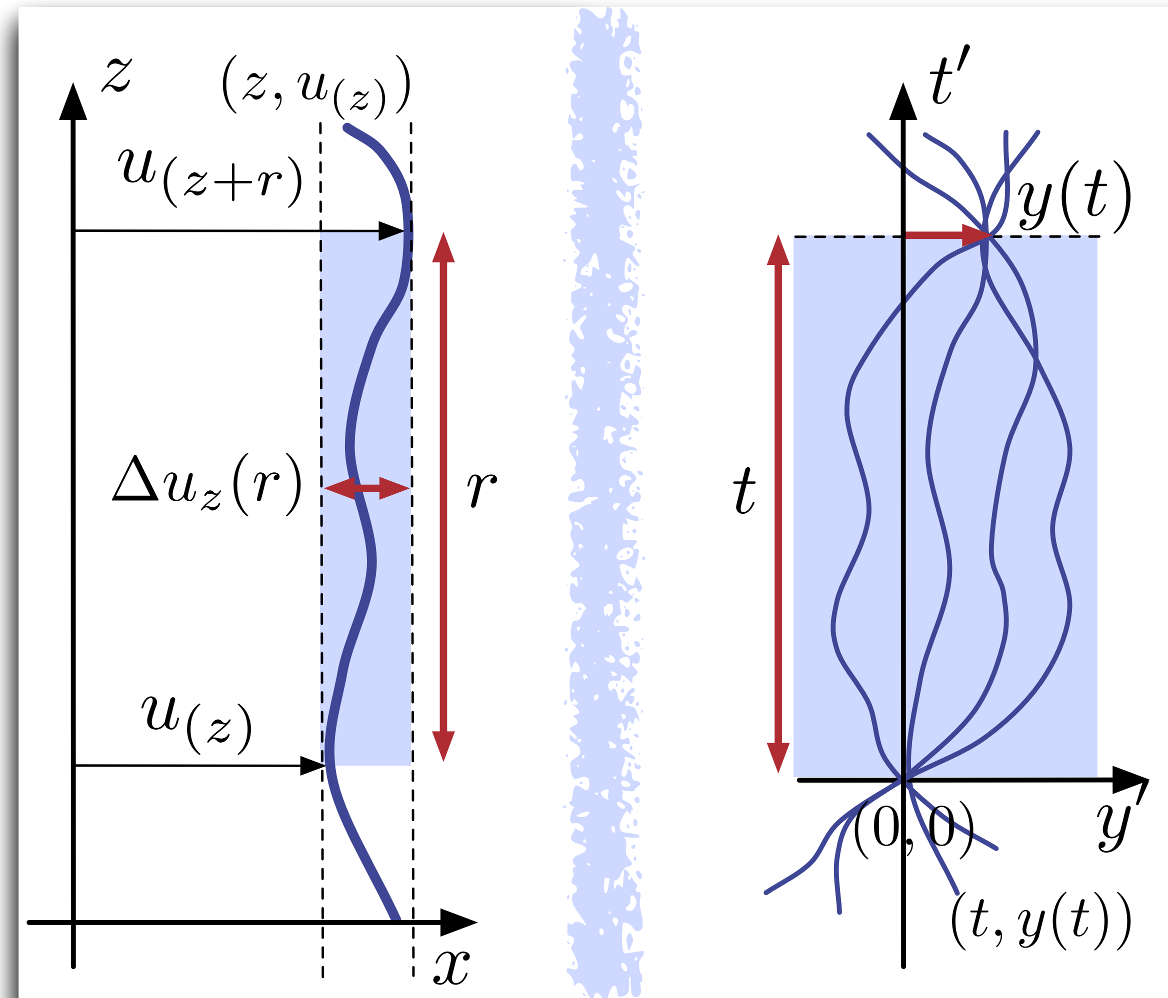
Model: static 1D interface & growing 1+1 directed polymer (DP)

- Observable: static geometrical fluctuations $\mathcal{P}(\Delta u(r))$

  & roughness $B(r) = \overline{\langle \Delta u(r)^2 \rangle} \sim A\, r^{2\zeta}$



- **Effective disorder** experienced by the 1D interface at a given lengthscale ↔ at fixed growing DP 'time'



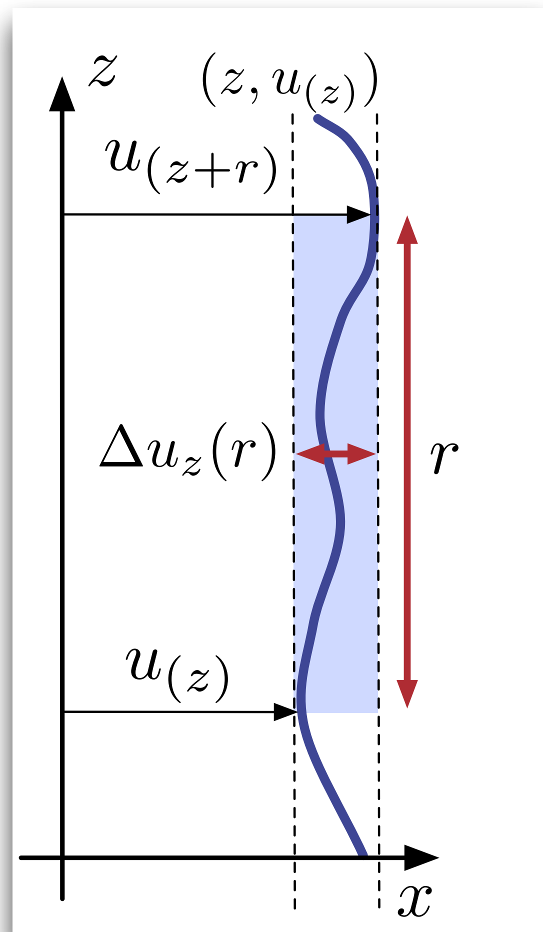
1D interface Lengthscale

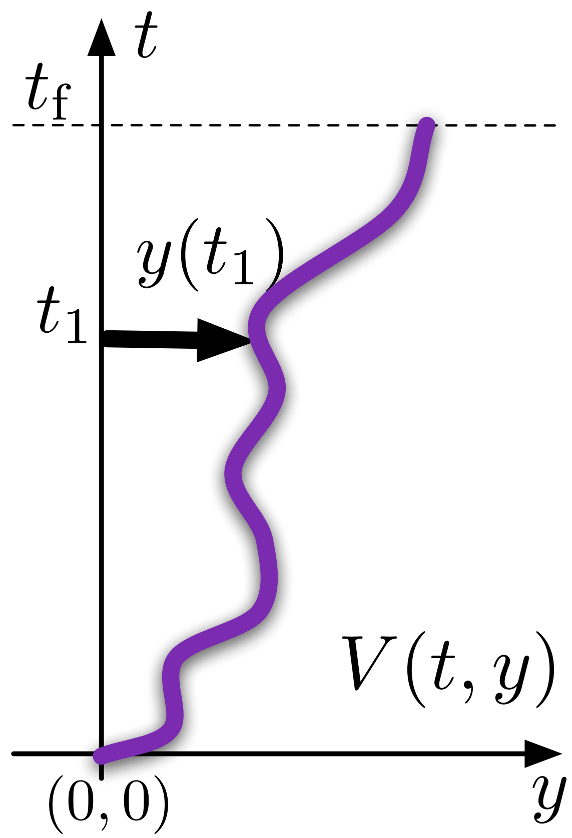
Directed Polymer Growing 'time'

Model: static 1D interface & growing 1+1 directed polymer (DP)

- Hamiltonian of the static 1D interface

$$\mathcal{H} = \mathcal{H}_{\rm el} + \mathcal{H}_{\rm dis}$$



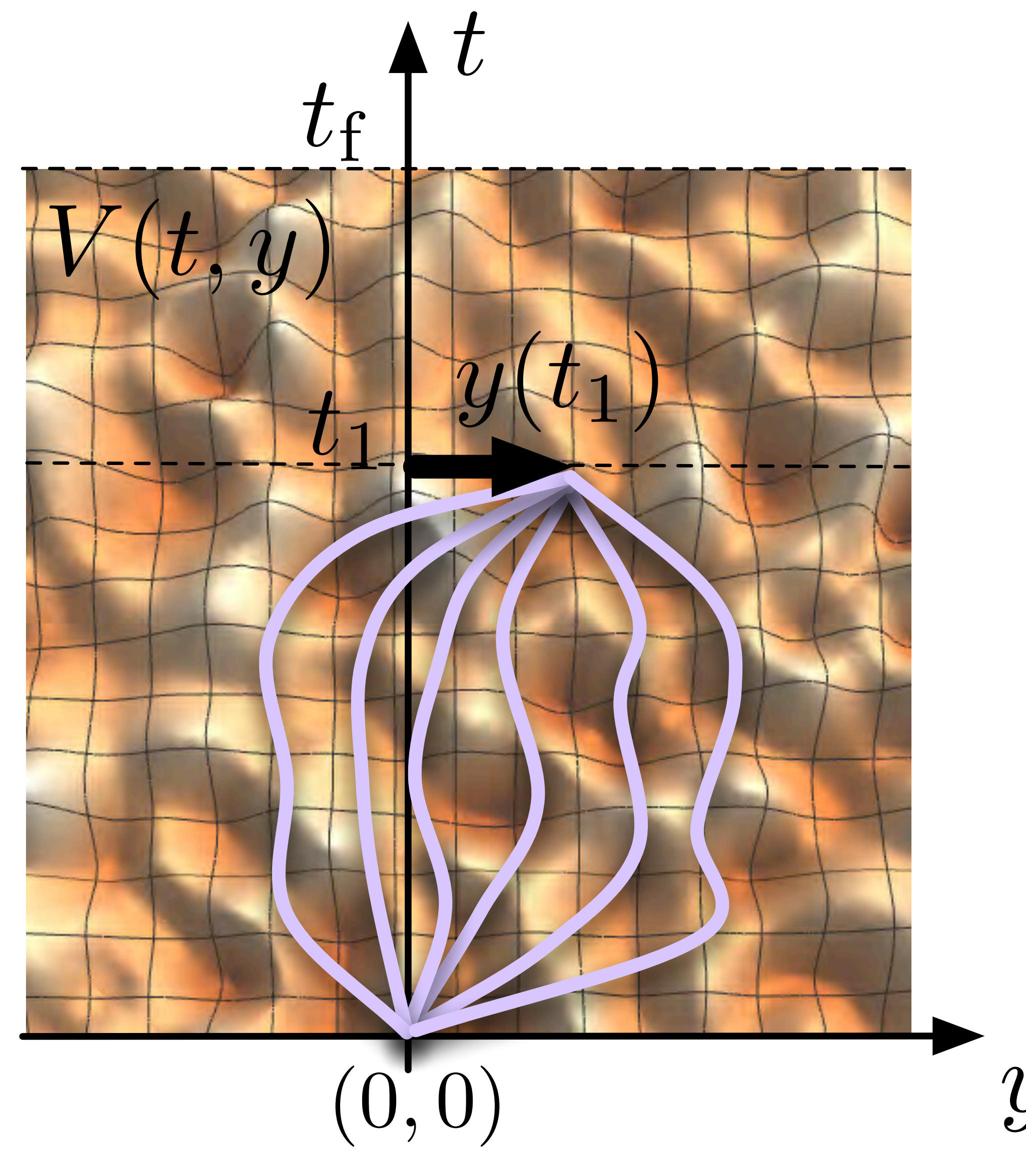


Boltzmann weight of a given trajectory $\propto e^{-\frac{1}{T}\mathcal{H}[y(t),V;t_{\rm f}]}$

Geometrical fluctuations & roughness?

- Free energy of the growing 1+1 DP

$$F_V = F_{V=0} + \bar{F}_V$$



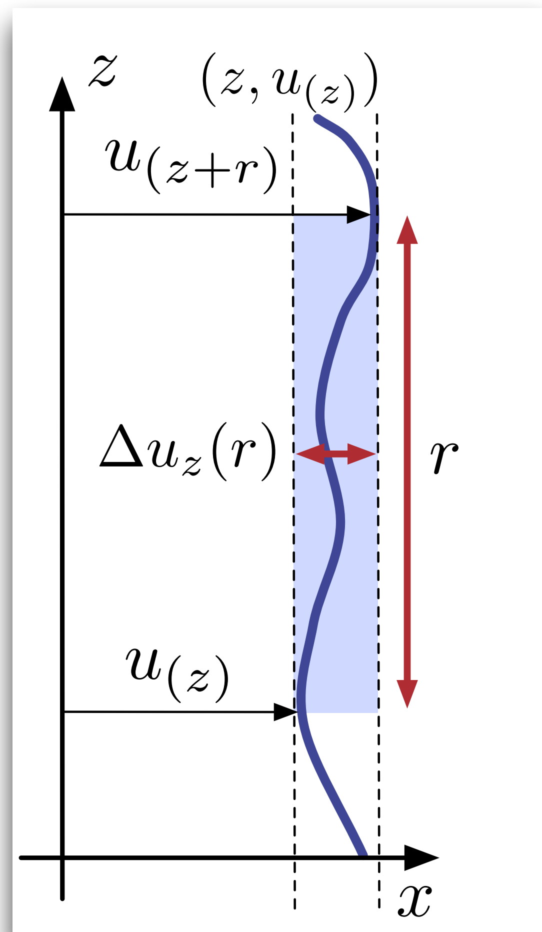
Boltzmann weight of a given endpoint $\propto e^{-\frac{1}{T}F_V(t_1,y)}$

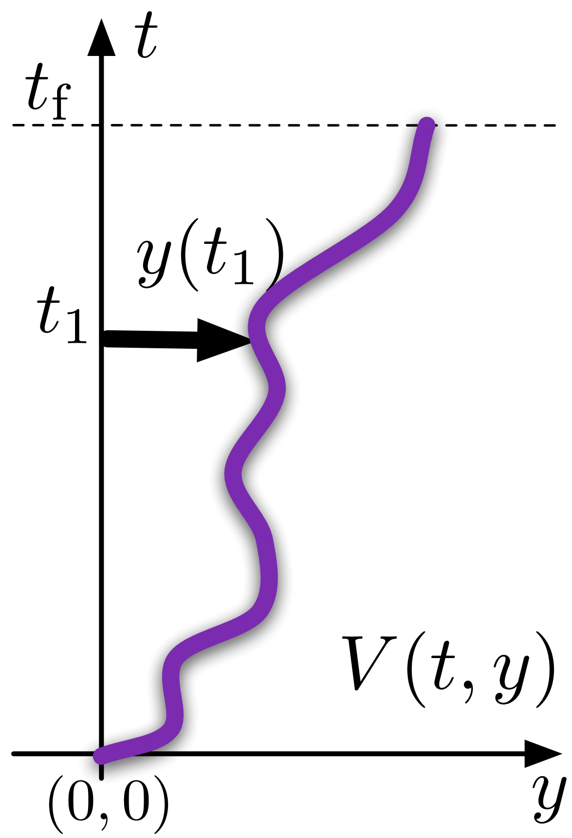
DP endpoint free-energy fluctuations?

Model: static 1D interface & growing 1+1 directed polymer (DP)

- Hamiltonian of the static 1D interface

$$\mathcal{H} = \mathcal{H}_{\mathrm{el}} + \mathcal{H}_{\mathrm{dis}}$$





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Geometrical fluctuations & roughness?

- Free energy of the growing 1+1 DP

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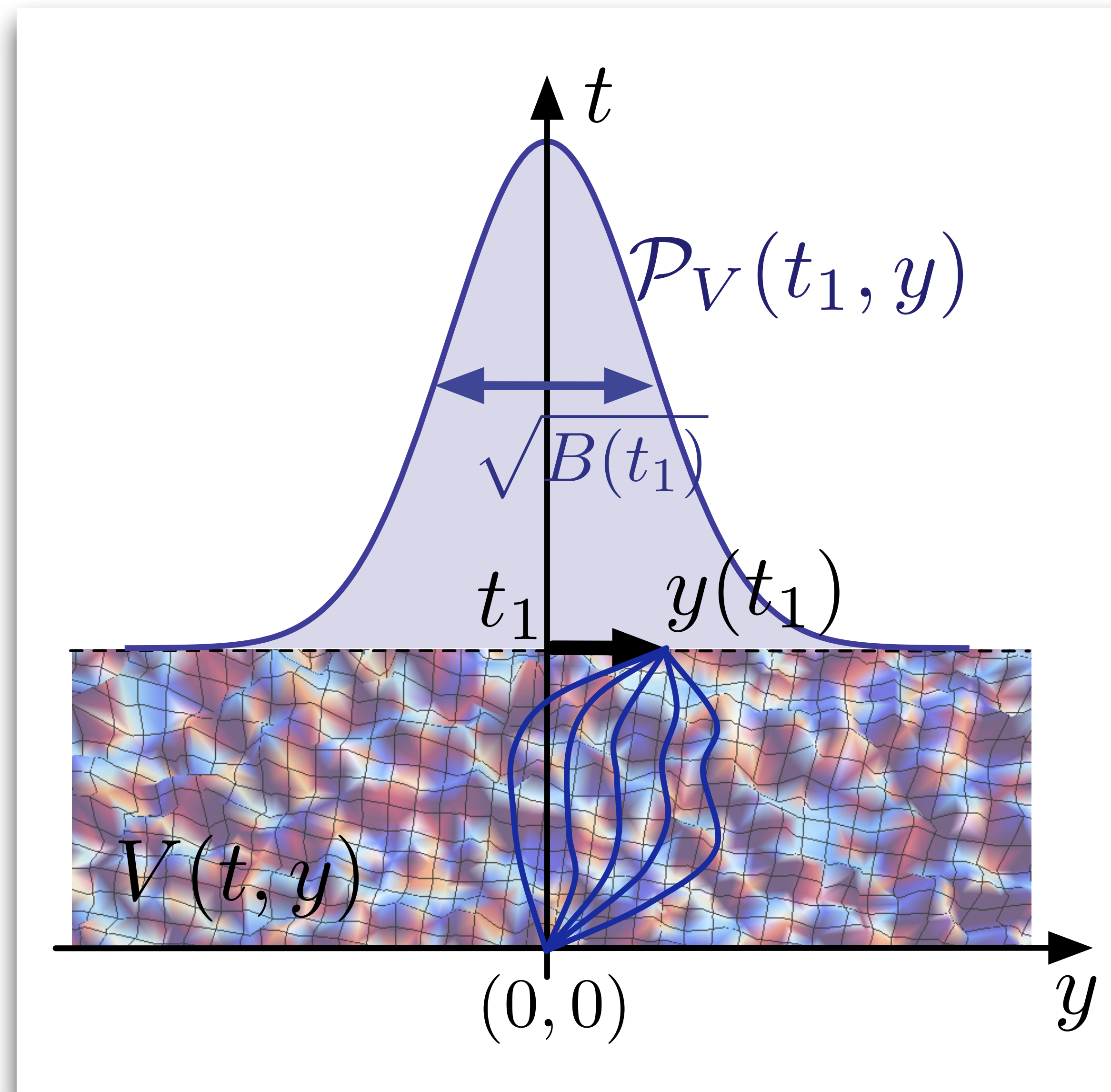

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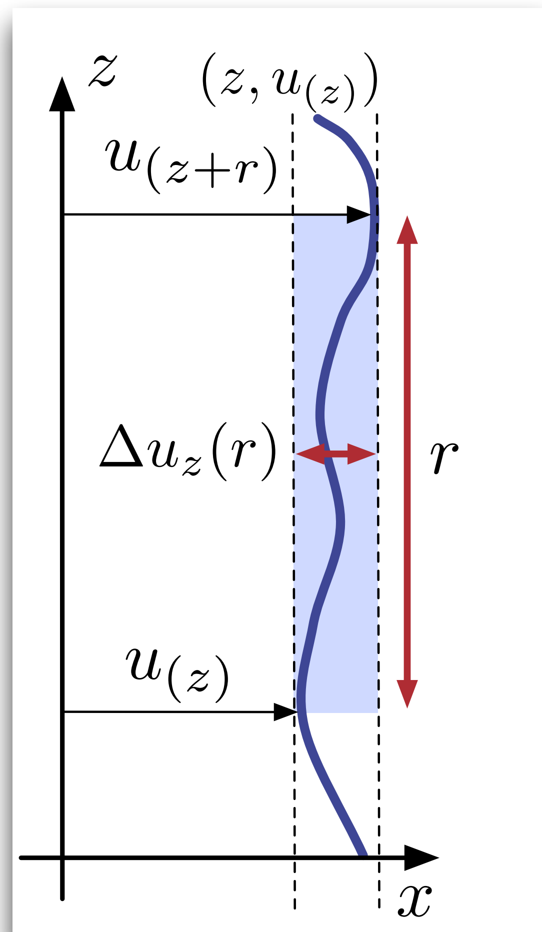
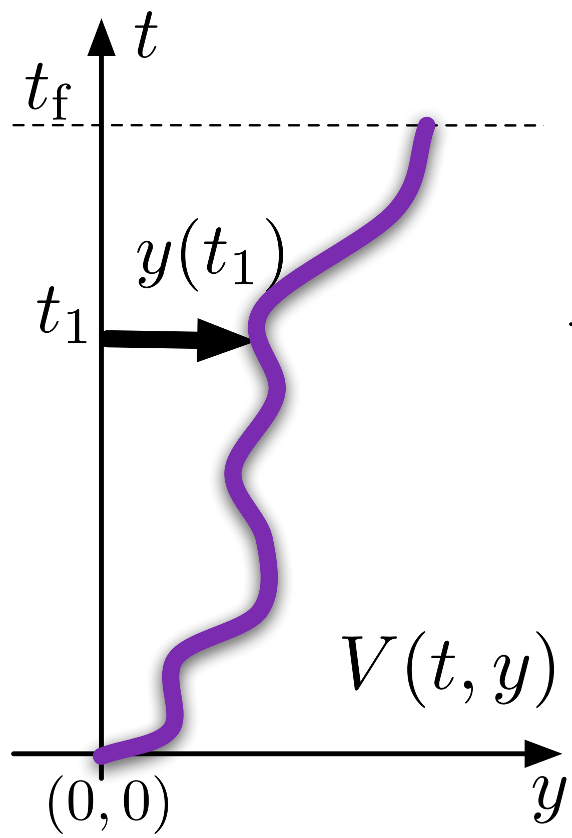
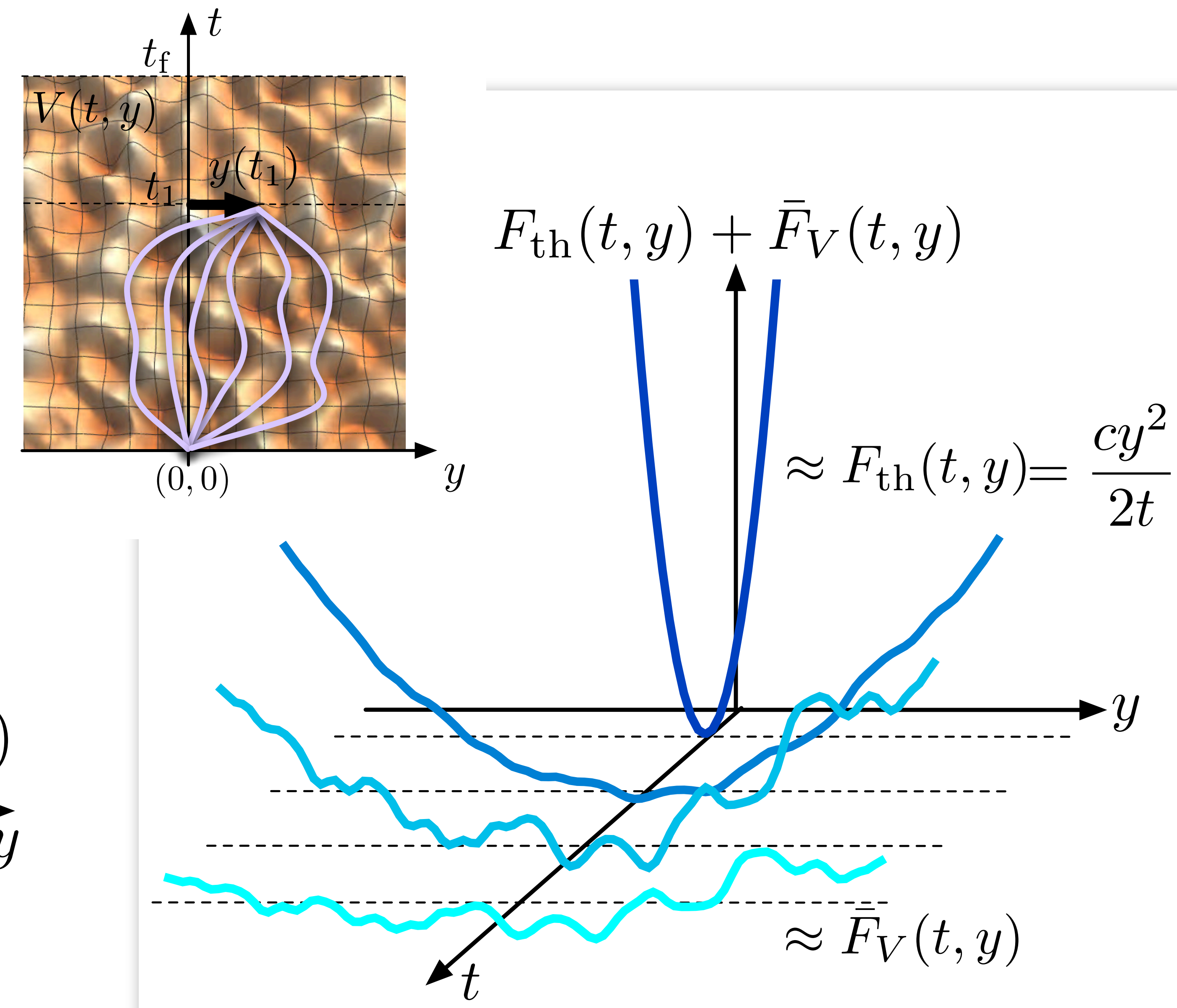






Boltzmann weight of a given trajectory $\propto e^{-\frac{1}{T}\mathcal{H}[y(t),V;t_{\mathrm{f}}]}$

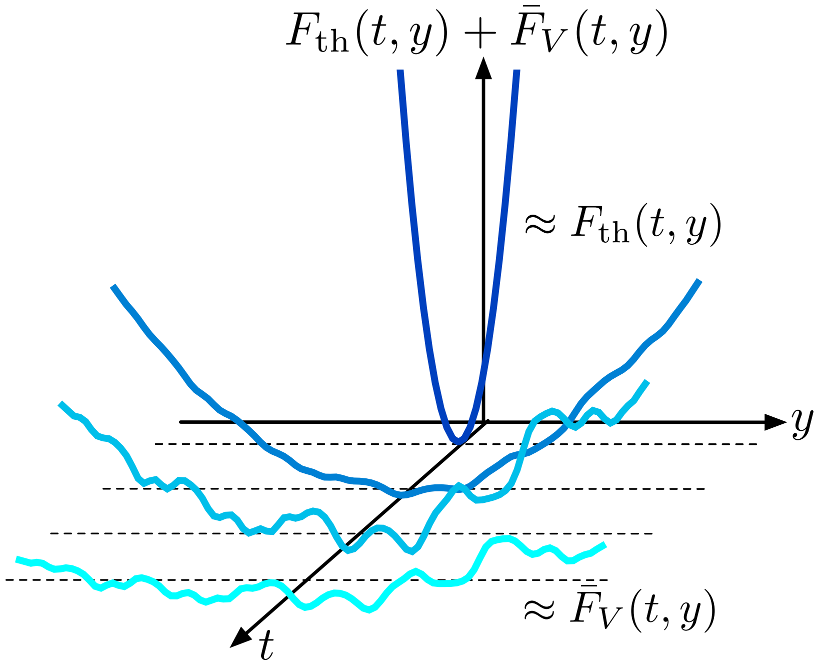
Boltzmann weight of a given endpoint $\propto e^{-\frac{1}{T}F_V(t_1,y)}$

Geometrical fluctuations & roughness?

DP endpoint free-energy fluctuations?

KPZ equation for the free-energy of the DP endpoint

- KPZ evolution equation for the total free-energy with 'sharp wedge' initial condition



D. Huse, C. L. Henley & D. S. Fisher, Phys. Rev. Lett. 55, 2924 (1985).

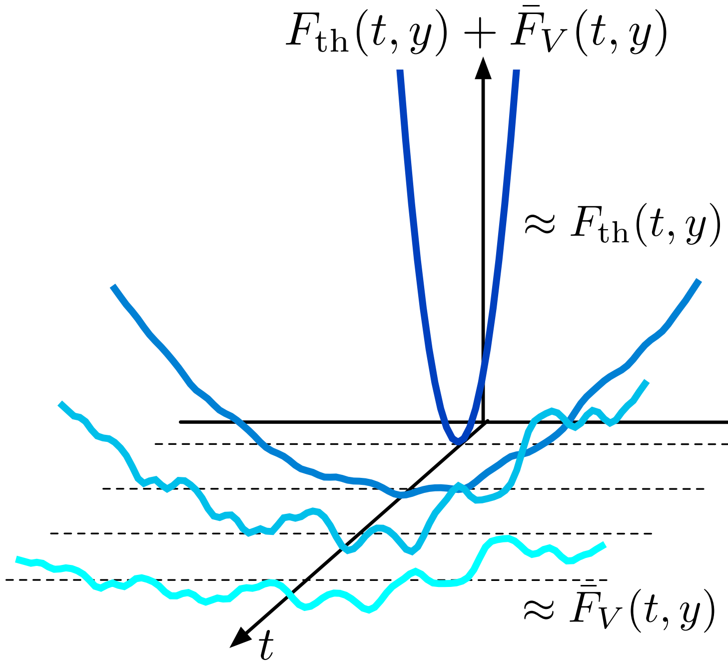
M. Kardar, G. Parisi & Y.-C. Zhang, Phys. Rev. Lett. 56, 889 (1986).

$$\begin{cases} \partial_t F_V(t,y) = \frac{T}{2c}\partial_y^2 F_V(t,y) - \frac{1}{2c}\left[\partial_y F_V(t,y)\right]^2 + V(t,y) \\ \mathcal{P}_V(0,y) = e^{-F_V(0,y)/T} = \delta(y) \end{cases}$$

$$\mathcal{P}_{V=0}(t,y) = e^{-F_{V=0}(t,y)} = \frac{e^{-\frac{1}{2}y^2/B_{\rm th}(t)}}{\sqrt{2\pi B_{\rm th}(t)}}\,, \quad B_{\rm th}(t) = \frac{Tt}{c}$$

KPZ equation for the free-energy of the DP endpoint

- KPZ evolution equation for the total free-energy with 'sharp wedge' initial condition



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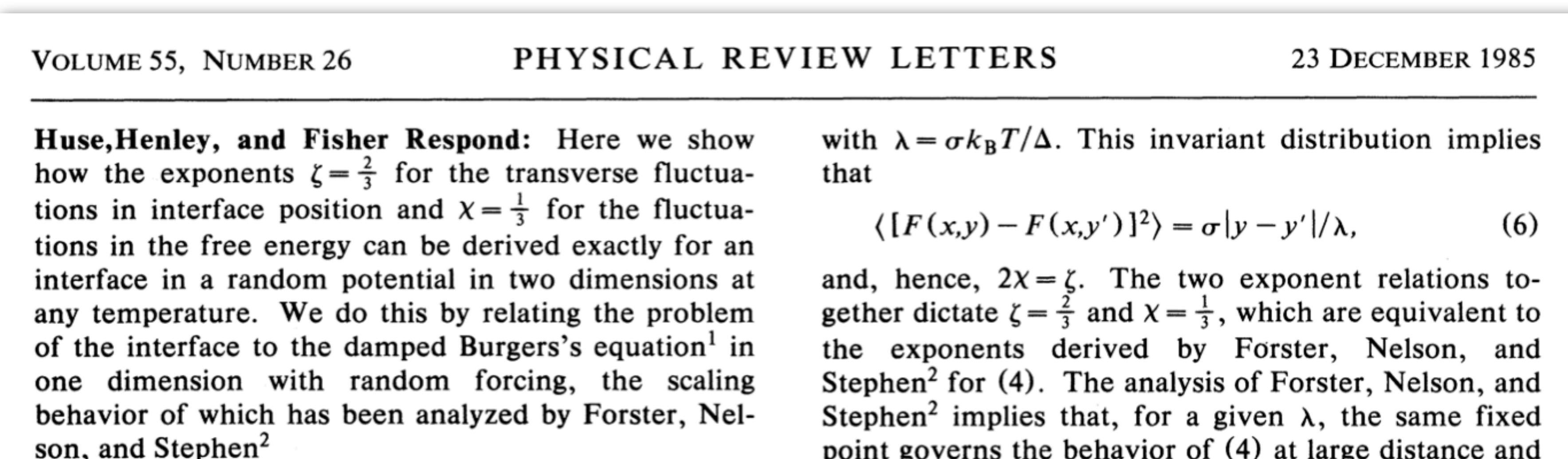


VOLUME 55, NUMBER 26 PHYSICAL REVIEW LETTERS 23 DECEMBER 1985

Huse, Henley, and Fisher Respond: Here we show how the exponents $\zeta=\frac{2}{3}$ for the transverse fluctuations in interface position and $\chi=\frac{1}{3}$ for the fluctuations in the free energy can be derived exactly for an interface in a random potential in two dimensions at any temperature. We do this by relating the problem of the interface to the damped Burgers's equation[1] in one dimension with random forcing, the scaling behavior of which has been analyzed by Forster, Nelson, and Stephen[2]

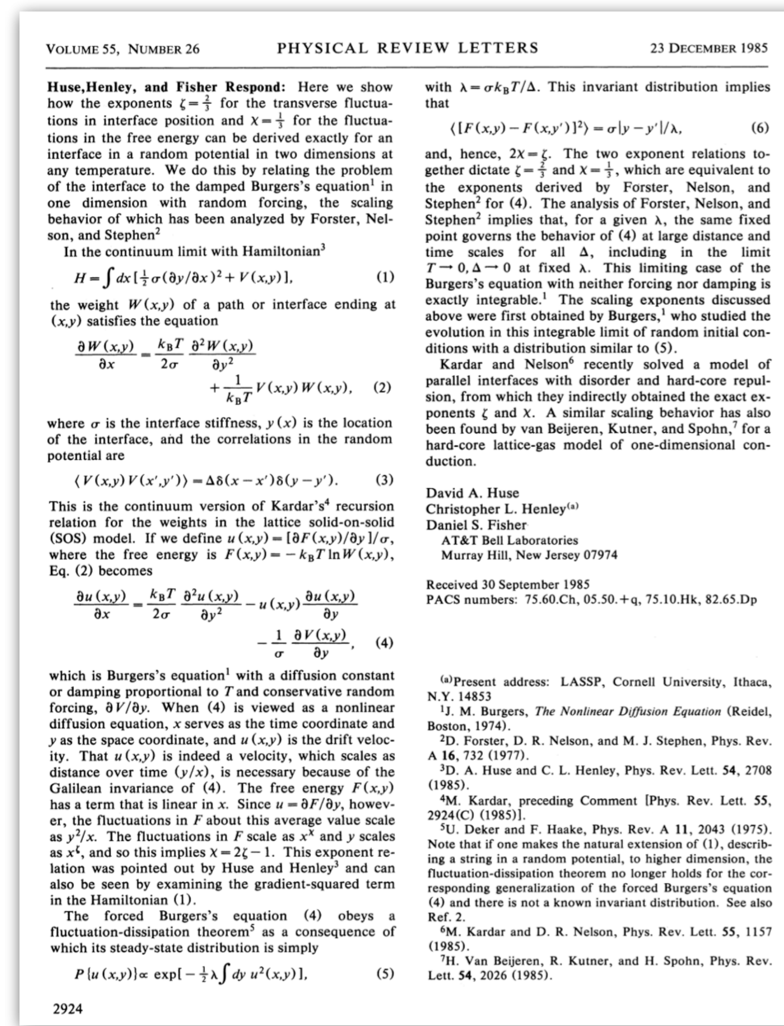
with $\lambda=\sigma k_B T/\Delta$. This invariant distribution implies that

$$\langle [F(x,y)-F(x,y')]^2\rangle = \sigma|y-y'|/\lambda, \qquad (6)$$

and, hence, $2\chi=\zeta$. The two exponent relations together dictate $\zeta=\frac{2}{3}$ and $\chi=\frac{1}{3}$, which are equivalent to the exponents derived by Forster, Nelson, and Stephen[2] for (4). The analysis of Forster, Nelson, and Stephen[2] implies that, for a given λ, the same fixed point governs the behavior of (4) at large distance and

Detailed derivation for instance in:

E. Agoritsas, V. Lecomte & T. Giamarchi, *Phys. Rev. E* 87, 042406 & 062405 (2013).



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In the continuum limit with Hamiltonian[3]

$$H=\int dx\,[\tfrac{1}{2}\sigma(\partial y/\partial x)^2+V(x,y)], \qquad (1)$$

the weight $W(x,y)$ of a path or interface ending at (x,y) satisfies the equation

$$\frac{\partial W(x,y)}{\partial x}=\frac{k_BT}{2\sigma}\frac{\partial^2 W(x,y)}{\partial y^2}+\frac{1}{k_BT}V(x,y)W(x,y), \qquad (2)$$

where σ is the interface stiffness, $y(x)$ is the location of the interface, and the correlations in the random potential are

$$\langle V(x,y)V(x',y')\rangle=\Delta\delta(x-x')\delta(y-y'). \qquad (3)$$

This is the continuum version of Kardar's[4] recursion relation for the weights in the lattice solid-on-solid (SOS) model. If we define $u(x,y)=[\partial F(x,y)/\partial y]/\sigma$, where the free energy is $F(x,y)=-k_BT\ln W(x,y)$, Eq. (2) becomes

$$\frac{\partial u(x,y)}{\partial x}=\frac{k_BT}{2\sigma}\frac{\partial^2 u(x,y)}{\partial y^2}-u(x,y)\frac{\partial u(x,y)}{\partial y}-\frac{1}{\sigma}\frac{\partial V(x,y)}{\partial y}, \qquad (4)$$

which is Burgers's equation[1] with a diffusion constant or damping proportional to T and conservative random forcing, $\partial V/\partial y$. When (4) is viewed as a nonlinear diffusion equation, x serves as the time coordinate and y as the space coordinate, and $u(x,y)$ is the drift velocity. That $u(x,y)$ is indeed a velocity, which scales as distance over time (y/x), is necessary because of the Galilean invariance of (4). The free energy $F(x,y)$ has a term that is linear in x. Since $u=\partial F/\partial y$, however, the fluctuations in F about this average value scale as y^2/x. The fluctuations in F scale as x^χ and y scales as x^ζ, and so this implies $\chi=2\zeta-1$. This exponent relation was pointed out by Huse and Henley[3] and can also be seen by examining the gradient-squared term in the Hamiltonian (1).

The forced Burgers's equation (4) obeys a fluctuation-dissipation theorem[5] as a consequence of which its steady-state distribution is simply

$$P\{u(x,y)\}\propto\exp[-\tfrac{1}{2}\lambda\int dy\,u^2(x,y)], \qquad (5)$$

with $\lambda=\sigma k_BT/\Delta$. This invariant distribution implies that

$$\langle [F(x,y)-F(x,y')]^2\rangle = \sigma|y-y'|/\lambda, \qquad (6)$$

and, hence, $2\chi=\zeta$. The two exponent relations together dictate $\zeta=\frac{2}{3}$ and $\chi=\frac{1}{3}$, which are equivalent to the exponents derived by Forster, Nelson, and Stephen[2] for (4). The analysis of Forster, Nelson, and Stephen[2] implies that, for a given λ, the same fixed point governs the behavior of (4) at large distance and time scales for all Δ, including in the limit $T\to 0, \Delta\to 0$ at fixed λ. This limiting case of the Burgers's equation with neither forcing nor damping is exactly integrable.[1] The scaling exponents discussed above were first obtained by Burgers,[1] who studied the evolution in this integrable limit of random initial conditions with a distribution similar to (5).

Kardar and Nelson[6] recently solved a model of parallel interfaces with disorder and hard-core repulsion, from which they indirectly obtained the exact exponents ζ and χ. A similar scaling behavior has also been found by van Beijeren, Kutner, and Spohn,[7] for a hard-core lattice-gas model of one-dimensional conduction.

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[1] J. M. Burgers, *The Nonlinear Diffusion Equation* (Reidel, Boston, 1974).
[2] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).
[3] D. A. Huse and C. L. Henley, Phys. Rev. Lett. **54**, 2708 (1985).
[4] M. Kardar, preceding Comment [Phys. Rev. Lett. **55**, 2924(C) (1985)].
[5] U. Deker and F. Haake, Phys. Rev. A **11**, 2043 (1975). Note that if one makes the natural extension of (1), describing a string in a random potential, to higher dimension, the fluctuation-dissipation theorem no longer holds for the corresponding generalization of the forced Burgers's equation (4) and there is not a known invariant distribution. See also Ref. 2.
[6] M. Kardar and D. R. Nelson, Phys. Rev. Lett. **55**, 1157 (1985).
[7] H. Van Beijeren, R. Kutner, and H. Spohn, Phys. Rev. Lett. **54**, 2026 (1985).

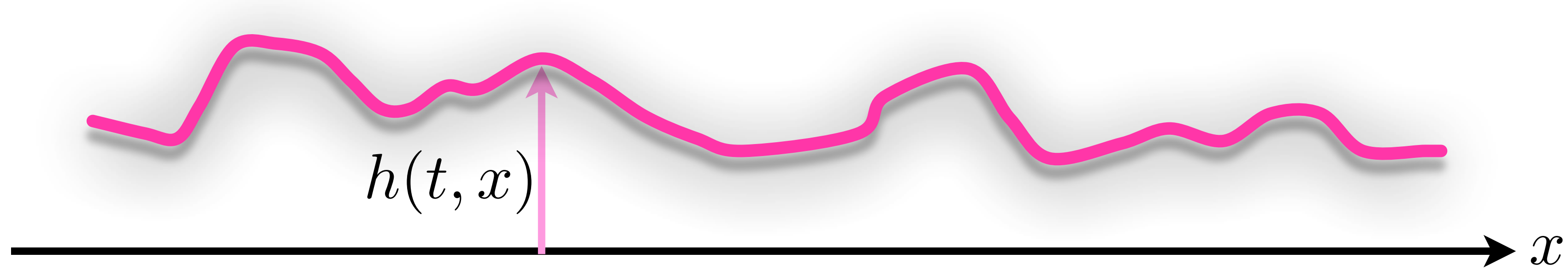
2924

KPZ equation for the free-energy of the DP endpoint

M. Kardar, G. Parisi & Y.-C. Zhang, "Dynamical Scaling of Growing Interfaces", *Phys. Rev. Lett.* 56 889 (1986).

- KPZ equation: model for the time-evolution of the profile of a growing interface

$$\partial_t h(t,\vec{x}) = \underbrace{\nu \nabla^2_{\vec{x}} h(t,\vec{x})}_{\substack{\text{relaxation}\\ \text{or diffusion}}} + \underbrace{\frac{\lambda}{2}\left[\nabla_{\vec{x}} h(t,\vec{x})\right]^2}_{\substack{\text{slope-dependent}\\ \text{lateral growth}}} + \underbrace{\eta(t,\vec{x})}_{\substack{\text{random}\\ \text{noise}}}$$



KITP, March 3, 2016.

- Playing with the initial condition (flat, stochastic, …)

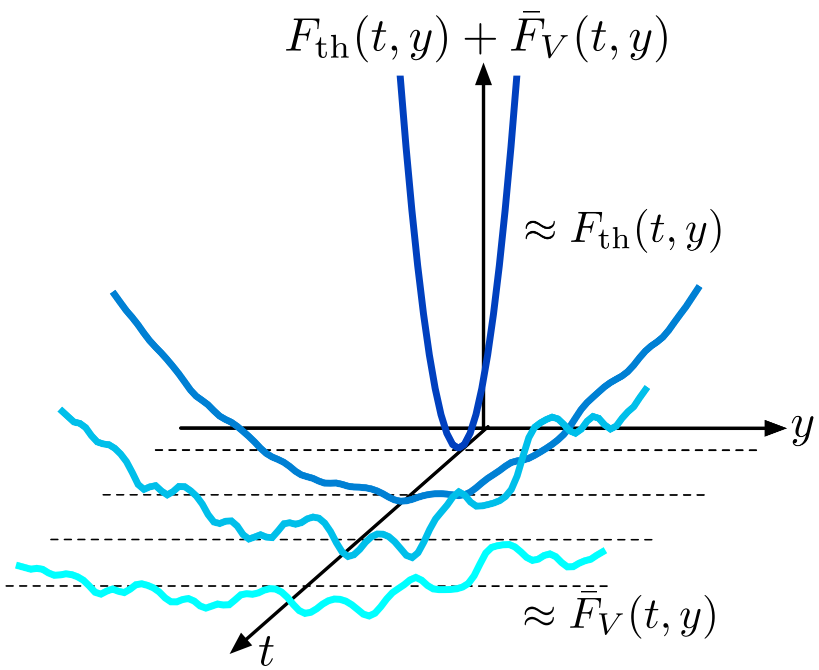
- 'Standard' random noise:
  Gaussian white spatially-uncorrelated noise

- At large times, fluctuations with power-law exponent
  $\Rightarrow \zeta_{\mathrm{KPZ}} = 2/3$

- For us: 'sharp-wedge' initial condition '$\log \delta(x)$'
  $2\nu = T/c \qquad \lambda = -1/c$

KPZ equation for the free-energy of the DP endpoint

- KPZ evolution equation for the total free-energy with 'sharp wedge' initial condition



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$$\mathcal{P}_{V=0}(t,y) = e^{-F_{V=0}(t,y)} = \frac{e^{-\frac{1}{2}y^2/B_{\rm th}(t)}}{\sqrt{2\pi B_{\rm th}(t)}}\,, \qquad B_{\rm th}(t) = \frac{Tt}{c}$$

- 1D Kardar-Parisi-Zhang universality class encompasses a wide range of problems:

Random matrices, Burgers equation in hydrodynamics, roughening phenomena & stochastic growth,

1+1 Directed Polymer (DP), our one-dimensional interface, ...

I. Corwin, "The Kardar-Parisi-Zhang equation and universality class", arXiv:1106.1596 [math.PR] (2011)

J. Quastel & H. Spohn, "The One-Dimensional KPZ Equation and Its Universality Class", *J. Stat. Mech.* 160, 965 (2015)

T. Halpin-Healy & K. A. Takeuchi., "A KPZ Cocktail-Shaken, not Stirred...", *J. Stat. Mech.* 160, 794 (2015)

K. A. Takeuchi., "An appetizer to modern developments on the Kardar-Parisi-Zhang universality class", *Physica A* (2018)

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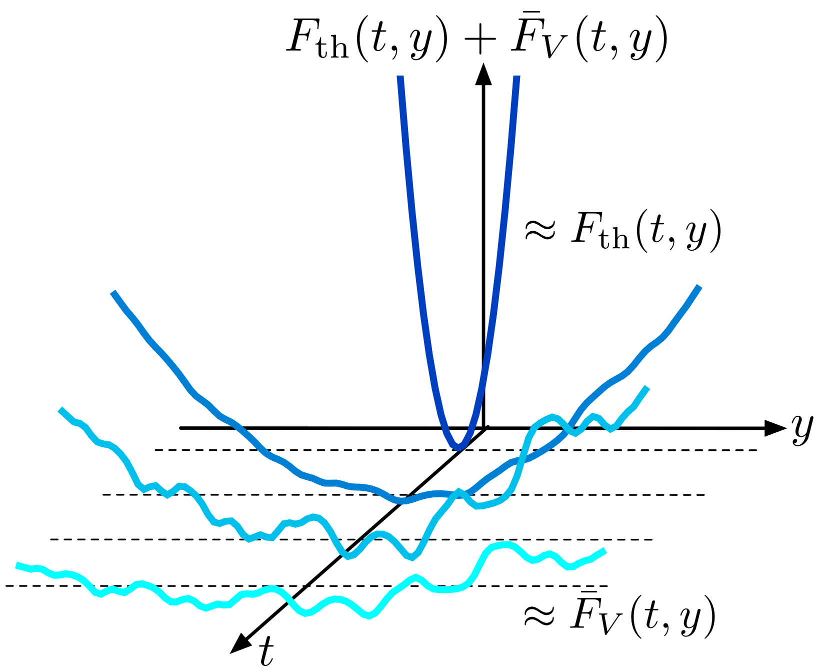
- KPZ evolution equation for the total free-energy with 'sharp wedge' initial condition



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$$\begin{cases} \bar C(t,y) = \overline{\left[\bar F_V(t,y) - \bar F_V(t,0)\right]^2} \\ \bar R(t,y) = \overline{\partial_y \bar F_V(t,y)\partial_y \bar F_V(t,0)} \end{cases}$$

- Fluctuations exactly known for an uncorrelated disorder / white noise (for this initial condition!):

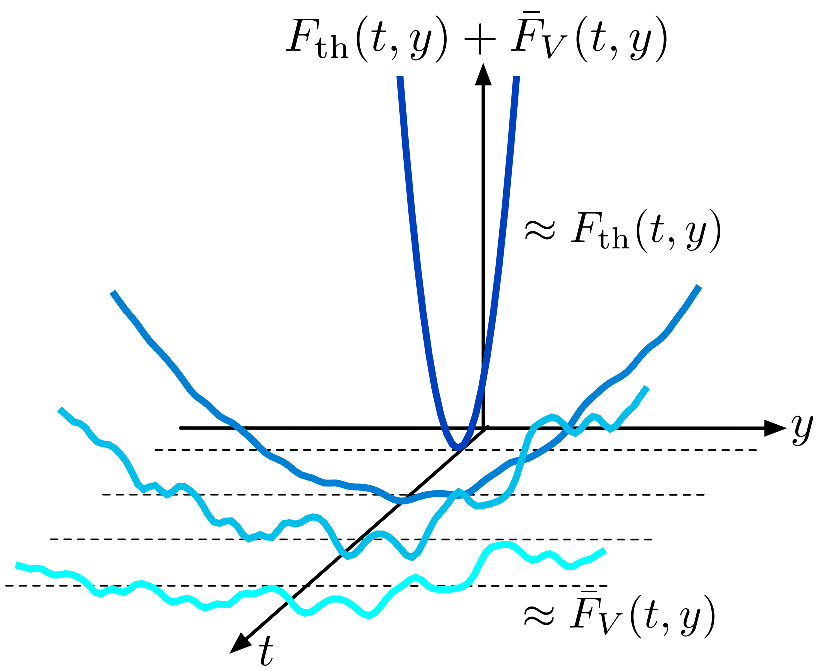
  - *Infinite-'time' limit:* Gaussian distribution, Brownian scaling: $\boxed{\bar C(\infty,y) = \frac{cD}{T}|y|} \Rightarrow \bar F_V \overset{(d)}{\sim} \left(\frac{cD}{T}\right)^{1/2} y^{1/2}$

  - *Asymptotically large-'time':* GUE Tracy-Widom distribution (non-Gaussian!)
    $\bar C(t,y) =$ 2-point correlator of Airy₂ process
    M. Prähofer & H. Spohn, *J. Stat. Phys.* 159 1071 (2002).

  - *At all 'times':*
    P. Calabrese, P. Le Doussal & A. Rosso, *Eur. Phys. Lett.* 90 20002 (2010).
    V. Dotsenko, *Eur. Phys. Lett.* 90 20003 (2010).
    T. Sasamoto & H. Spohn, *Nucl. Phys. B* 834 523 (2010).
    G. Amir, I. Corwin, J. Quastel., *Comm. Pure Appl. Math.* 64 466 (2011).

KPZ equation for the free-energy of the DP endpoint

- KPZ evolution equation for the total free-energy with 'sharp wedge' initial condition



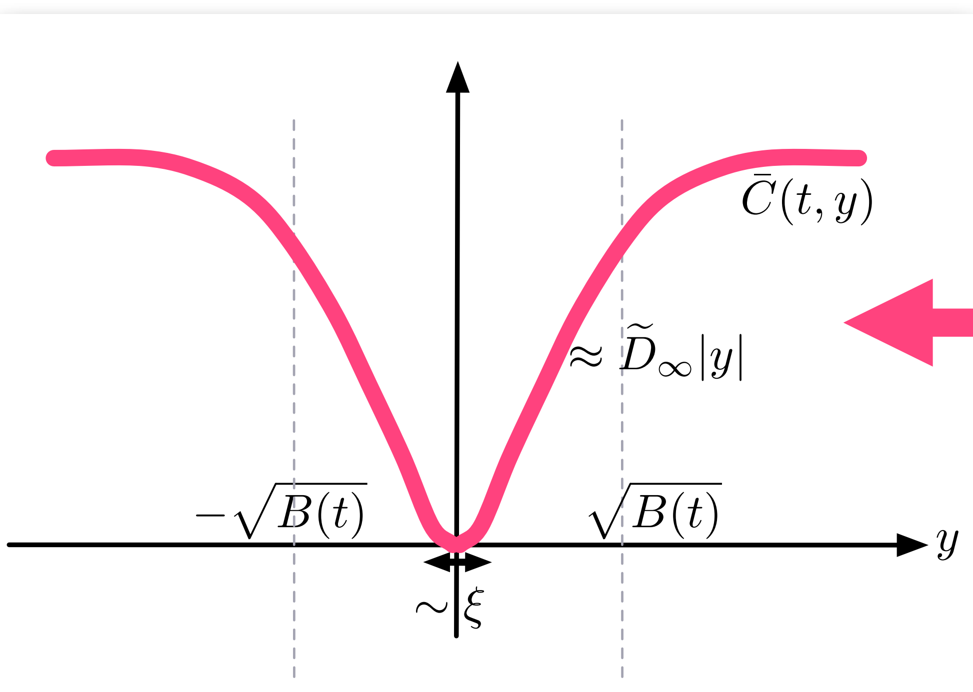
D. Huse, C. L. Henley & D. S. Fisher, Phys. Rev. Lett. 55, 2924 (1985).

M. Kardar, G. Parisi & Y.-C. Zhang, Phys. Rev. Lett. 56, 889 (1986).

$$\begin{cases} \partial_t F_V(t,y) = \frac{T}{2c}\partial_y^2 F_V(t,y) - \frac{1}{2c}\left[\partial_y F_V(t,y)\right]^2 + V(t,y) \\ \mathcal{P}_V(0,y) = e^{-F_V(0,y)/T} = \delta(y) \end{cases}$$

$$\begin{cases} \bar{C}(t,y) = \overline{\left[\bar{F}_V(t,y) - \bar{F}_V(t,0)\right]^2} \\ \bar{R}(t,y) = \overline{\partial_y \bar{F}_V(t,y)\partial_y \bar{F}_V(t,0)} \end{cases}$$

- Fluctuations exactly known for an uncorrelated disorder / white noise (for this initial condition!):



Gaussian distribution
Brownian scaling

$$\boxed{\bar{C}(\infty,y) = \frac{cD}{T}|y|} \Rightarrow \bar{F}_V \overset{(d)}{\sim} \left(\frac{cD}{T}\right)^{1/2} y^{1/2}$$

GUE Tracy-Widom distribution (non-Gaussian!)

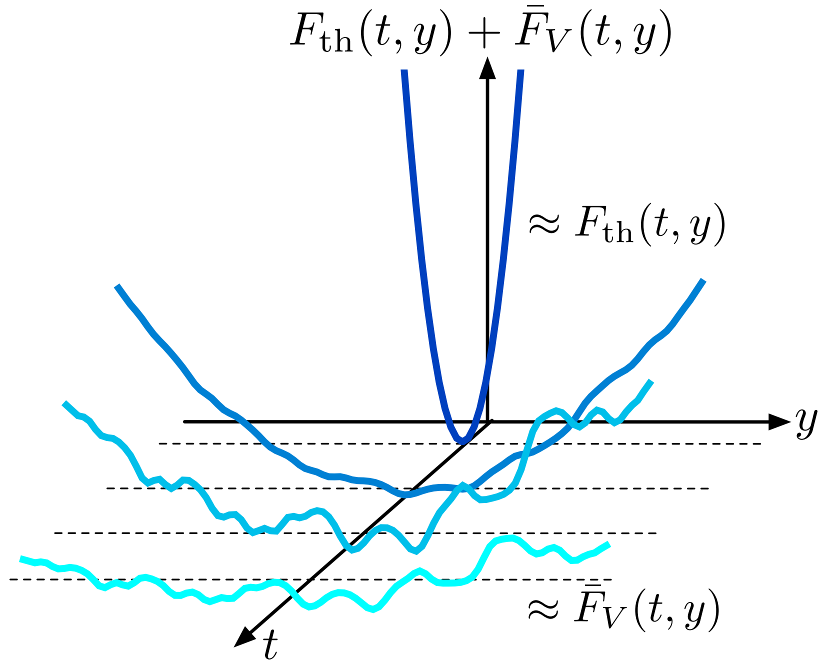
$\bar{C}(t,y) =$ 2-point correlator of Airy$_2$ process

M. Prähofer & H. Spohn, *J. Stat. Phys.* 159 1071 (2002).

P. Calabrese, P. Le Doussal & A. Rosso, *Eur. Phys. Lett.* 90 20002 (2010).
V. Dotsenko, *Eur. Phys. Lett.* 90 20003 (2010).
T. Sasamoto & H. Spohn, *Nucl. Phys. B* 834 523 (2010).
G. Amir, I. Corwin, J. Quastel., *Comm. Pure Appl. Math.* 64 466 (2011).

Tilted KPZ equation for the disorder free-energy

- KPZ evolution equation for the total free-energy with ‘sharp wedge’ initial condition



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$$\begin{cases} \bar{C}(t,y) = \overline{\left[\bar{F}_V(t,y) - \bar{F}_V(t,0)\right]^2} \\ \bar{R}(t,y) = \overline{\partial_y \bar{F}_V(t,y)\partial_y \bar{F}_V(t,0)} \end{cases}$$

- Tilted KPZ equation for the disorder contribution to the free-energy (its excess part!)

E. Agoritsas, V. Lecomte & T. Giamarchi, *Phys. Rev. E* 87, 042406 & 062405 (2013).

$$\begin{cases} \partial_t \bar{F}_V(t,y) = \frac{T}{2c}\partial_y^2 \bar{F}_V(t,y) \boxed{- \frac{1}{2c}\left[\partial_y \bar{F}_V(t,y)\right]^2 - \frac{y}{t}\partial_y \bar{F}_V(t,y)} + V(t,y) \\ \bar{F}_V(0,y) = 0 \qquad \text{(‘flat’ initial condition)} \end{cases}$$

Translational-invariant distribution at fixed time: $\bar{\mathcal{P}}\left[\bar{F}_V(t,y+Y)\right] = \bar{\mathcal{P}}\left[\bar{F}_V(t,y)\right]$

$$\overline{V(t,y)V(t',y')} = D\cdot\delta(t-t')\,R_\xi(y-y')$$
$$R_\xi(y) = \xi^{-1}R_1(y/\xi)$$

Two-point correlators of the disorder free-energy

- Focus on the two-point correlators:

$$\begin{cases} \bar{C}(t,y) = \overline{\left[\bar{F}_V(t,y) - \bar{F}_V(t,0)\right]^2} \\ \bar{R}(t,y) = \overline{\partial_y \bar{F}_V(t,y)\partial_y \bar{F}_V(t,0)} \end{cases}$$

- Spatially-correlated disorder/noise:

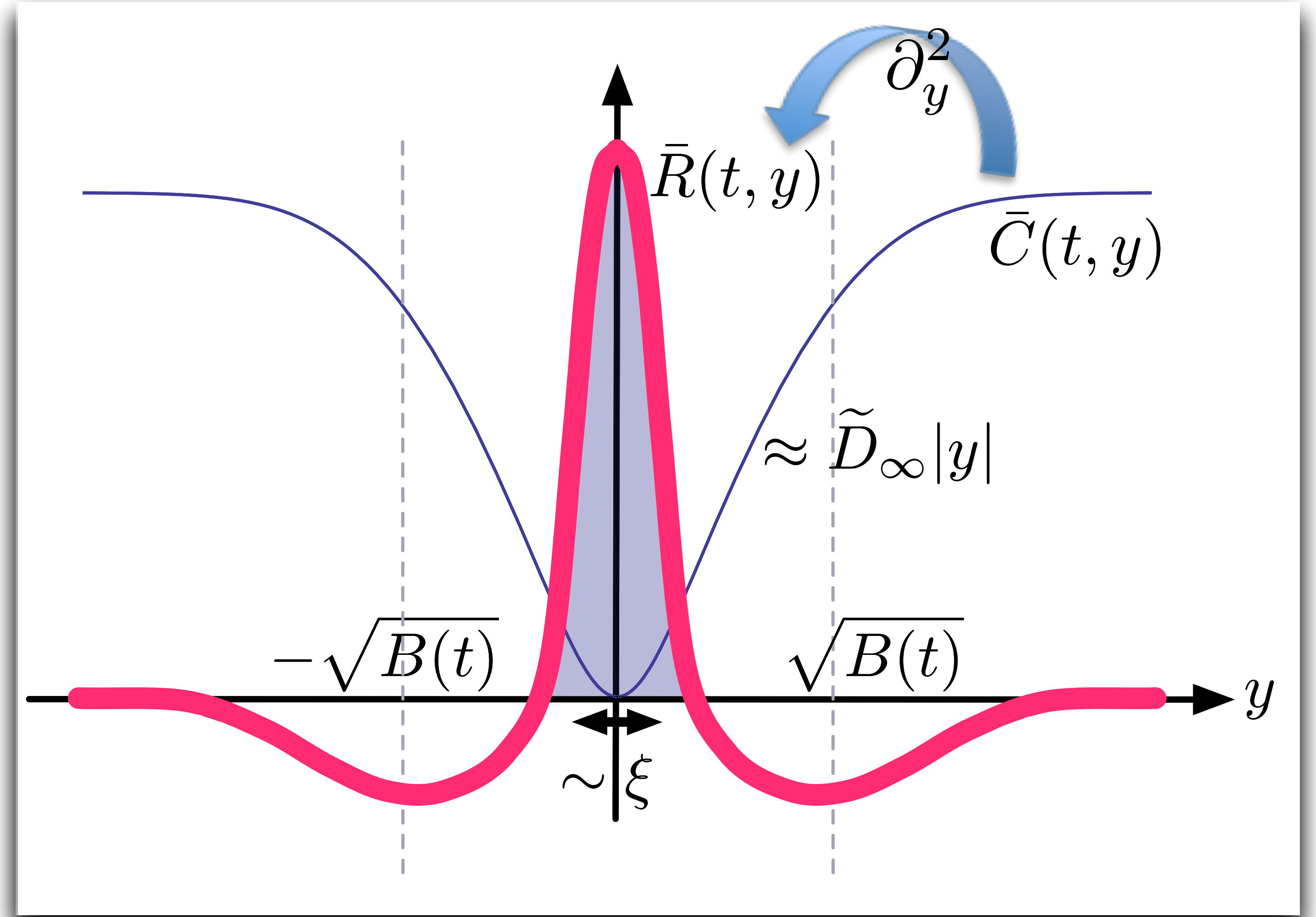
$$\overline{V(t,y)V(t',y')} = D\cdot\delta(t-t')\,R_\xi(y-y')$$

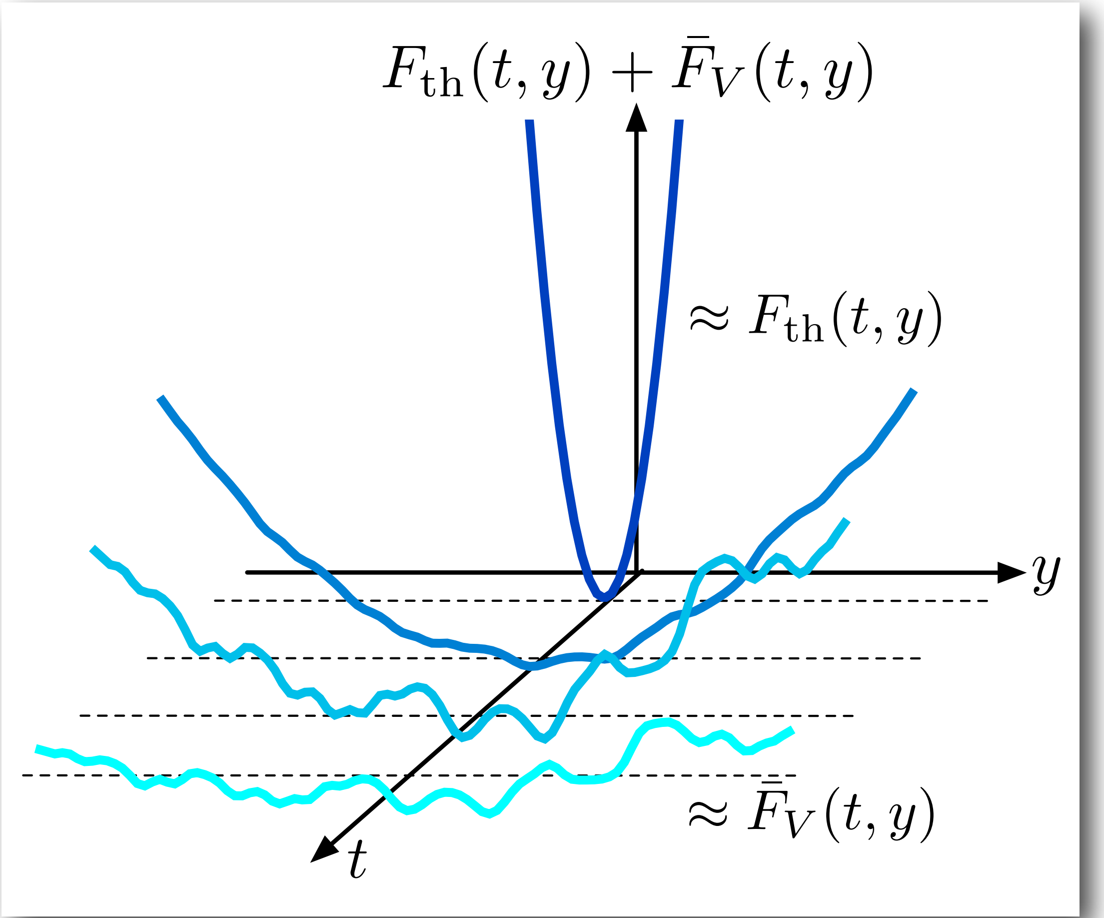
$$R_\xi(y) = \xi^{-1}R_1(y/\xi)$$

Mean evolution: $\partial_t \overline{\bar{F}_V(t,y)} = -\frac{1}{2c}\bar{R}(t,y=0)$

$$\bar{R}_3(t,y) = \overline{[\partial_y \bar{F}_V(t,y)]^2\partial_y \bar{F}(t,0)}$$

2-pt correlator evolution: $\partial_t \bar{R}(t,y) = \frac{T}{c}\partial_y^2 \bar{R}(t,y) - \frac{1}{t}\{\bar{R}(t,y) + \partial_y[y\bar{R}(t,y)]\} - \frac{1}{c}\partial_y \bar{R}_3(t,y) - DR''_\xi(y)$





E. Agoritsas, V. Lecomte & T. Giamarchi, *Phys. Rev. E* 87, 042406 & 062405 (2013).

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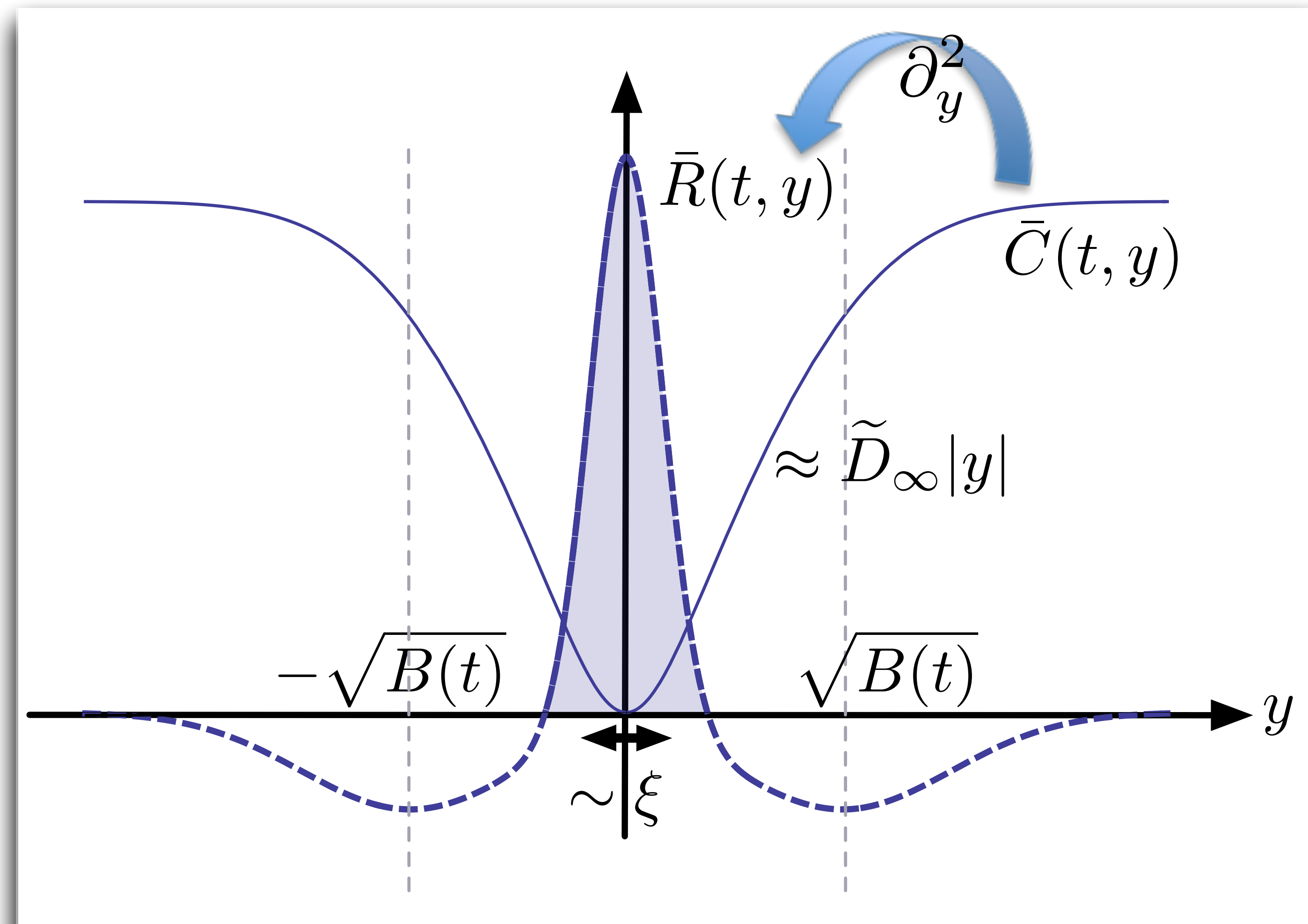
$$R_\xi(y) = \xi^{-1} R_1(y/\xi)$$

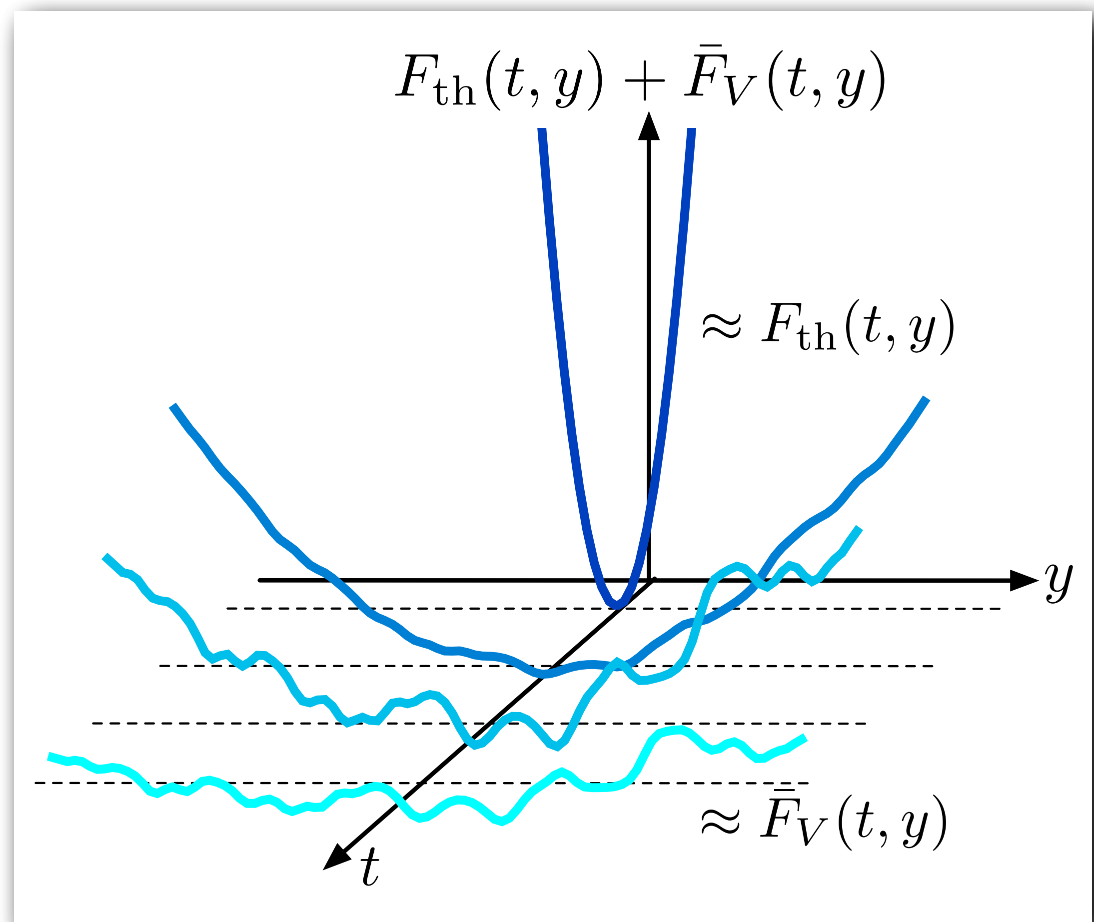
$$\boxed{\bar{R}(t,y) = \widetilde{D}\left[\mathcal{R}_\xi(y) - b(t,y,\xi)\right]}$$

amplitude $\widetilde{D}(T,\xi)$

asymptotic shape $\mathcal{R}_\xi(y) = \mathcal{F}\left[R_\xi(y), T\right]$

time-dependence (encoding the roughness?)





E. Agoritsas, V. Lecomte & T. Giamarchi, *Phys. Rev. E* 87, 042406 & 062405 (2013).

Full evolution of free-energy of the DP endpoint

- Stochastic heat equation for the partition function $\mathcal{Z}_V(t,y)$

$$\begin{cases} \partial_t \mathcal{Z}_V(t,y) = \left[\dfrac{T}{2c}\partial_y^2 - \dfrac{1}{T}V(t,y)\right]\mathcal{Z}_V(t,y) \\ \mathcal{Z}_V(0,y) = \delta(y) \end{cases}$$

$$\boxed{\mathcal{Z}_V(t,y) \equiv e^{-F_V(t,y)/T}}$$

- KPZ evolution equation for the total free-energy $F_V(t,y)$

$$\begin{cases} \partial_t F_V(t,y) = \dfrac{T}{2c}\partial_y^2 F_V(t,y) - \dfrac{1}{2c}\left[\partial_y F_V(t,y)\right]^2 + V(t,y) \\ \mathcal{Z}_V(0,y) = e^{-F_V(0,y)/T} = \delta(y) \qquad \text{('sharp wedge' initial condition)} \end{cases}$$

$$\boxed{F_V(t,y) \equiv F_{V=0}(t,y) + \bar{F}_V(t,y)}$$

- Tilt KPZ evolution equation for the disorder free-energy $\bar{F}_V(t,y)$

$$\begin{cases} \partial_t \bar{F}_V(t,y) = \dfrac{T}{2c}\partial_y^2 \bar{F}_V(t,y) - \dfrac{1}{2c}\left[\partial_y \bar{F}_V(t,y)\right]^2 - \dfrac{y}{t}\partial_y \bar{F}_V(t,y) + V(t,y) \\ \bar{F}_V(0,y) \equiv 0 \qquad \text{(flat initial condition)} \end{cases}$$

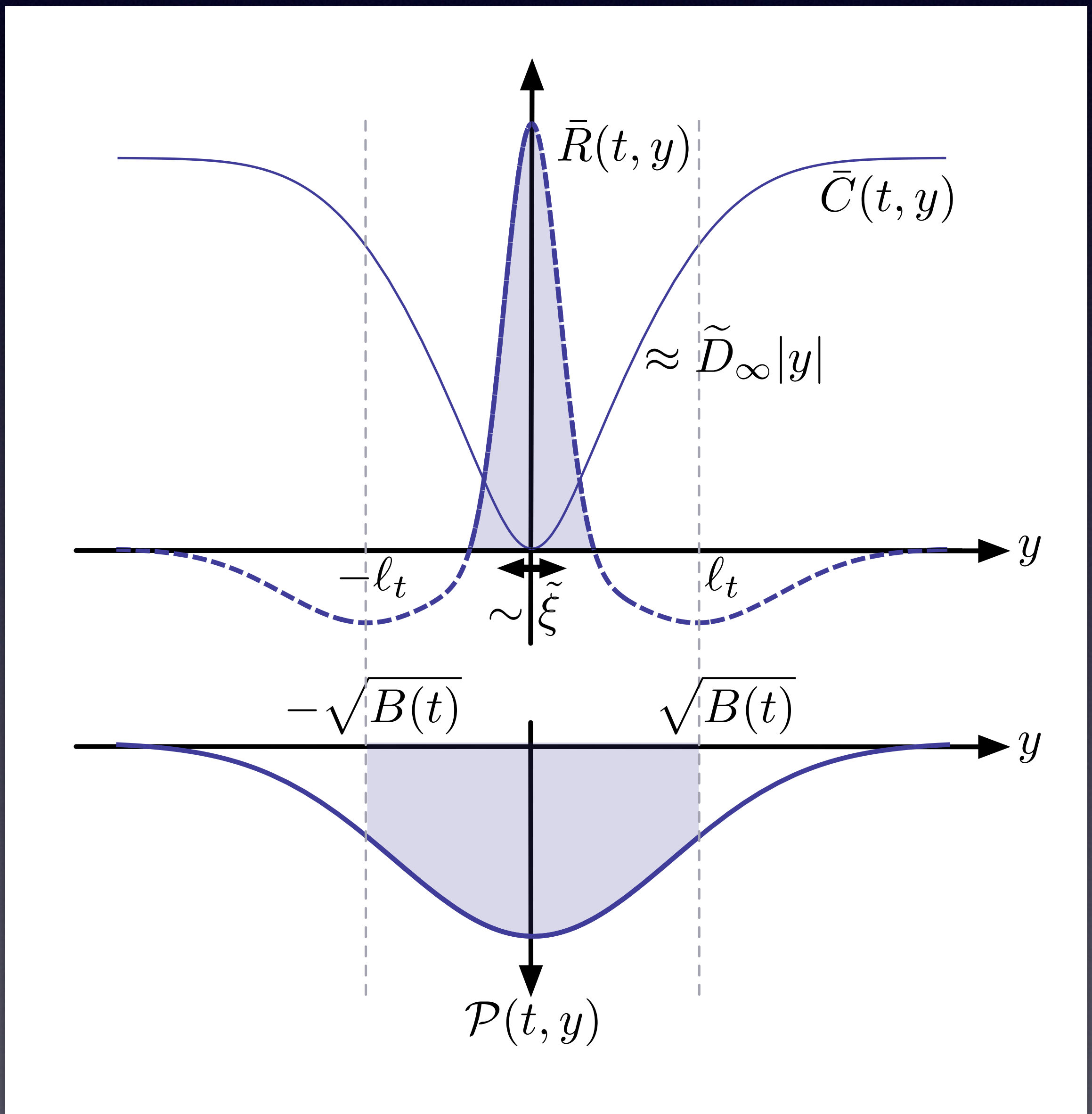
Two-point correlators of the disorder free-energy

- Focus on the two-point correlators:

$$\begin{cases} \bar{C}(t,y) \equiv \overline{\left[\bar{F}_V(t,y) - \bar{F}_V(t,0)\right]^2} \\ \bar{R}(t,y) \equiv \overline{\partial_y \bar{F}_V(t,y)\, \partial_y \bar{F}_V(t,0)} \end{cases}$$

- *Correlated* disorder (colored-noise):

$$\boxed{R_{\xi>0}(y) = \xi^{-1} R_1(y/\xi)}$$



- Non-Gaussian distribution of $\bar{F}_V(t,y)$
- Normalization: $\int_{\mathbb{R}} dy\, \bar{R}(t < \infty, y) = 0$
- Infinite-'time' limit: $\begin{cases} \bar{R}(\infty,y) = \widetilde{D}_\infty\, \mathcal{R}_\xi(y) \\ \int_{\mathbb{R}} dy\, \mathcal{R}_\xi(y) \equiv 1 \end{cases}$
- DP 'toymodel': $\begin{cases} \text{Gaussian fluctuations} \\ \bar{R}(t,y) \approx \widetilde{D}_t \cdot R_{\tilde{\xi}_t}(y) \\ \widetilde{D}_\infty(T,\xi) = f(T,\xi)\dfrac{cD}{T} \end{cases}$

E. Agoritsas, V. Lecomte & T. Giamarchi, Phys. Rev. E **87**, 042406 (2013).

Two-point correlators of the disorder free-energy

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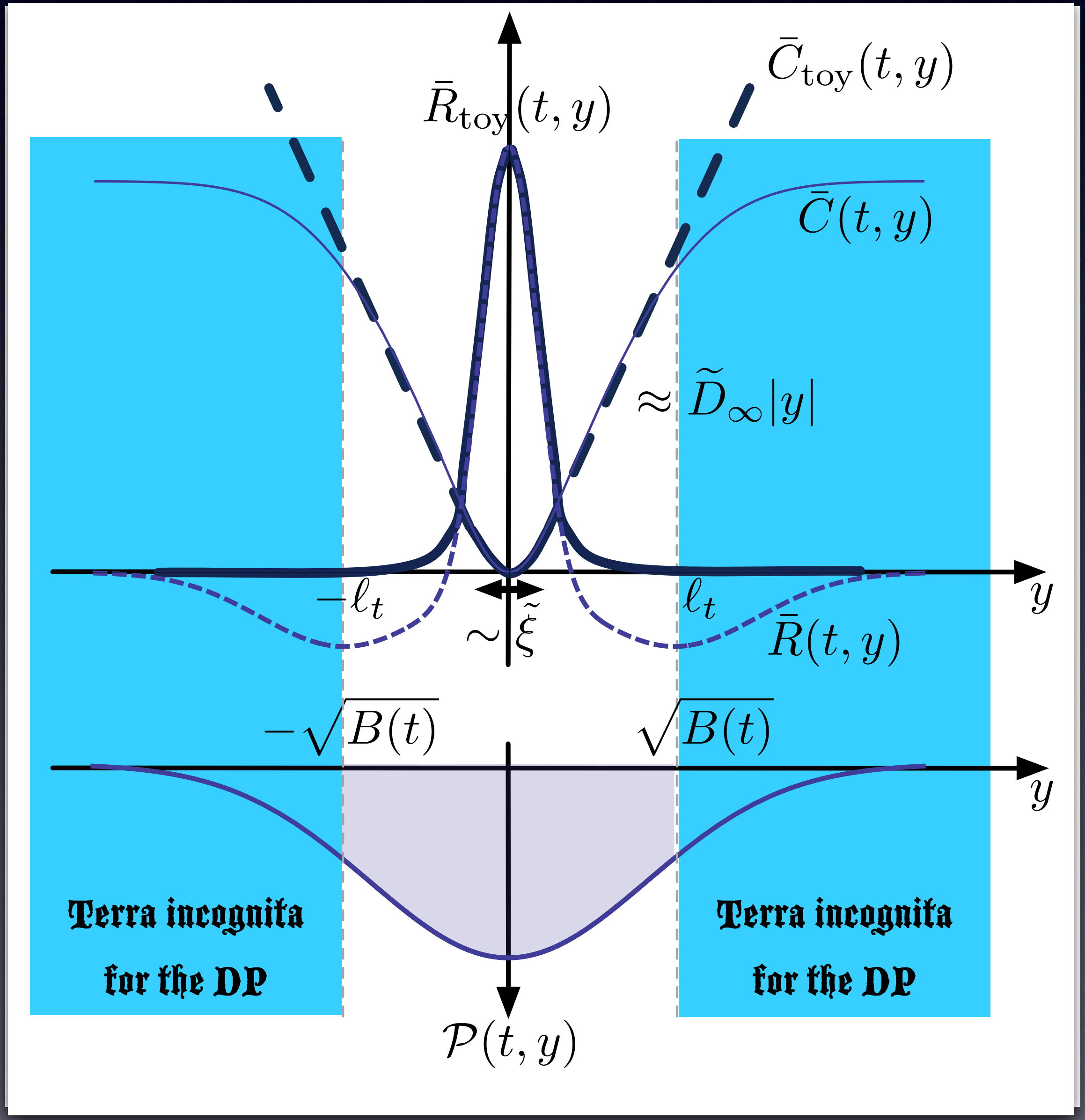

- Non-Gaussian distribution of $\bar{F}_V(t,y)$
- Normalization: $\int_{\mathbb{R}} dy\, \bar{R}(t<\infty, y) = 0$
- Infinite-'time' limit: $\begin{cases} \bar{R}(\infty,y) = \widetilde{D}_\infty \mathcal{R}_\xi(y) \\ \int_{\mathbb{R}} dy\, \mathcal{R}_\xi(y) \equiv 1 \end{cases}$
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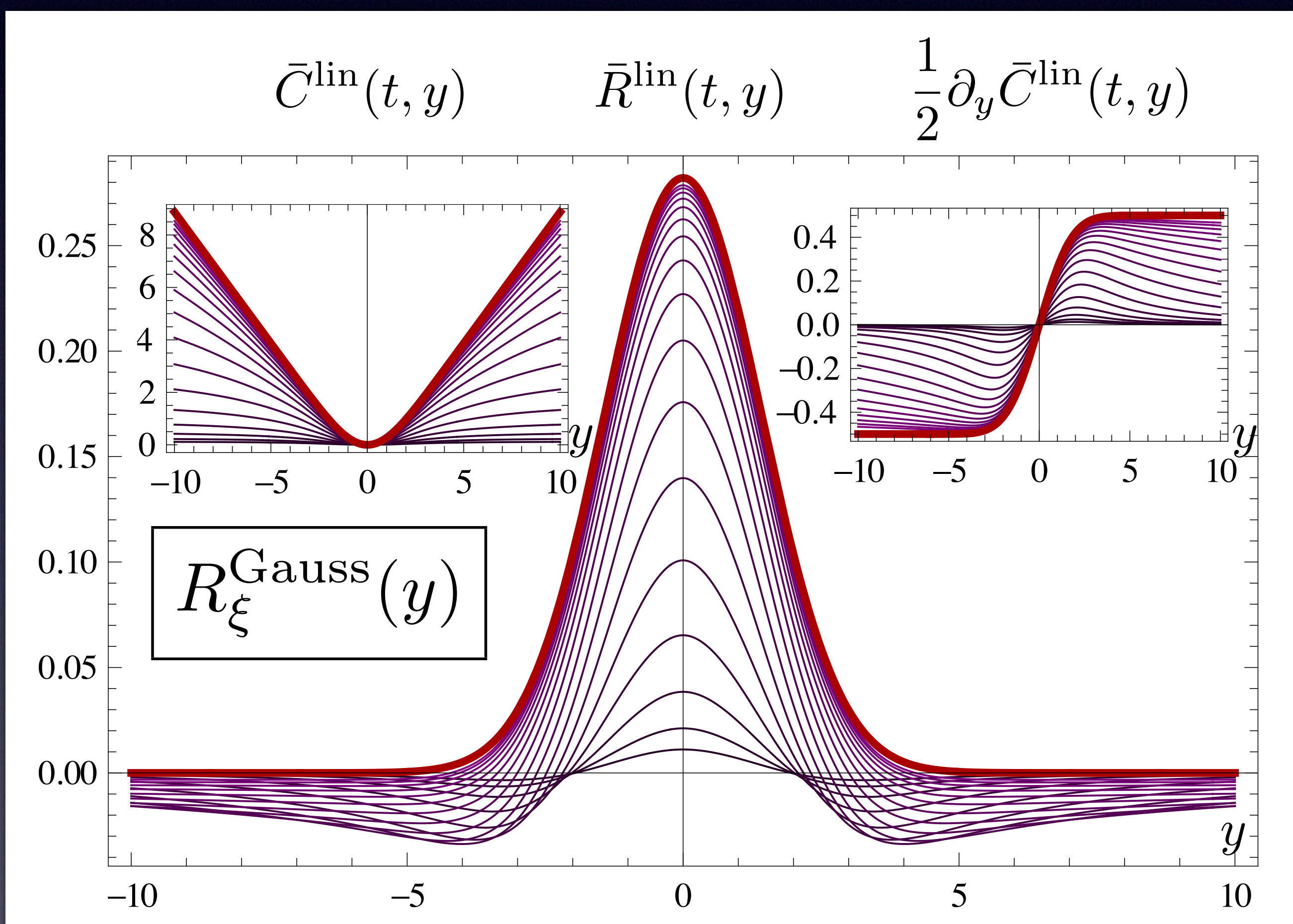
E. Agoritsas, V. Lecomte & T. Giamarchi, Phys. Rev. E **87**, 042406 (2013).

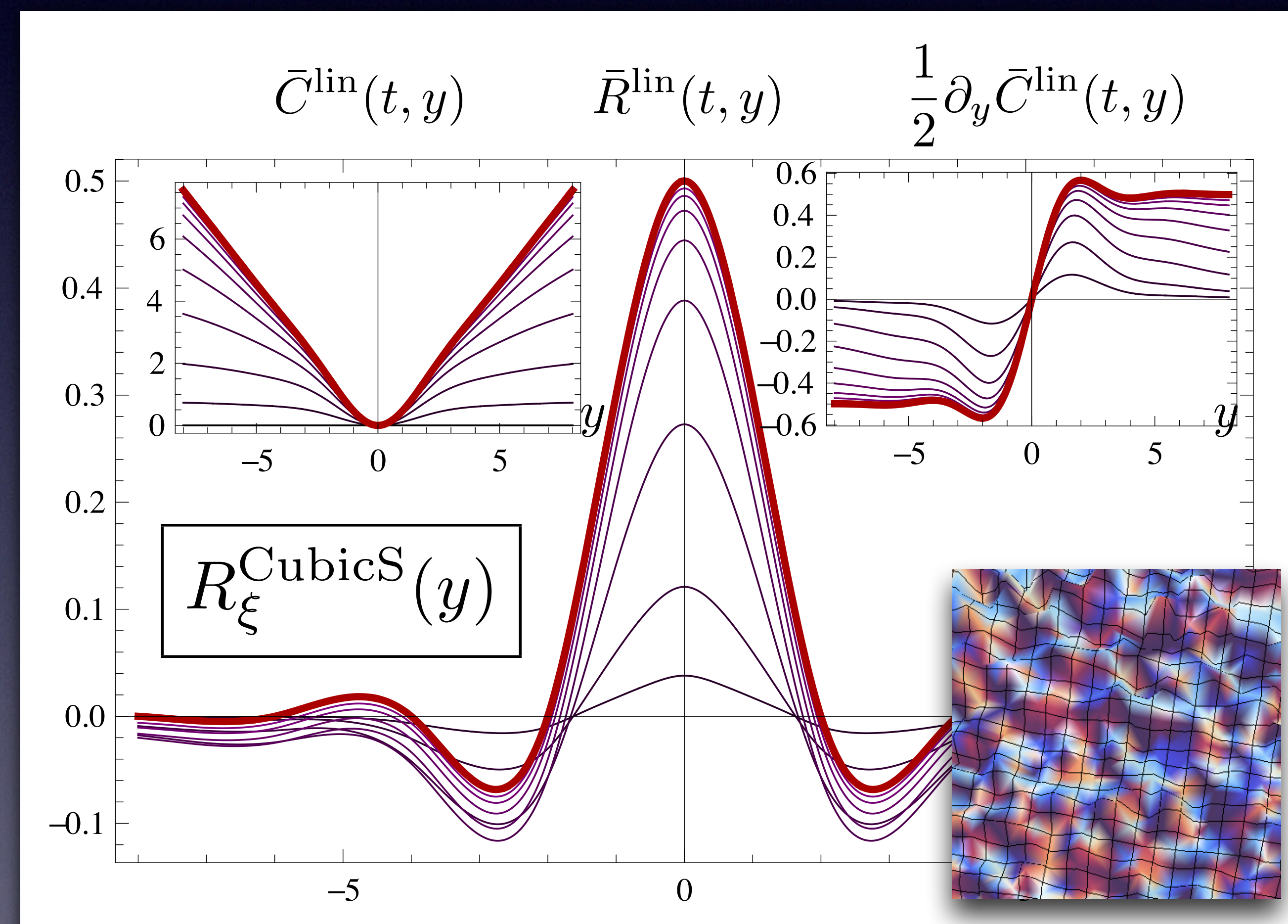
Linearised evolution of free-energy of the DP endpoint

$$\partial_t \bar{F}_V(t,y) = \frac{T}{2c}\partial_y^2 \bar{F}_V(t,y) - \cancel{\frac{1}{2c}\left[\partial_y \bar{F}_V(t,y)\right]^2} - \frac{y}{t}\partial_y \bar{F}_V(t,y) + V(t,y)$$

- Fluctuations are exactly Gaussian at all \`times' ⇒ fully characterized by: $\bar{R} = \overline{\partial \bar{F} \partial \bar{F}}$

$$\bar{R}^{\text{lin}}(t,y) = \frac{cD}{T}\left[R_\xi(y) - b^{\text{lin}}(t,y,\xi)\right]$$





- Normalization:

$$\int_{\mathbb{R}} dy\, \bar{R}^{\text{lin}}(t,y) = 0$$

- Scaling with the diffusive roughness:

$$b^{\text{lin}}(t,y,\xi) = \frac{\tilde{b}(y/\sqrt{B_{\text{th}}(t)}, \xi/\sqrt{B_{\text{th}}(t)})}{\sqrt{B_{\text{th}}(t)}}$$

- Asymptote:

$$\bar{R}^{\text{lin}}(\infty,y) = \frac{cD}{T} R_\xi(y)$$

Linearised evolution of free-energy of the DP endpoint

$$\partial_t \bar{F}_V(t,y) = \frac{T}{2c}\partial_y^2 \bar{F}_V(t,y) - \cancel{\frac{1}{2c}\left[\partial_y \bar{F}_V(t,y)\right]^2} - \frac{y}{t}\partial_y \bar{F}_V(t,y) + V(t,y)$$

- Fluctuations are exactly Gaussian at all `times' ⇒ fully characterized by:

$$\bar{R}^{\mathrm{lin}}(t,y) = \frac{cD}{T}\left[R_\xi(y) - b^{\mathrm{lin}}(t,y,\xi)\right]$$

$$b^{\mathrm{lin}}(t,y,\xi) = \frac{\tilde{b}(y/\sqrt{B_{\mathrm{th}}(t)}, \xi/\sqrt{B_{\mathrm{th}}(t)})}{\sqrt{B_{\mathrm{th}}(t)}}$$

$$\begin{aligned}\tilde{b}^{\mathrm{lin}}(\bar{y}_t,\bar{\xi}_t) = -\bar{y}_t R^{(-1)}_{\bar{\xi}_t}(\bar{y}_t) &+ \int_0^\infty dw\, w^2 e^{-w(w+\bar{y}_t)} R^{(-1)}_{\bar{\xi}_t}(w) \\ &+ \int_{\bar{y}_t}^\infty dw\, w^2 e^{-w(w-\bar{y}_t)} R^{(-1)}_{\bar{\xi}_t}(w)\end{aligned} \qquad (5.26)$$

where $R^{(-1)}_\xi(y)$ denotes the primitive of the disorder correlator. The subscript 't' is kept in $\bar{y}_t$ and $\bar{\xi}_t$ to emphasize where the purely diffusive rescaling $B_{\mathrm{th}}(t) = \frac{Tt}{c}$ intervenes.

Full vs linearised evolution of free-energy of the DP endpoint

- Tilt KPZ evolution equation for the disorder free-energy $\bar{F}_V(t,y)$

$$\begin{cases} \partial_t \bar{F}_V(t,y) = \dfrac{T}{2c}\partial_y^2 \bar{F}_V(t,y) - \cancel{\dfrac{1}{2c}\left[\partial_y \bar{F}_V(t,y)\right]^2} - \dfrac{y}{t}\partial_y \bar{F}_V(t,y) + V(t,y) \\ \partial_t \bar{F}_V^{\mathrm{lin}}(t,y) = \dfrac{T}{2c}\partial_y^2 \bar{F}_V^{\mathrm{lin}}(t,y) - \dfrac{y}{t}\partial_y \bar{F}_V^{\mathrm{lin}}(t,y) + V(t,y) \end{cases}$$

$$\boxed{F_V^{\mathrm{lin}}(t,y) \equiv F_{V=0}(t,y) + \bar{F}_V^{\mathrm{lin}}(t,y)}$$

- KPZ evolution equation for the total free-energy $F_V(t,y)$

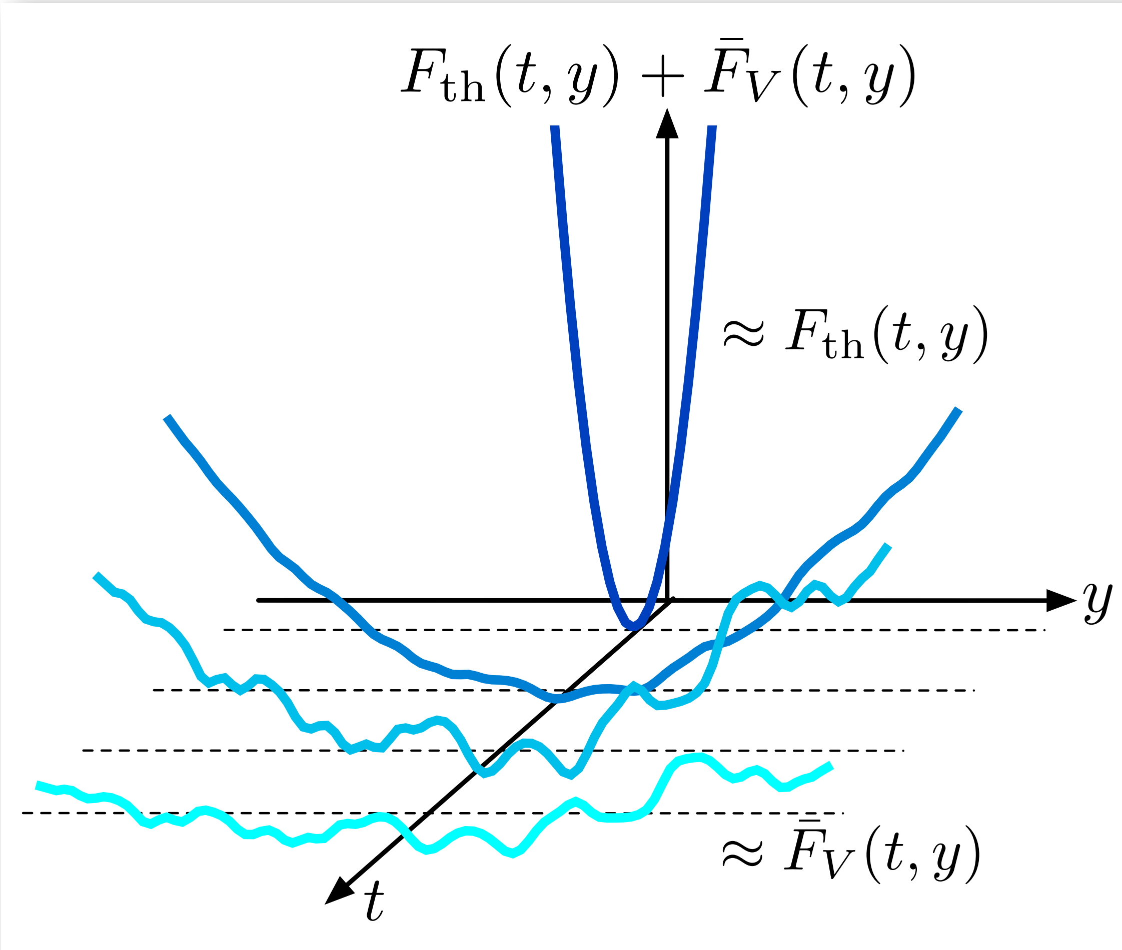
$$\begin{cases} \partial_t F_V(t,y) = \dfrac{T}{2c}\partial_y^2 F_V(t,y) - \dfrac{1}{2c}\left[\partial_y F_V(t,y)\right]^2 + V(t,y) \\ \partial_t F_V^{\mathrm{lin}}(t,y) = \left[\dfrac{T}{2c}\partial_y^2 - \dfrac{y}{t}\partial_y\right] F_V^{\mathrm{lin}}(t,y) + \dfrac{cy^2}{2t^2} + V(t,y) \end{cases}$$

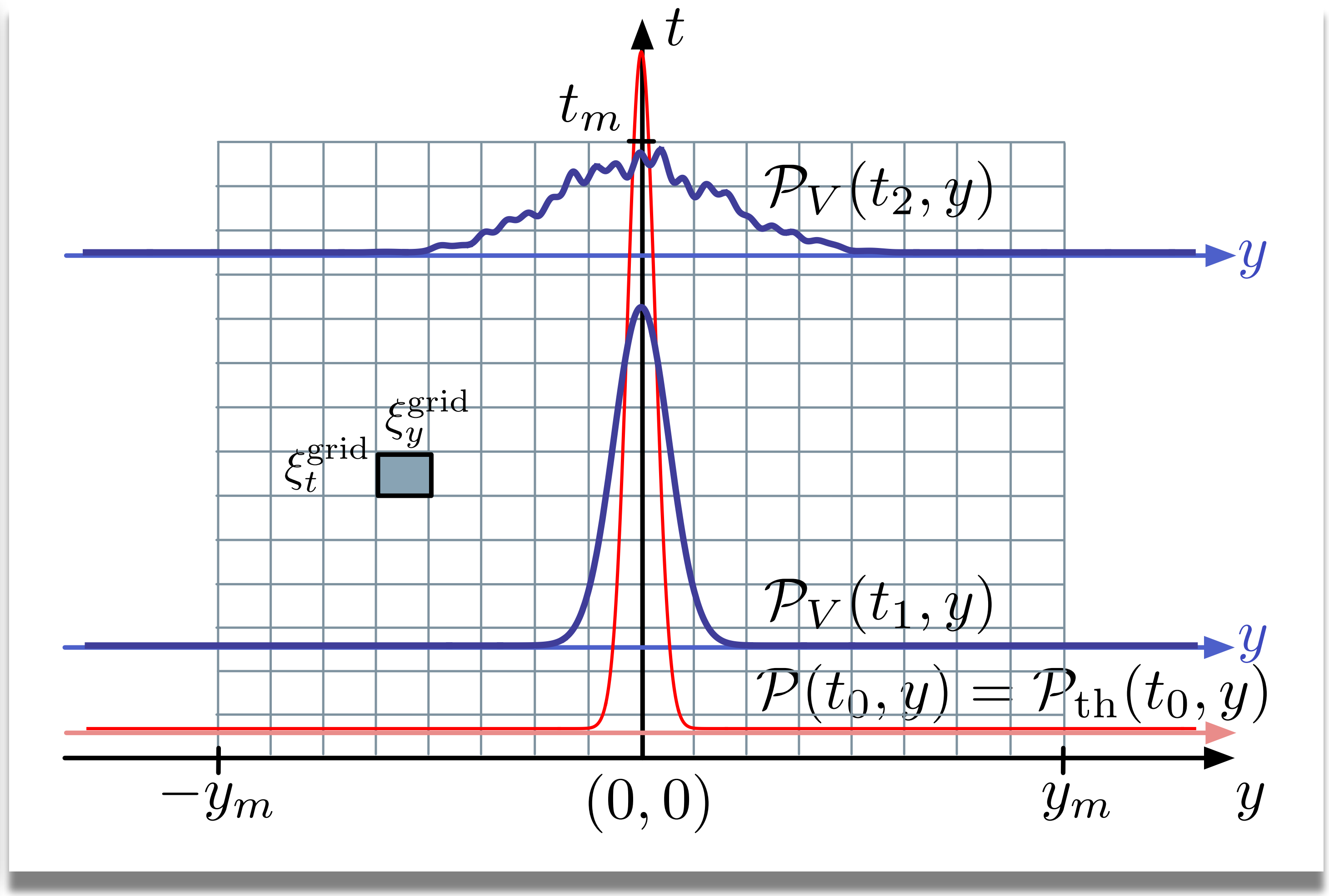
$$\boxed{\mathcal{Z}_V^{\mathrm{lin}}(t,y) \equiv e^{-F_V^{\mathrm{lin}}(t,y)/T}}$$

- Stochastic heat equation for the partition function $\mathcal{Z}_V(t,y)$

$$\begin{cases} \partial_t \mathcal{Z}_V(t,y) = \left[\dfrac{T}{2c}\partial_y^2 - \dfrac{1}{T}V(t,y)\right]\mathcal{Z}_V(t,y) \\ \partial_t \mathcal{Z}_V^{\mathrm{lin}}(t,y) = \left\{\dfrac{T}{2c}\partial_y^2 - \dfrac{1}{T}\left[V(t,y) + \dfrac{1}{2c}\left(\partial_y \bar{F}_V^{\mathrm{lin}}(t,y)\right)^2\right]\right\}\mathcal{Z}_V^{\mathrm{lin}}(t,y) \end{cases}$$

Numerics: evolution of disorder free-energy of the DP endpoint



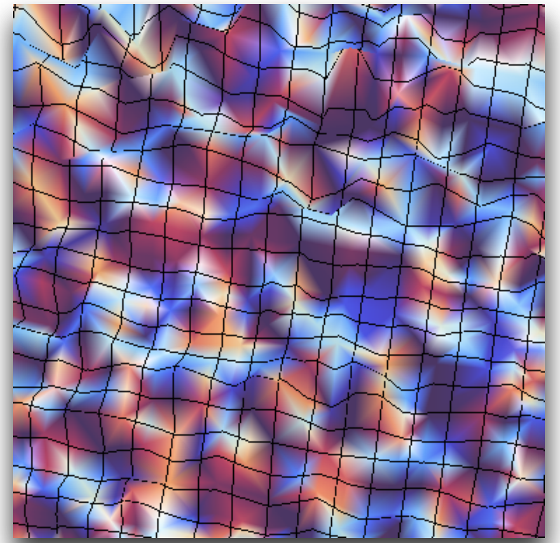


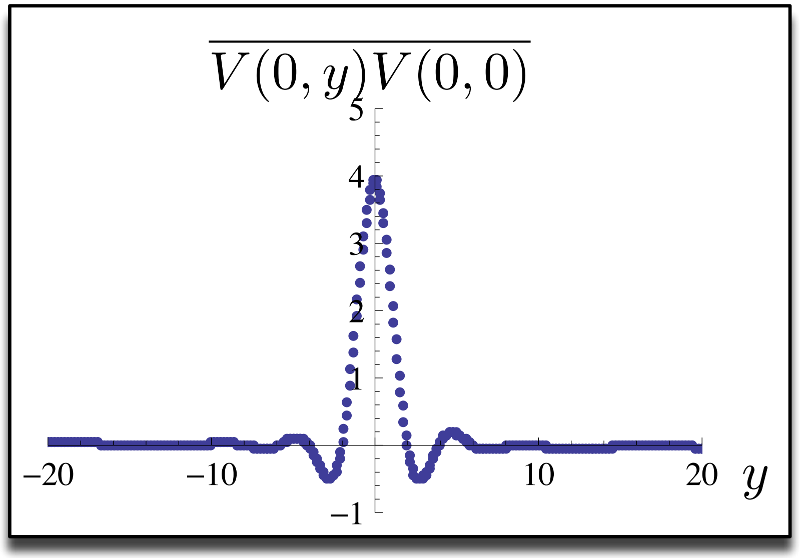
$$\partial_t \bar{F}_V(t,y) = \frac{T}{2c}\partial_y^2 \bar{F}_V(t,y) - \frac{1}{2c}\left[\partial_y \bar{F}_V(t,y)\right]^2 - \frac{y}{t}\partial_y \bar{F}_V(t,y) + V(t,y)$$

Initial condition: $\bar{F}_V(t_0,y)\equiv 0$

Boundary condition: $\partial_y \bar{F}_V(t_0,\pm y_m) = 0$

$$\overline{V(t,y)V(t,0)} = D\,\xi_t^{\mathrm{grid}}\,R^{\mathrm{CubicS}}(y)$$





E. Agoritsas, V. Lecomte & T. Giamarchi, *Phys. Rev. E* 87, 062405 (2013).

Numerics: evolution of disorder free-energy of the DP endpoint

- Focus on the two-point correlators:

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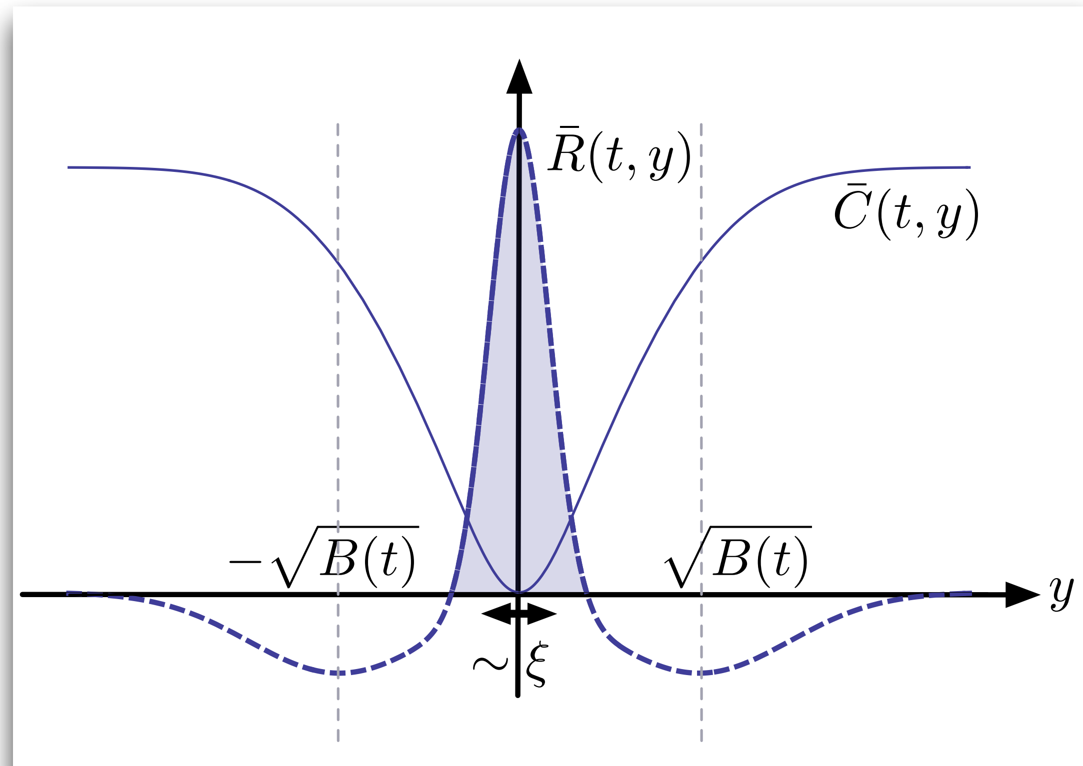

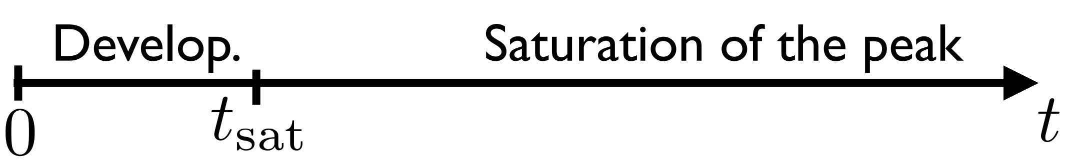

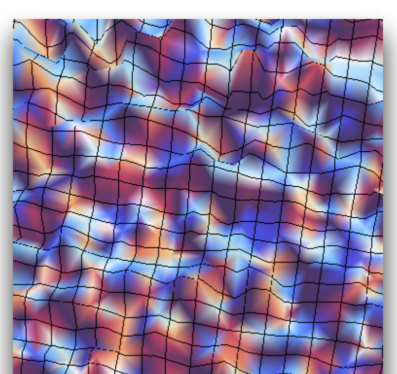

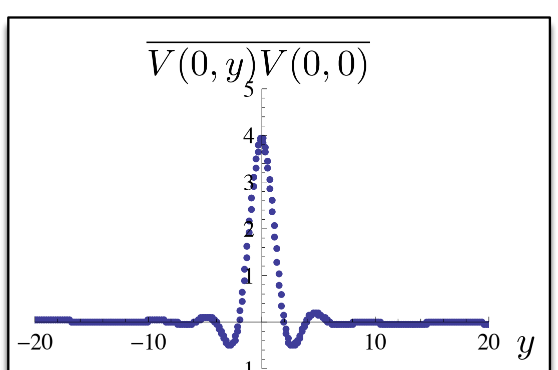

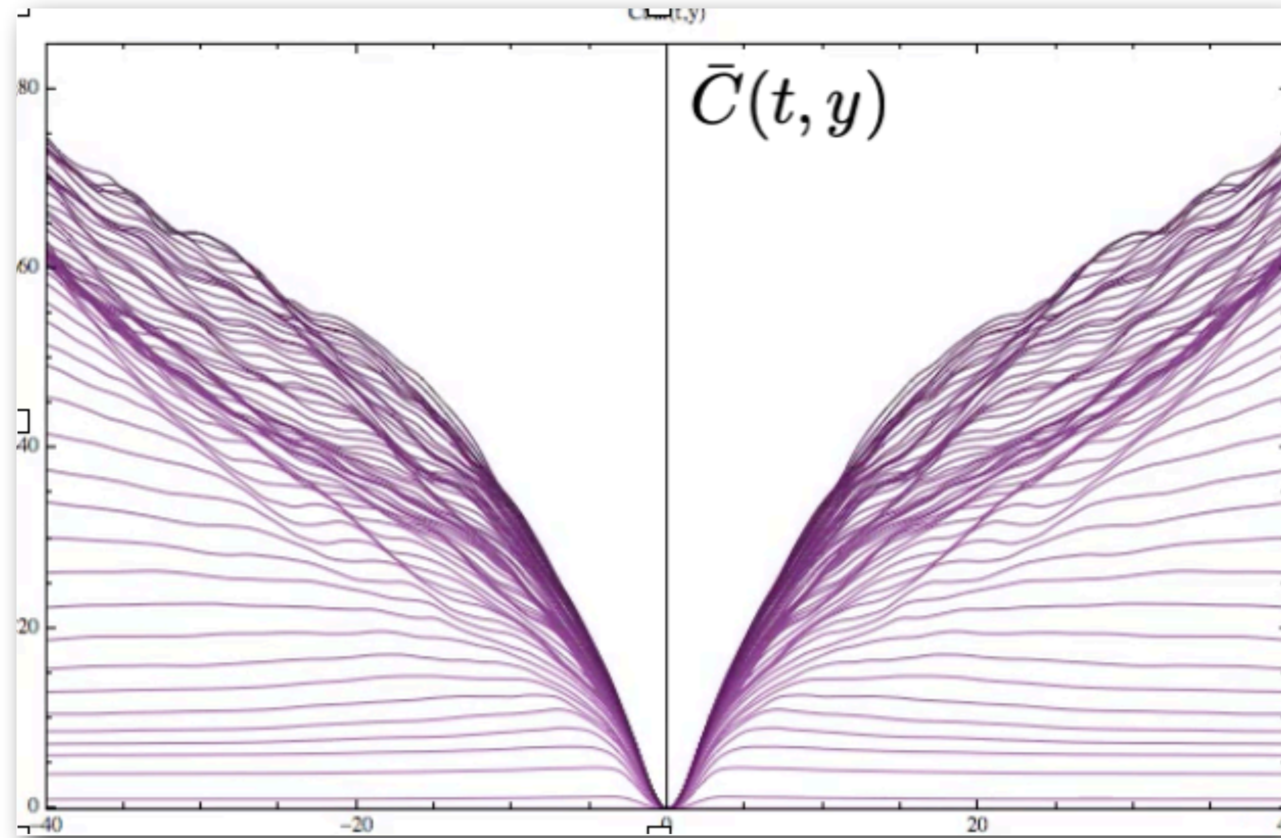

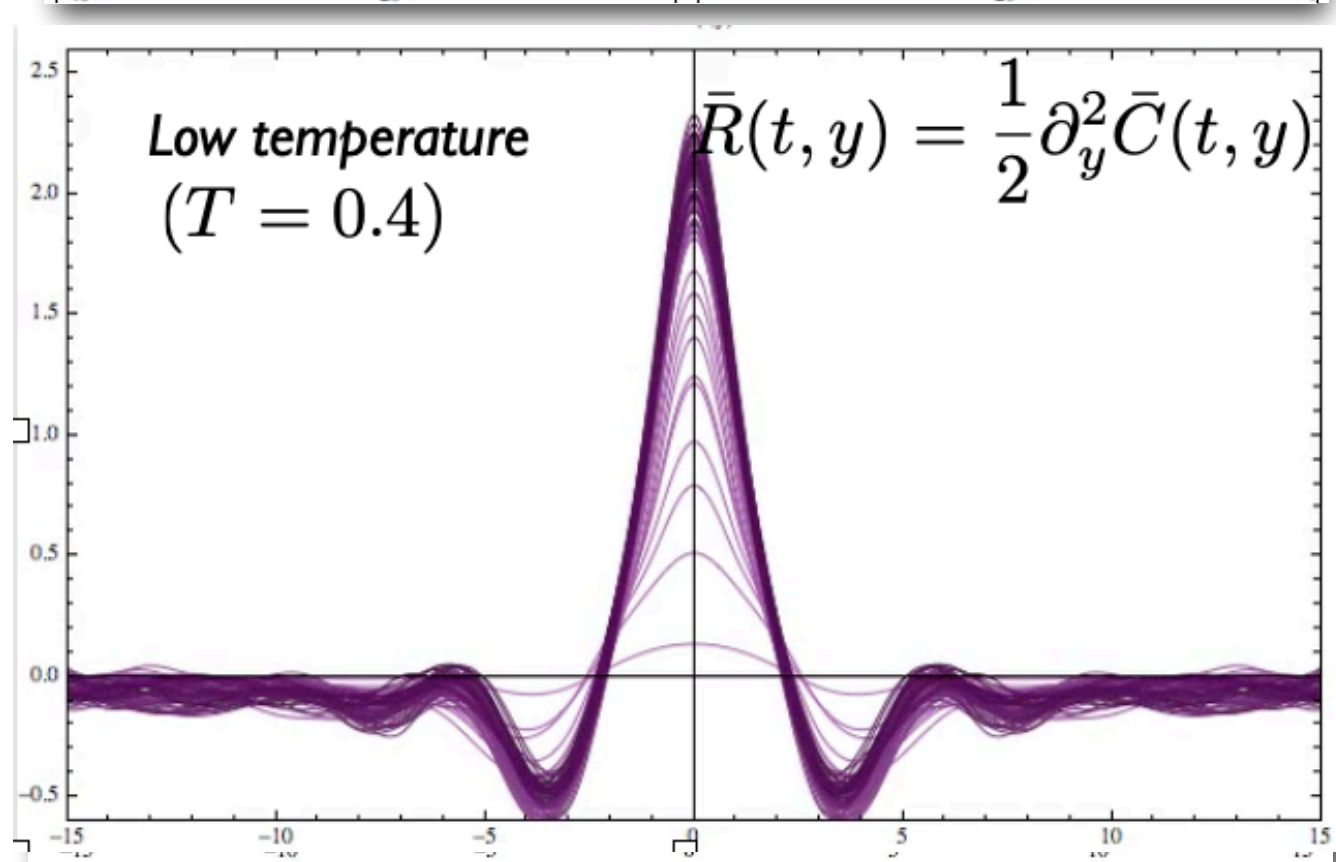

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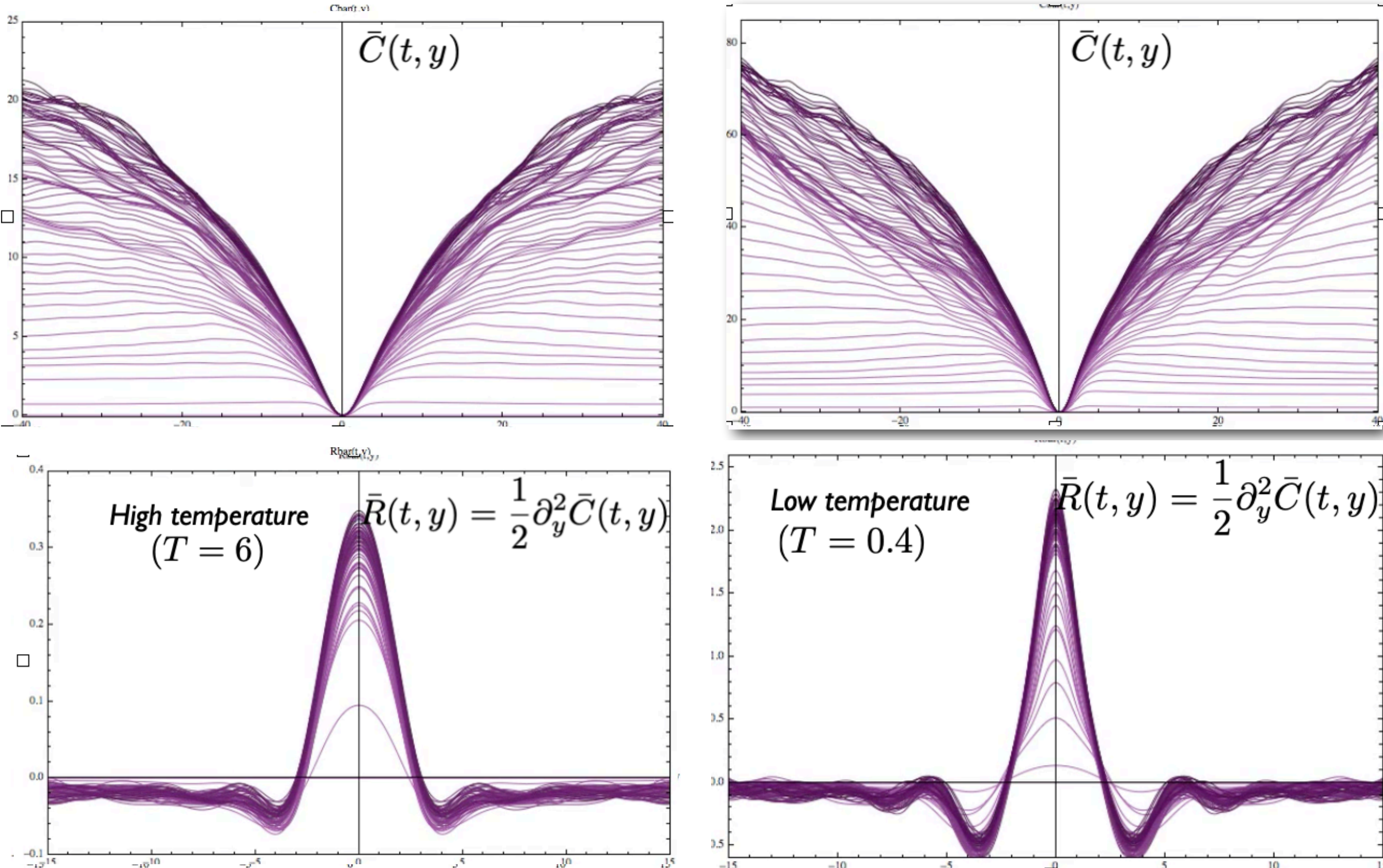
Numerics: evolution of disorder free-energy of the DP endpoint



E. Agoritsas, V. Lecomte & T. Giamarchi, *Phys. Rev. E* 87, 062405 (2013).

Numerics: asymptotic 2-pt correlator of disorder free-energy

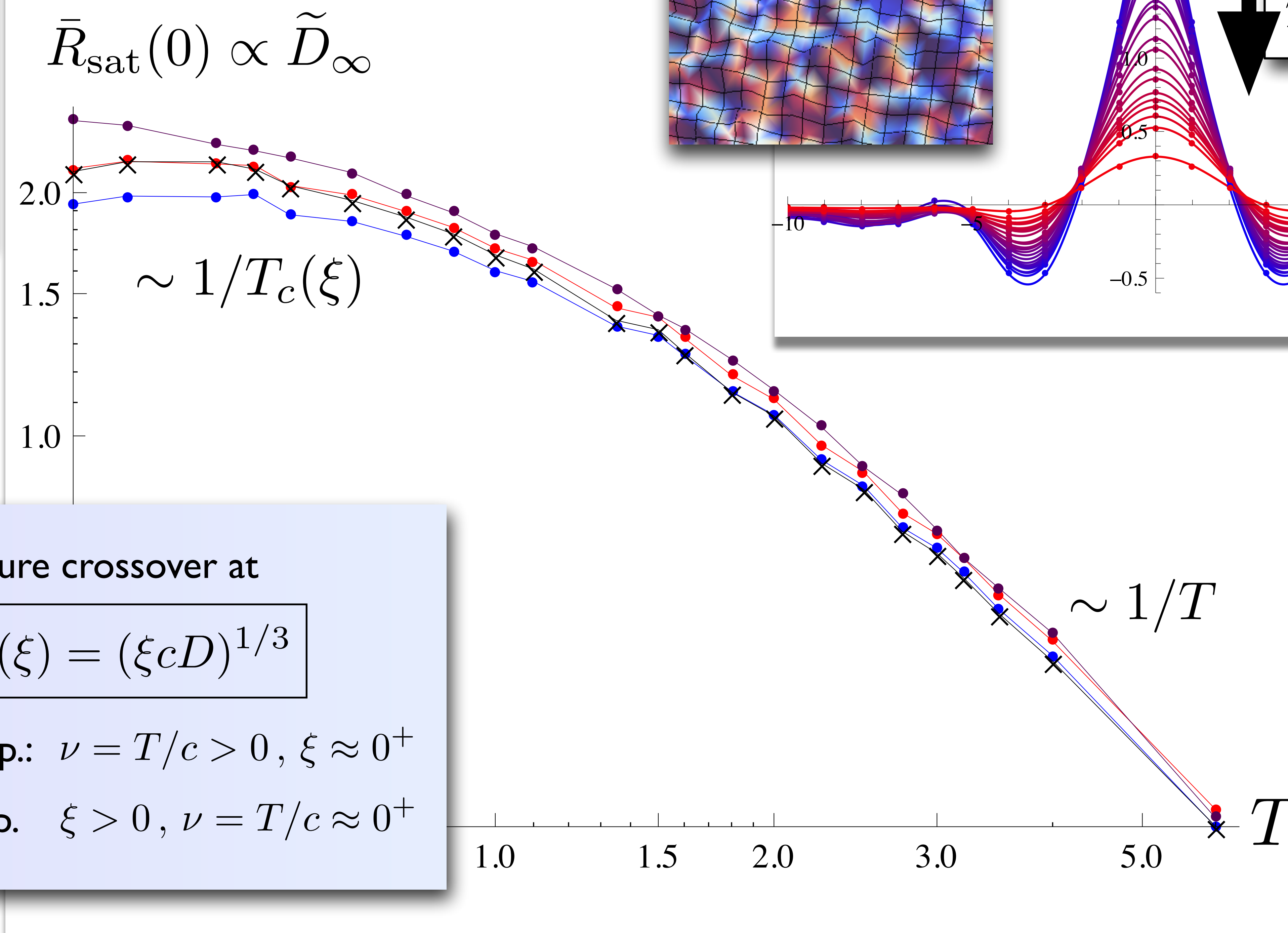
$$\bar{R}_{\mathrm{sat}}(y) \approx \bar{R}(\infty, y) = \frac{1}{2}\partial_y^2 \bar{C}(\infty, y)$$

- Amplitude of the correlator / Maximum value:

$T \approx 0$

$\widetilde{D}_\infty \sim \frac{cD}{T_c}$

$\mathcal{R}_{\tilde{\xi}} \approx ??$



- Temperature crossover at

$$\boxed{T_c(\xi) = (\xi c D)^{1/3}}$$

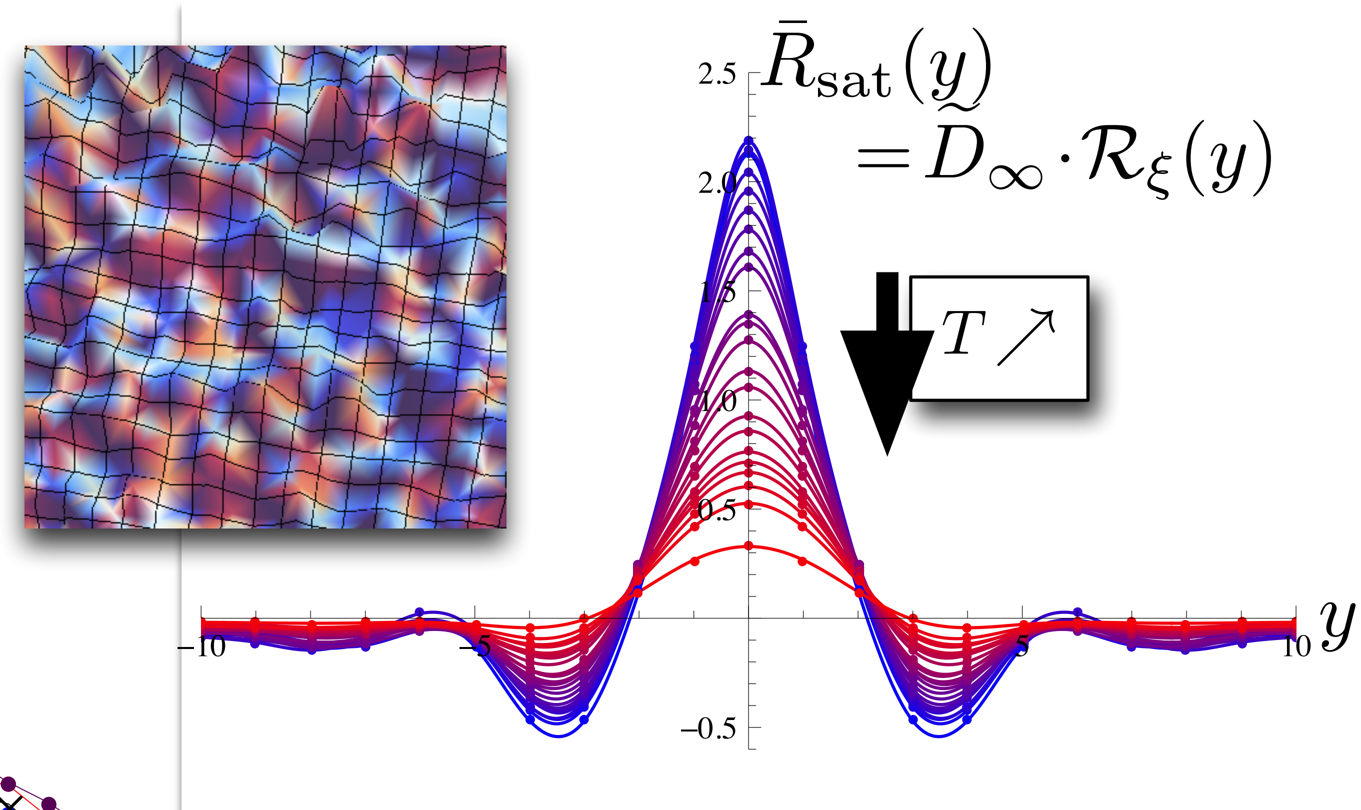
- High temp.: $\nu = T/c > 0\,,\ \xi \approx 0^+$
- Low temp. $\xi > 0\,,\ \nu = T/c \approx 0^+$

$\xi \approx 0$

$\widetilde{D}_\infty \lesssim \frac{cD}{T}$

$\mathcal{R}_{\tilde{\xi}} \approx R_\xi$

- Shape of the asymptotic correlator:



E. Agoritsas, V. Lecomte & T. Giamarchi, *Phys. Rev. E* 87, 042406 & 062405 (2013).

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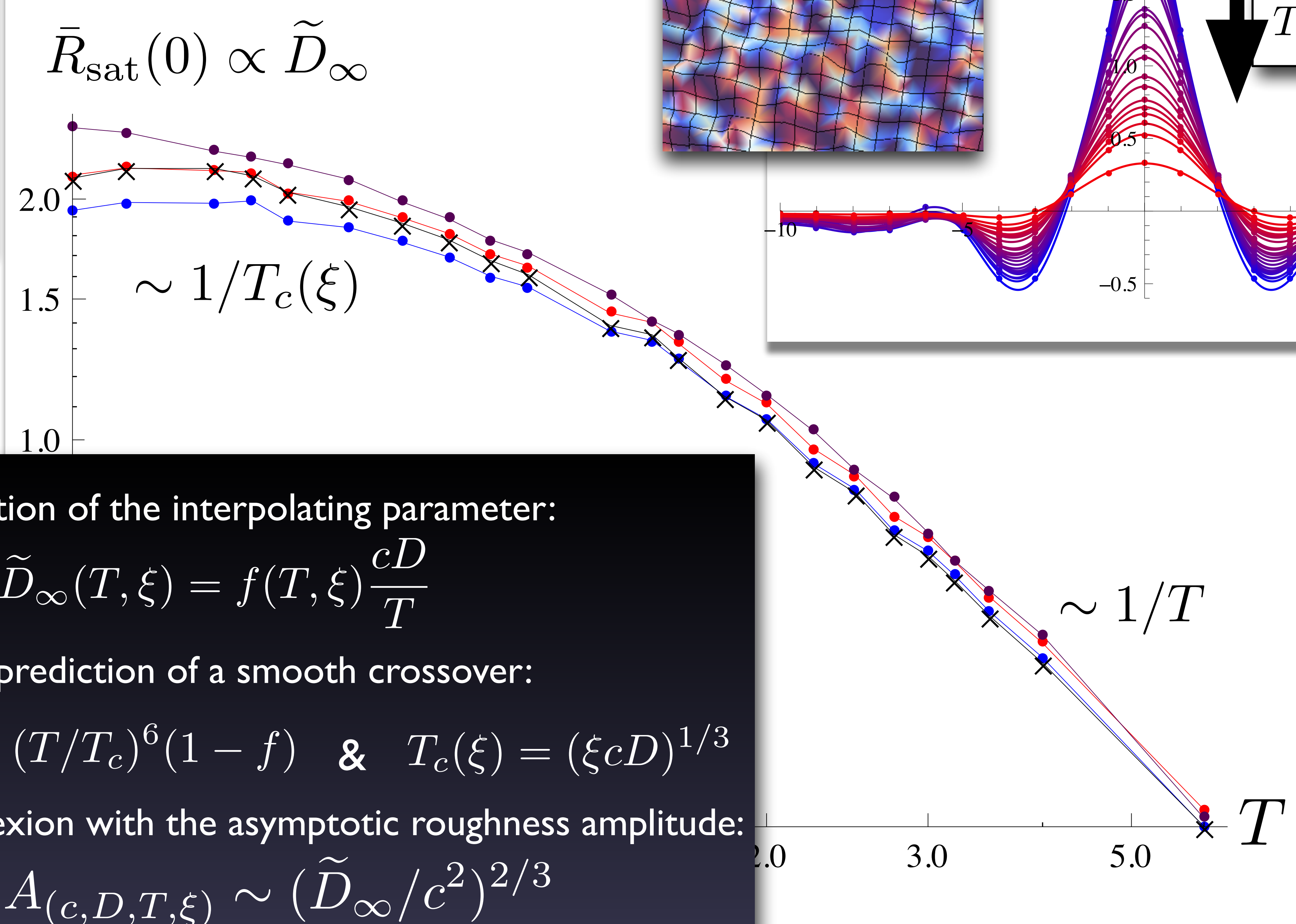
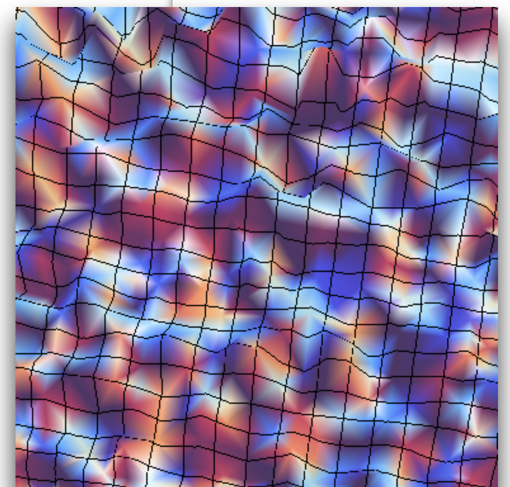
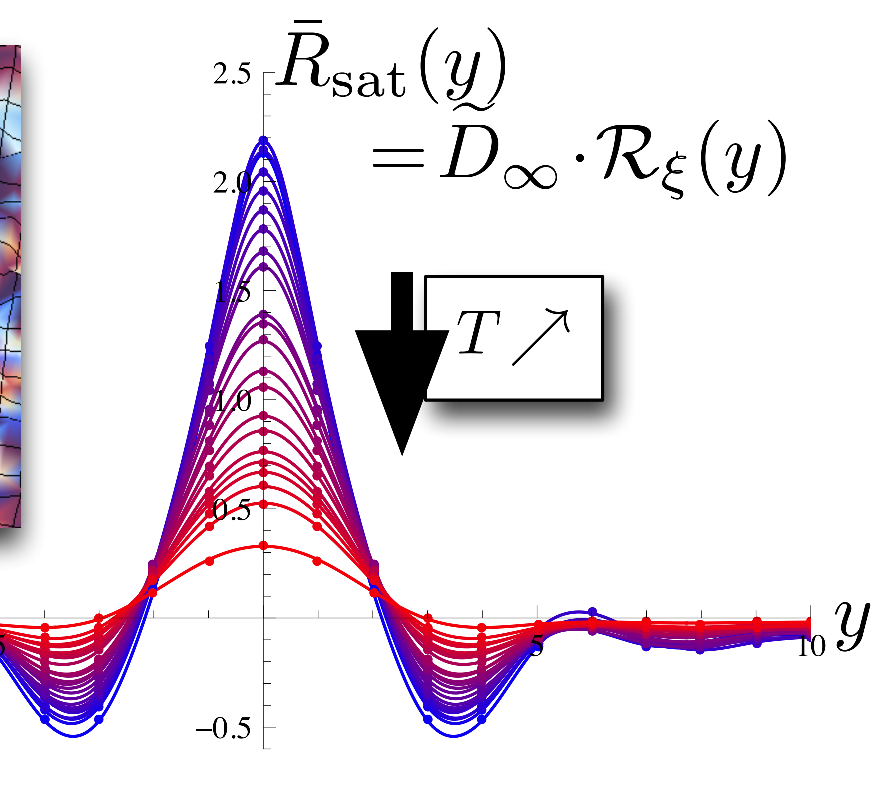
$\widetilde{D}_\infty \sim \dfrac{cD}{T_c}$

$\mathcal{R}_{\tilde{\xi}} \approx ??$



- Shape of the asymptotic correlator:





- Definition of the interpolating parameter:

$$\widetilde{D}_\infty(T,\xi) = f(T,\xi)\frac{cD}{T}$$

- GVM prediction of a smooth crossover:

$$f^6 \propto (T/T_c)^6(1-f) \quad \& \quad T_c(\xi) = (\xi c D)^{1/3}$$

- Connexion with the asymptotic roughness amplitude:

$$A_{(c,D,T,\xi)} \sim (\widetilde{D}_\infty/c^2)^{2/3}$$

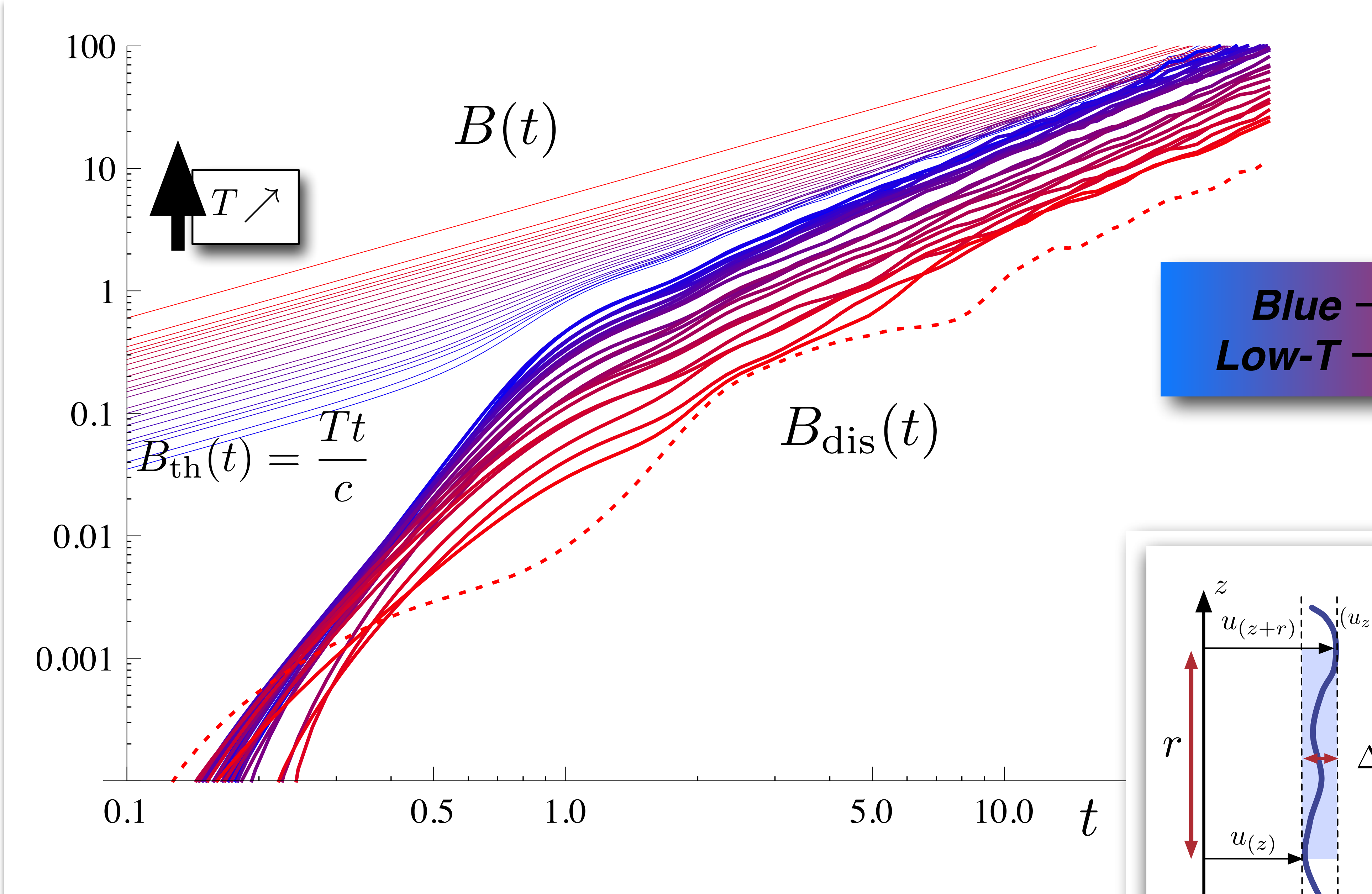
$\xi \approx 0$

$\widetilde{D}_\infty \lesssim \dfrac{cD}{T}$

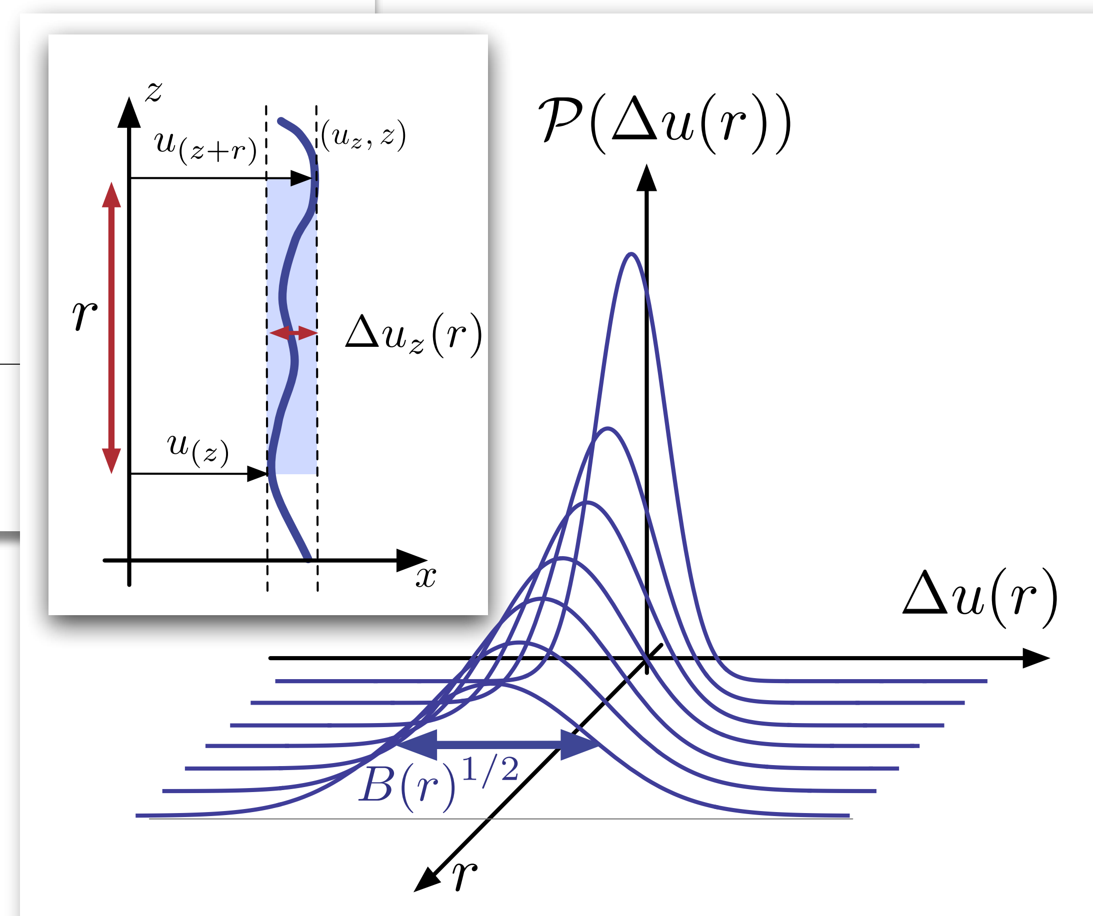
$\mathcal{R}_{\tilde{\xi}} \approx R_\xi$

E. Agoritsas, V. Lecomte & T. Giamarchi, *Phys. Rev. E* 87, 042406 & 062405 (2013).

Numerics: temperature-dependent roughness



Blue → Red:
Low-T → High-T

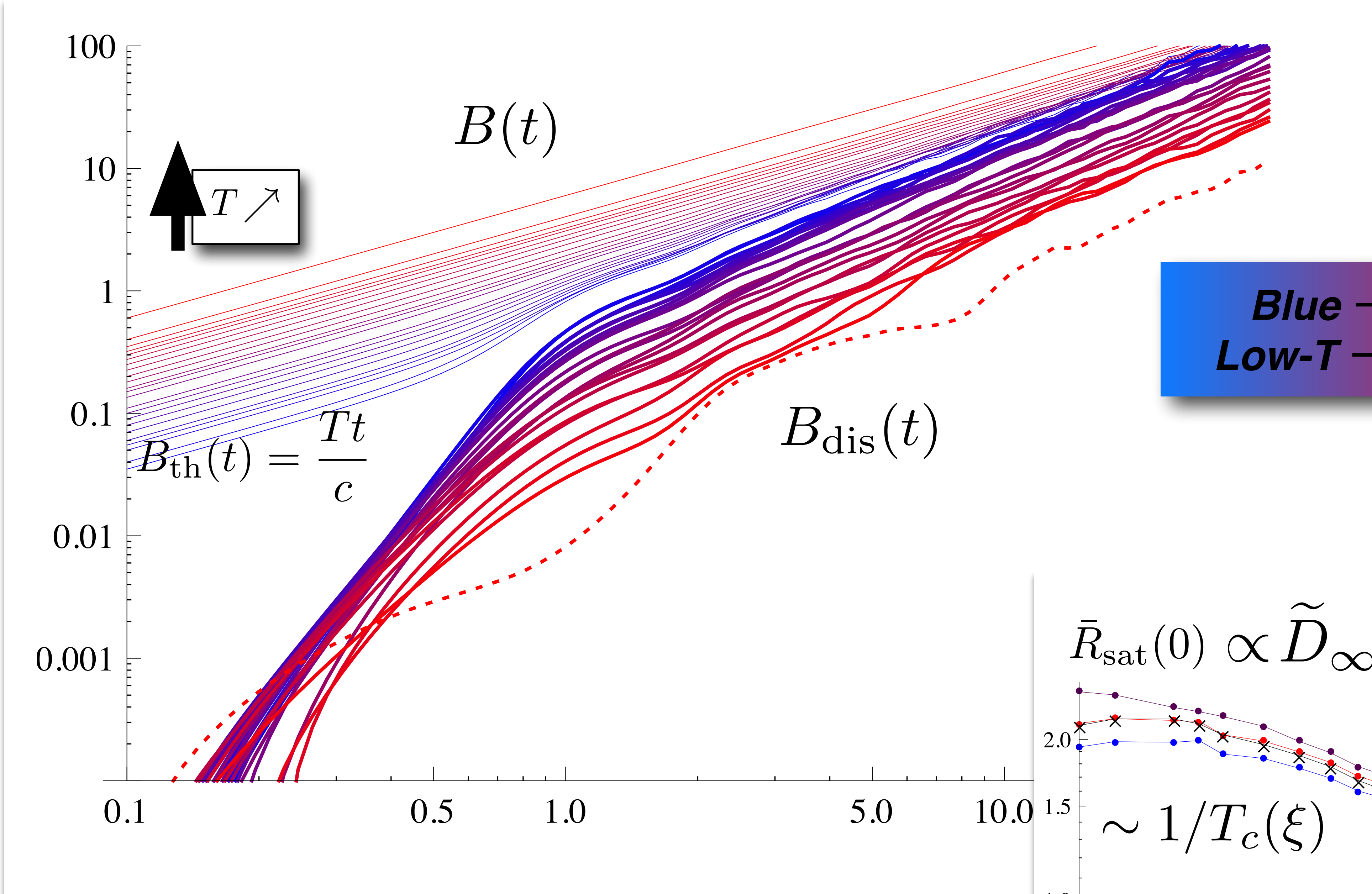


$$B(t) \equiv \overline{\langle y(t)^2 \rangle} = B_{\mathrm{thermal}}(t) + B_{\mathrm{dis}}(t)$$

$$\begin{cases} B(t) \overset{(t\to\infty)}{\sim} A_{(c,D,T,\xi)}\, t^{4/3} \\ A_{(c,D,T,\xi)} \sim (\widetilde{D}_\infty/c^2)^{2/3} \end{cases}$$

E. Agoritsas, V. Lecomte & T. Giamarchi, *Phys. Rev. E* 87, 042406 & 062405 (2013).

Numerics: temperature-dependent roughness



Blue → Red:
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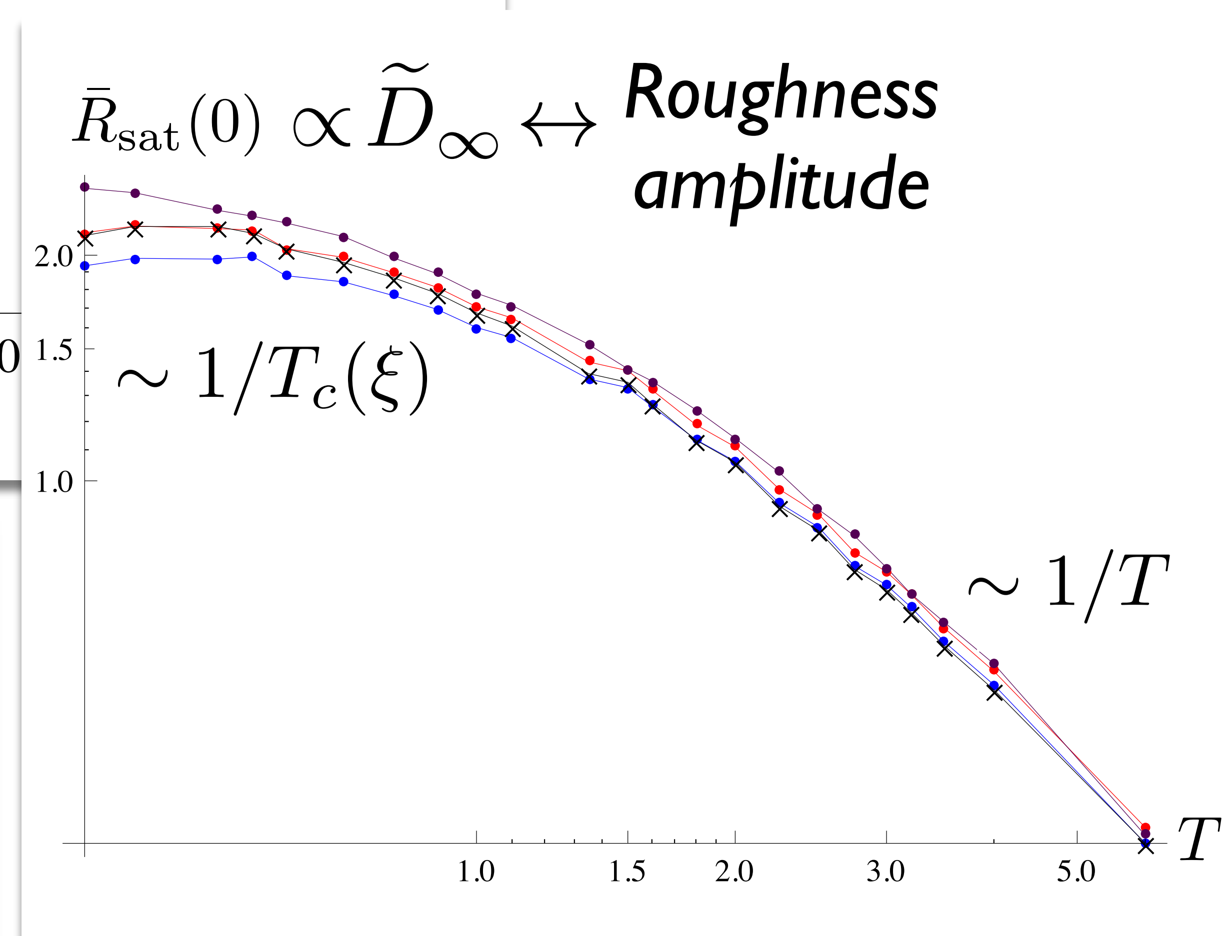

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E. Agoritsas, V. Lecomte & T. Giamarchi, *Phys. Rev. E* 87, 042406 & 062405 (2013).

Roughness regimes & characteristic crossover scales

ROUGHNESS $B(t)$

THERMAL $B_{\rm th}(t) = \dfrac{Tt}{c}$

THERMAL $(T \gg T_c)$

MODIFIED LARKIN $(T \ll T_c)$

RANDOM MANIFOLD $B_{\rm RM}(t) \sim (\widetilde{D}_\infty/c^2)^{\frac{2}{3}}\, t^{\frac{4}{3}}$

0 $L_c(T=0)$ $L_c \sim \dfrac{T^5}{cD^2}/f^5$ ‘time’ t

$T \nearrow$

0 $t_{\rm sat}$ $L_c(T=0)$ ‘time’ t

DISORDER FREE-ENERGY

SHORT 'TIMES'

SATURATION at $|y| \lesssim \xi$

$\bar{\mathcal{P}}\left[\bar{F}_V, t\right]$

Feedback of the KPZ nonlinearity $-\dfrac{1}{2c}\left[\partial_y \bar{F}_V\right]^2$

Non-Gaussian features: $\partial_t \bar{R}(t,y),\ \partial_t \bar{R}_3(t,y),\ \dots$

Non-Gaussian steady-state: $\bar{R}(\infty,y) = \widetilde{D}_\infty \cdot \mathcal{R}_\xi(y)$

Linearized evolution: Gaussian $\bar{\mathcal{P}}\left[\bar{F}_V^{\rm lin}, t\right]$

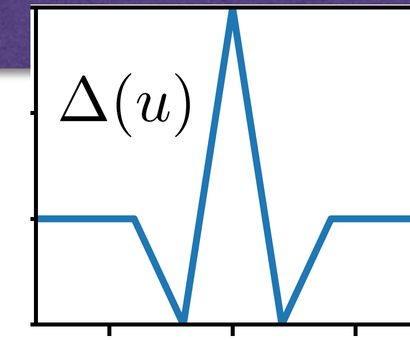
$B_{\rm dis}(t)$

Powerlaw? $(\zeta_{\rm num} \cancel{\approx 2 -} 2.5\,?)$

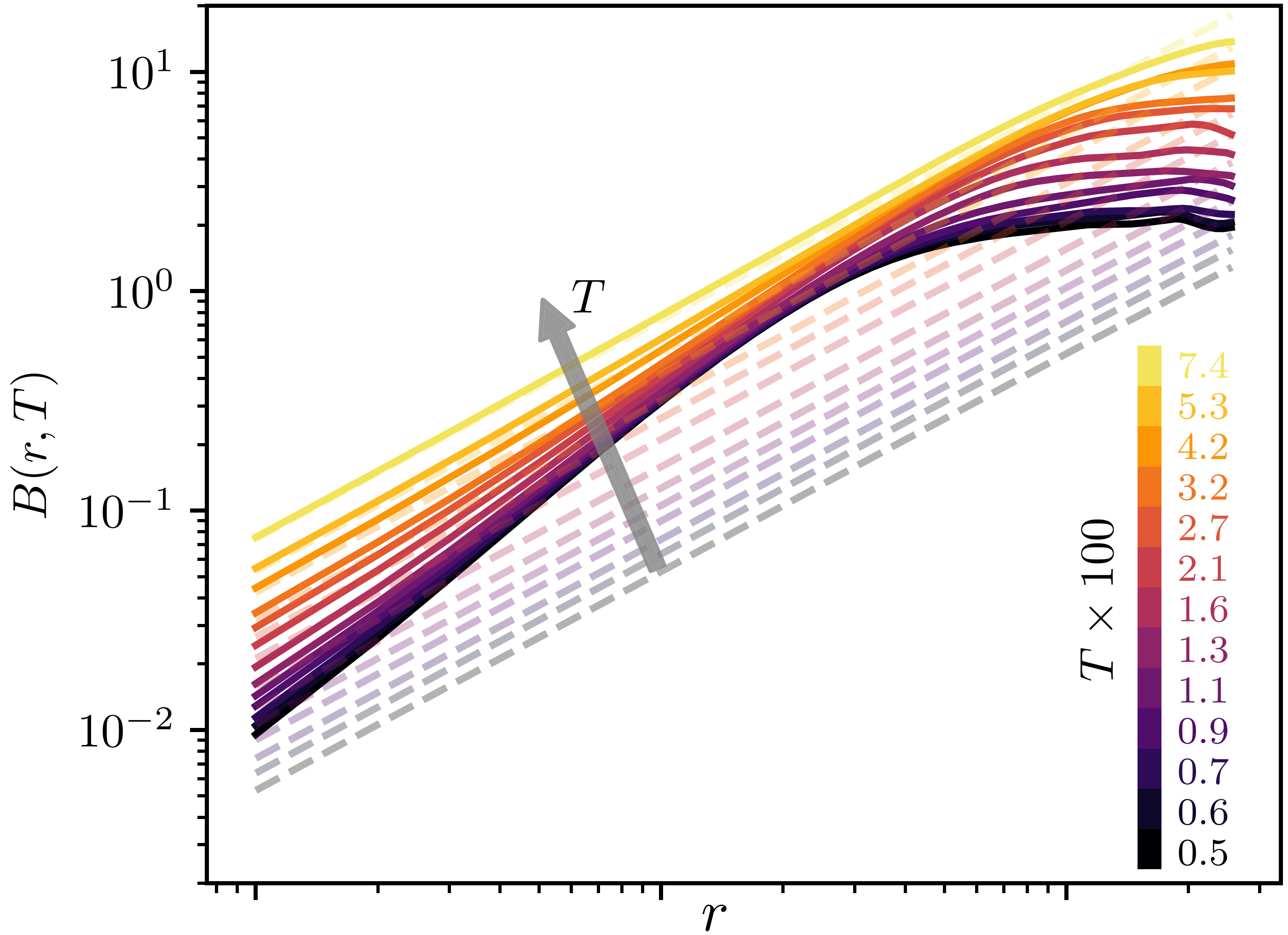
$B_{\rm dis}(t) \sim (\widetilde{D}_\infty/c^2)^{\frac{2}{3}}\, t^{\frac{4}{3}}$ versus $B_{\rm th}(t) = \dfrac{Tt}{c}$

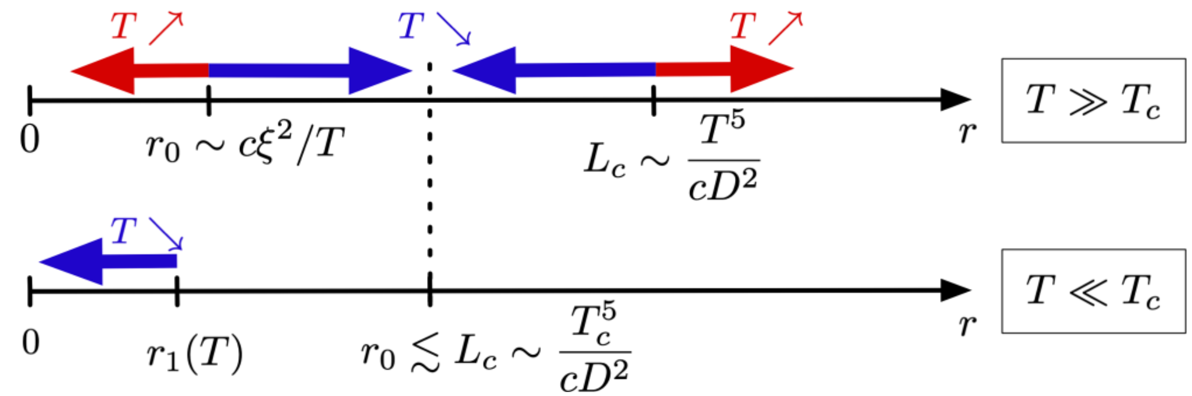
in the total roughness: $B(t) = B_{\rm th}(t) + B_{\rm dis}(t)$

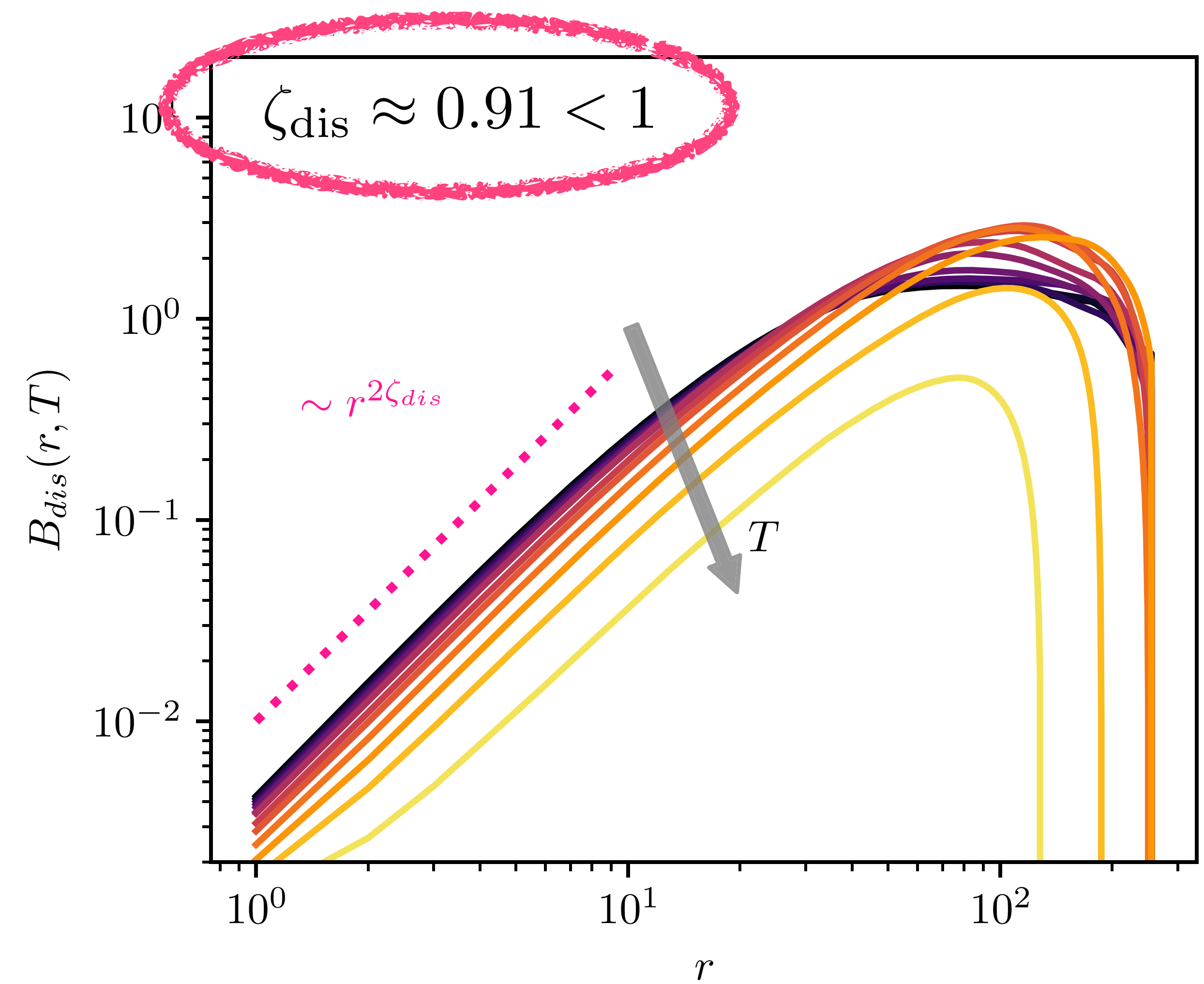
Numerics: temperature-dependent roughness



N. Caballero, T. Giamarchi, V. Lecomte & E. Agoritsas *Phys. Rev. E* 105, 044138 (2022).
"Microscopic interplay of temperature and disorder of a one-dimensional elastic interface"







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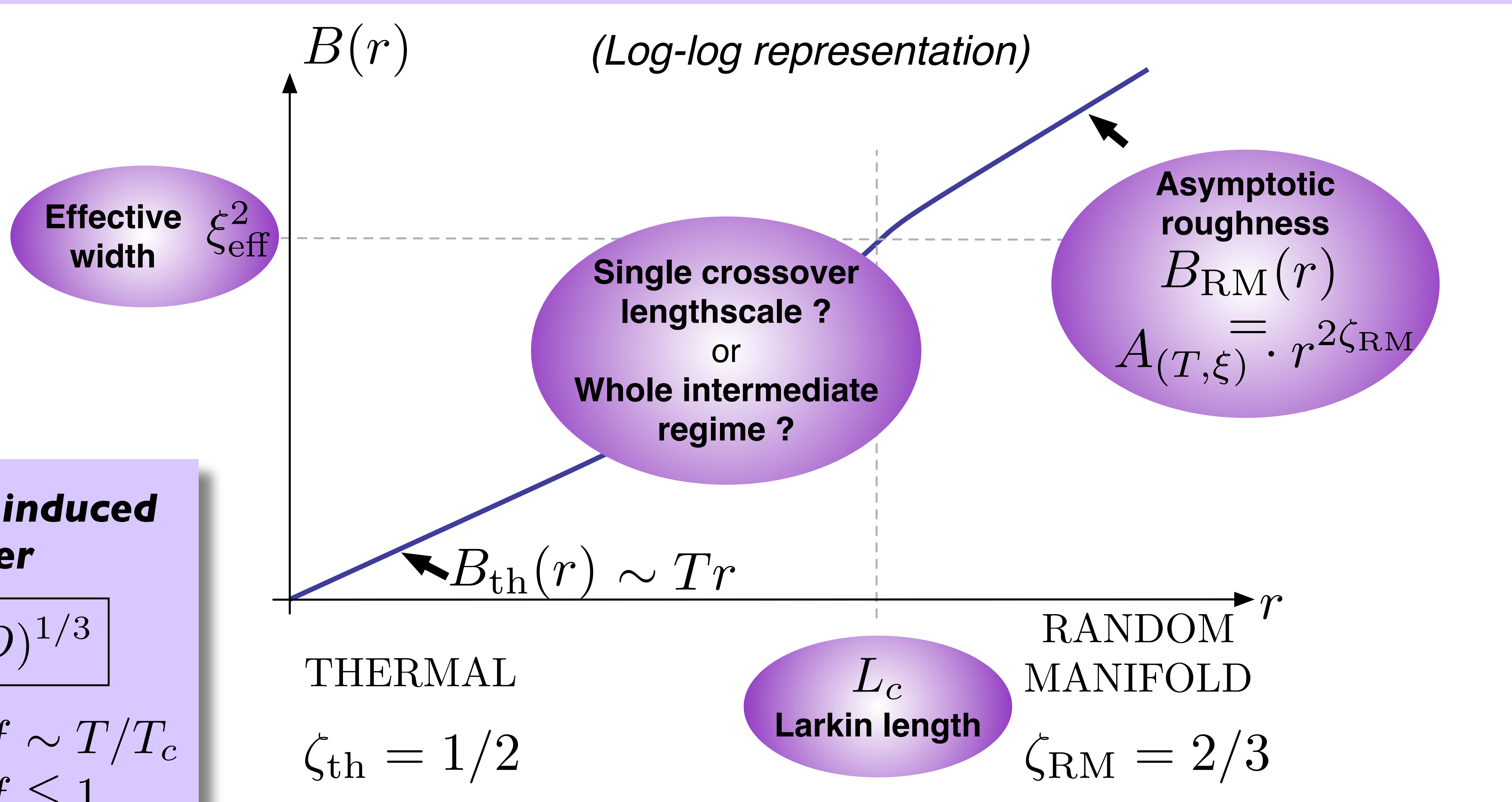


Temperature-induced crossover

$$\boxed{T_c = (\xi c D)^{1/3}}$$

$$\begin{cases} T \ll T_c : & f \sim T/T_c \\ T \gg T_c : & f \lesssim 1 \end{cases}$$

Universal scaling

KPZ scaling for the asymptotic roughness

$$B(r) \overset{(r\to\infty)}{\sim} \left(\frac{D}{cT/f}\right)^{2/3} r^{4/3}$$

Non-universal features

Roughness amplitude: $A_{(T,\xi)} \sim \left(\frac{D}{cT/f}\right)^{2/3}$

Larkin length: $L_c = \frac{(T/f)^5}{cD^2}$

- ***Physical motivation***: study of **interfaces**, roughness regimes (exponents, amplitude, crossover scales)

  ⇒ how to extract information on the **microscopic disorder, beyond universal scalings**

- ***Exact mapping***: static 1D interface with short-range elasticity & spatially-correlated disorder

  ⇒ 1+1 Directed Polymer (DP) free-energy endpoint

  ⇒ 1D KPZ equation with spatially-correlated noise & 'sharp-wedge' initial condition

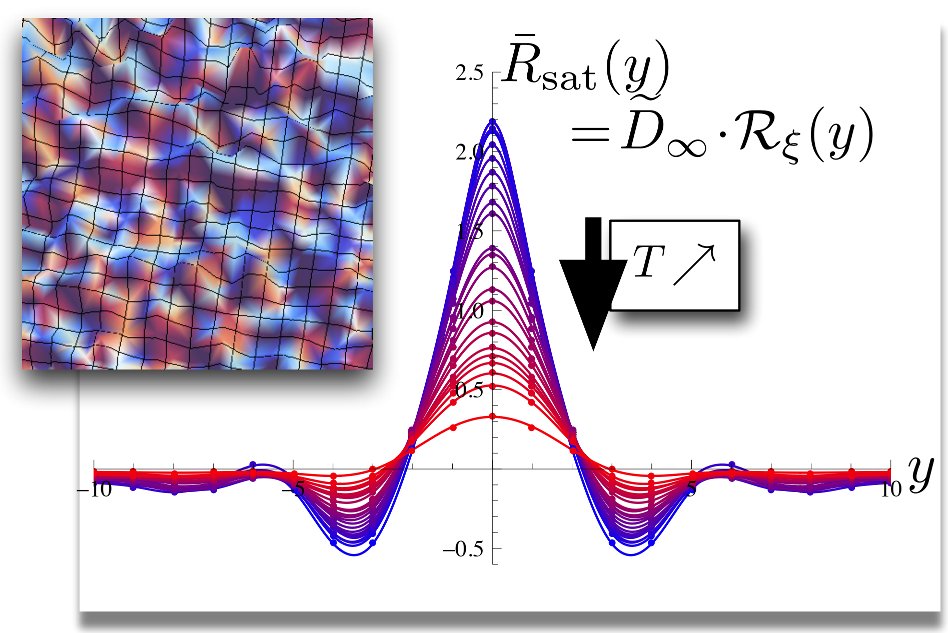
---

- ***Focus on disorder free-energy***: effective disorder at a given lengthscale or DP 'time'

  ⇒ Tilted 1D KPZ equation with spatially-correlated noise & flat initial condition

  ⇒ 'Time'-dependence & asymptote of the 2-pt correlator $\bar{R}(t,y) = \overline{\partial_y \bar{F}_V(t,y)\partial_y \bar{F}_V(t,0)}$

$$\boxed{\bar{R}(t,y) = \widetilde{D}\left[\mathcal{R}_\xi(y) - b(t,y,\xi)\right]}$$



  ⇒ Still open issue (from an <u>exact</u> perspective):

  - Asymptotic shape $\mathcal{R}_\xi(y) = \mathcal{F}\left[R_\xi(y), T\right]$
  - Asymptotic amplitude $\widetilde{D}(T,\xi)$
  - Full 'time'-dependence

- ***Complementary studies***: scaling analysis & non-perturbative function renormalisation approach (NP-FRG)

E. Agoritsas & V. Lecomte, *J. Phys. A* <u>50</u>, 104001 (2017)
S. Mathey, E. Agoritsas, T. Kloss, V. Lecomte, & L. Canet, *Phys. Rev. E* <u>95</u>, 032117 (2017)

Non-perturbative functional renormalization group (NP-FRG) study

- NP-FRG for 1D KPZ in a nutshell: ⇒ ***Cf. Léonie Canet's talk***

- RG 'à la Wilson', scale-dependent effective action: $$e^{-\Gamma_k[\varphi,\tilde\varphi]} = \int_{p>k} D[h, I\tilde h] e^{-S[h,\tilde h]}$$
- Symmetries of the microscopic model explicitly preserved throughout the flow, thanks to choice of cutoff/regulator $$\mathcal{R}_k(\boldsymbol{p}) = \frac{\alpha}{e^{p^2/k^2}-1}\begin{pmatrix} 0 & \nu(k)p^2 \\ \nu(k)p^2 & -2D(k) \end{pmatrix}$$
- <u>Exact</u> RG flow: $$k\partial_k \Gamma_k[\varphi,\tilde\varphi] = \frac{1}{2}\mathrm{Tr}\left[\frac{k\partial_k \mathcal{R}_k}{\Gamma_k^{(2)}[\varphi,\tilde\varphi]+\mathcal{R}_k}\right]$$ with $$\Gamma^{(2)}_{k,ij}(t-t',\boldsymbol{x}-\boldsymbol{x}') = \frac{\delta^2\Gamma_k[\varphi,\tilde\varphi]}{\delta\varphi_i(t,\boldsymbol{x})\delta\varphi_j(t',\boldsymbol{x}')}$$

- Symmetry-preserving ansatz: $$\Gamma_k[\varphi,\tilde\varphi] = \int_{t,\boldsymbol{x}}\left[\tilde\varphi \boxed{f_k^\lambda}\left(\partial_t\varphi - \frac{1}{2}[\nabla\varphi]^2\right) - \frac{1}{2}(\nabla^2\varphi \boxed{f_k^\nu}\tilde\varphi + \tilde\varphi \boxed{f_k^\nu}\nabla^2\varphi) + \tilde\varphi \boxed{f_k^D}\tilde\varphi\right]$$
- ⇒ RG flow $k\partial_k f_k^X(\omega,p)$ for the effective nonlinearity/dissipation/noise
- Some additional quite technical approximations ⇒ Numerically-obtained fixed points

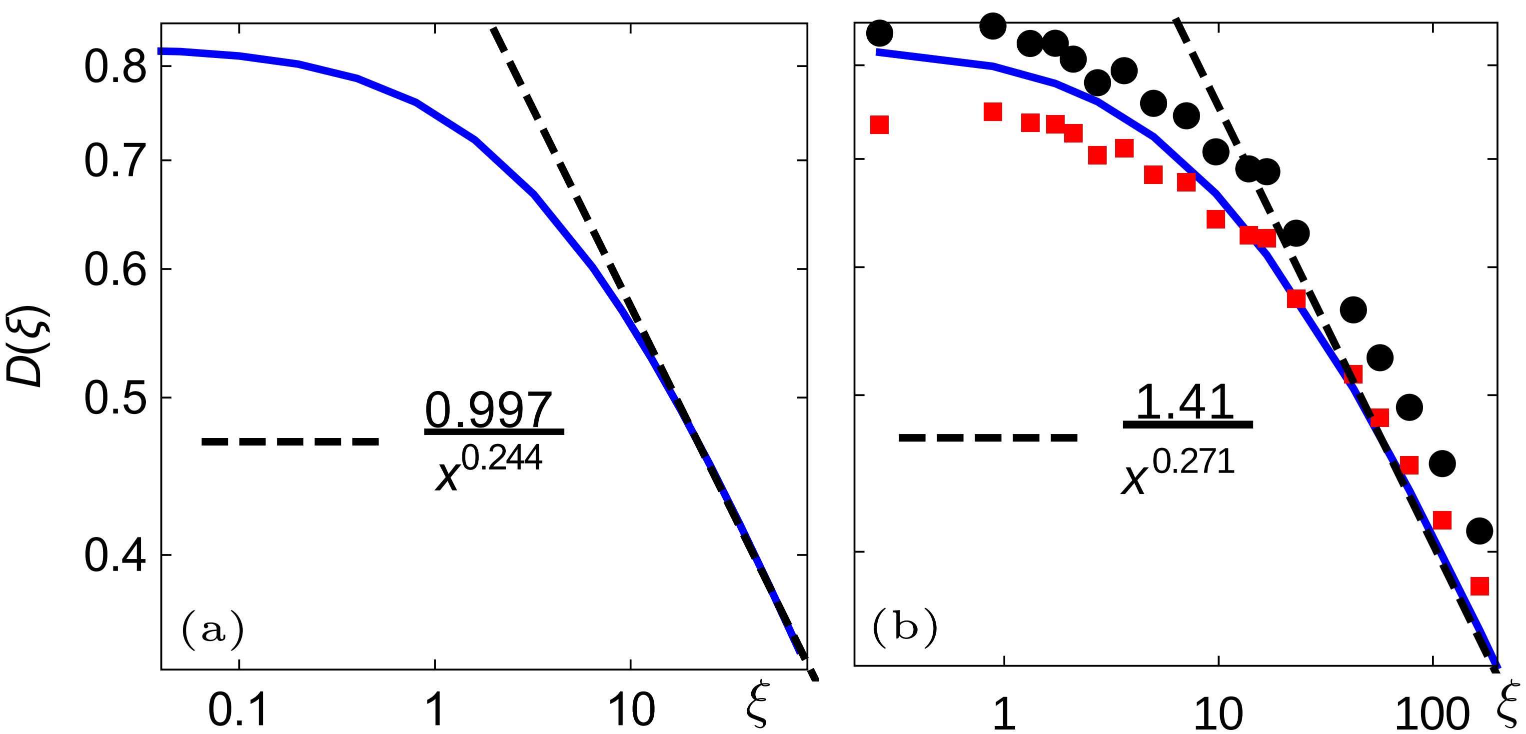


FIG. 5. Nonuniversal amplitude $\widetilde{D}$, in log-log scale, from NPFRG and from numerical simulations: (a) $\widetilde{D}$ computed with spatial noise correlations (2), with the NPFRG, and for different values of ξ (blue solid line), as well as a power-law fit of the tail of the data (for $\xi \geqslant 18$) (black dashed line), and (b) $\widetilde{D}$ computed with spatiotemporal correlations (55), with the NPFRG for different values of ξ and $\xi_\tau \cong 0.539\xi^{5/3}$ (blue solid line) with its power-law fit (for $\xi \geqslant 43$) (black dashed line) as well as two different estimations of $\widetilde{D}$ from the numerical computation of [66] (black circles and red squares).

S. Mathey, E. Agoritsas, T. Kloss, V. Lecomte, & L. Canet, *Phys. Rev. E* <u>95</u>, 032117 (2017)

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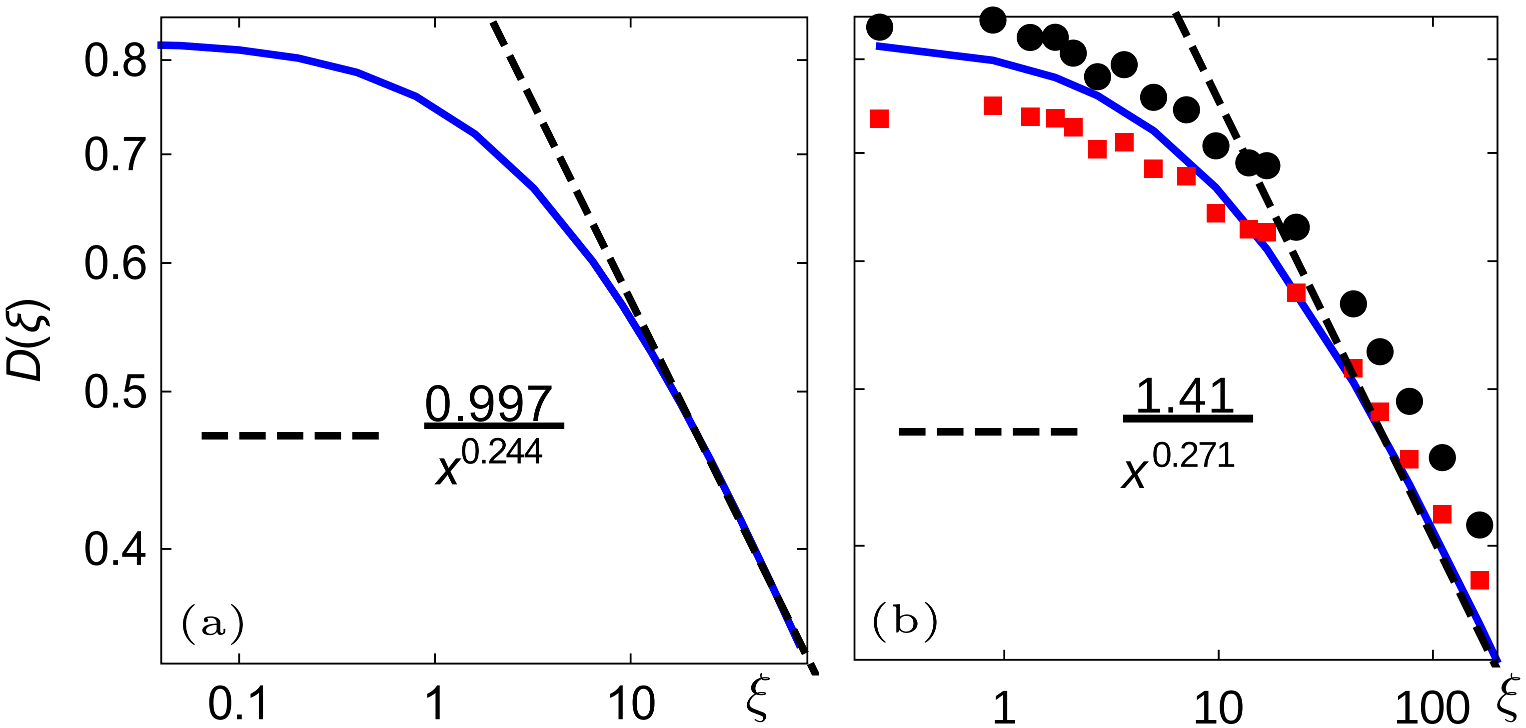


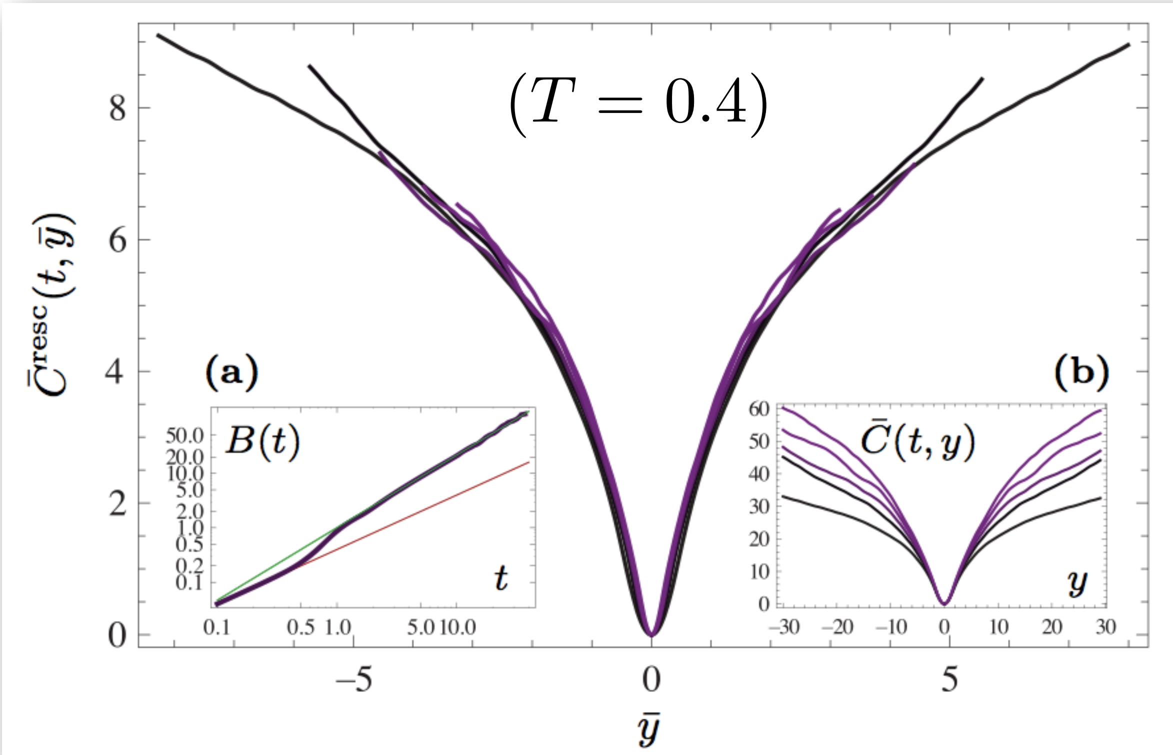
⚠️ Pending discrepancy for the exponent, that we expect via other approaches (in our low-temperature or KPZ inviscid limit) to be instead

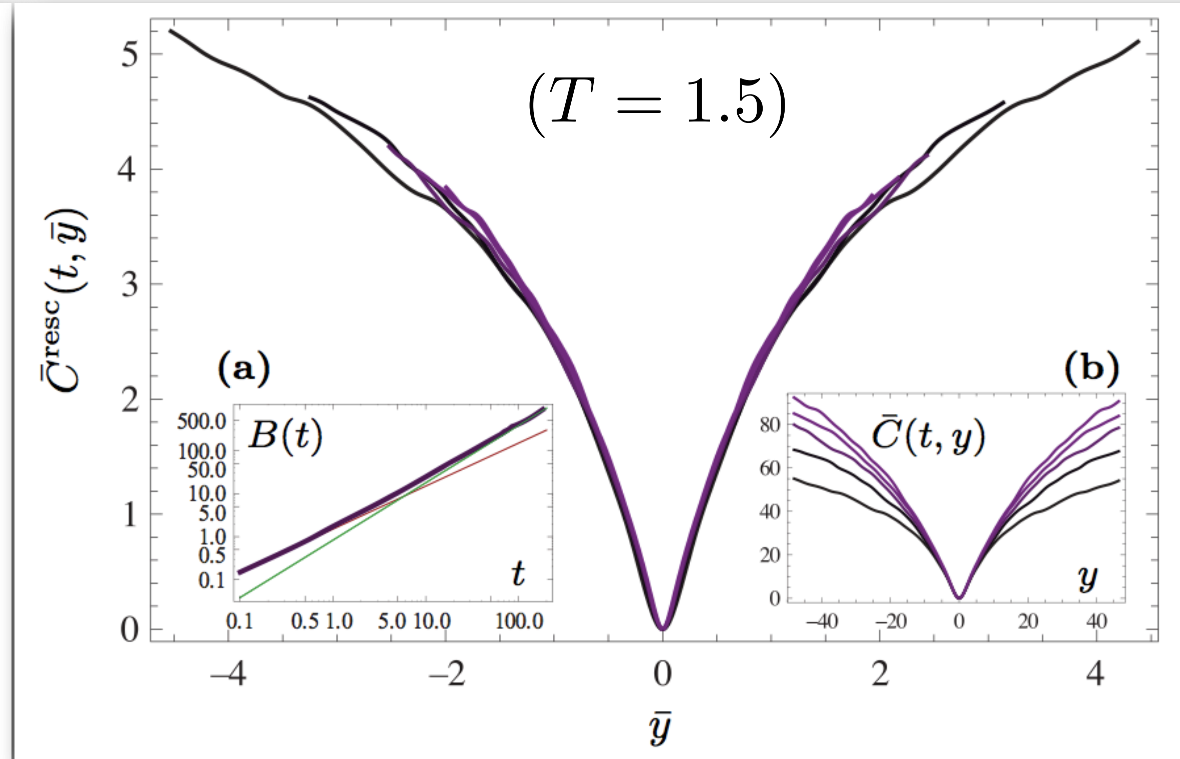
$$\widetilde{D}(\xi) \sim 1/\xi^{1/3}$$

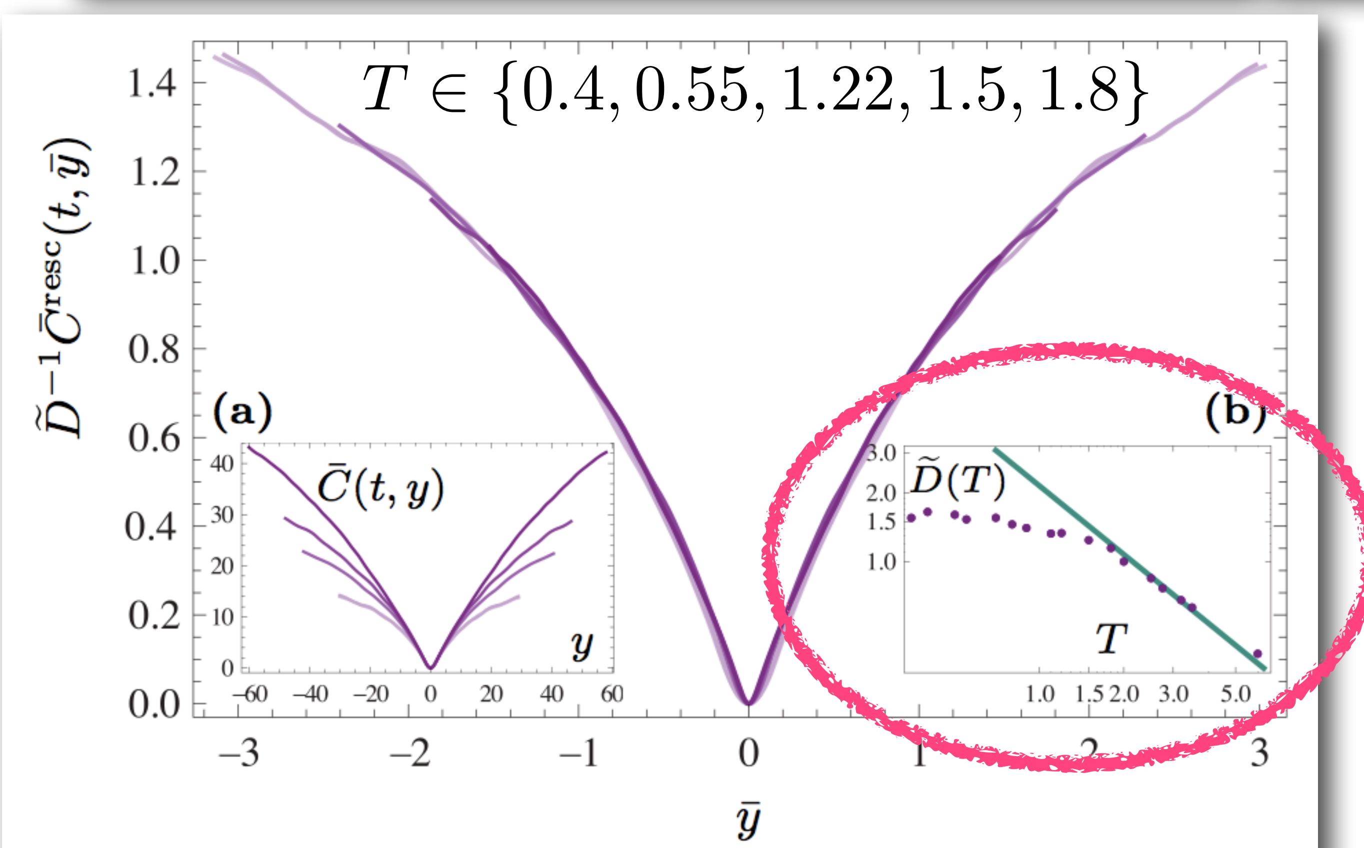
S. Mathey, E. Agoritsas, T. Kloss, V. Lecomte, & L. Canet, *Phys. Rev. E* <u>95</u>, 032117 (2017)

Numerics: continuous vs discrete Directed Polymer

- Airy-like 2-pt correlators:







- Scaling law for the correlator:

$$\bar{C}_\xi(t, y) = \widetilde{D}\sqrt{B(t)}\,\hat{C}_{\frac{\xi}{\sqrt{B(t)}}}\left(\frac{y}{\sqrt{B(t)}}\right)$$

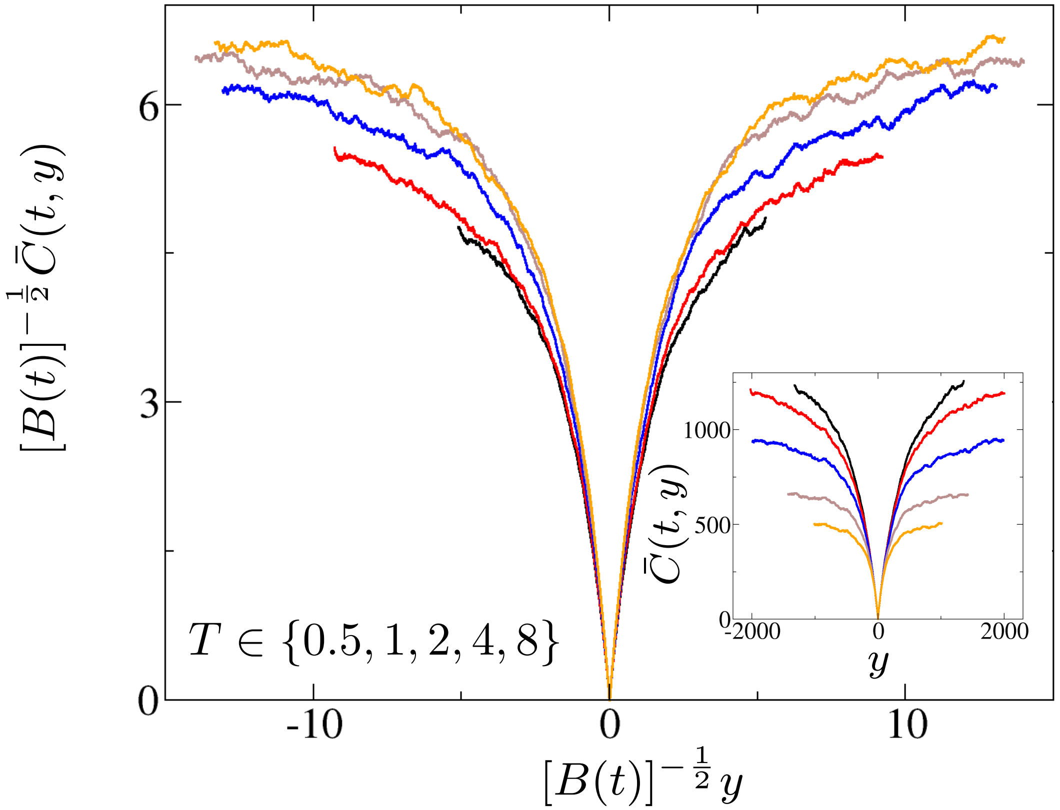
with the roughness $B(t; c, D, T, \xi)$

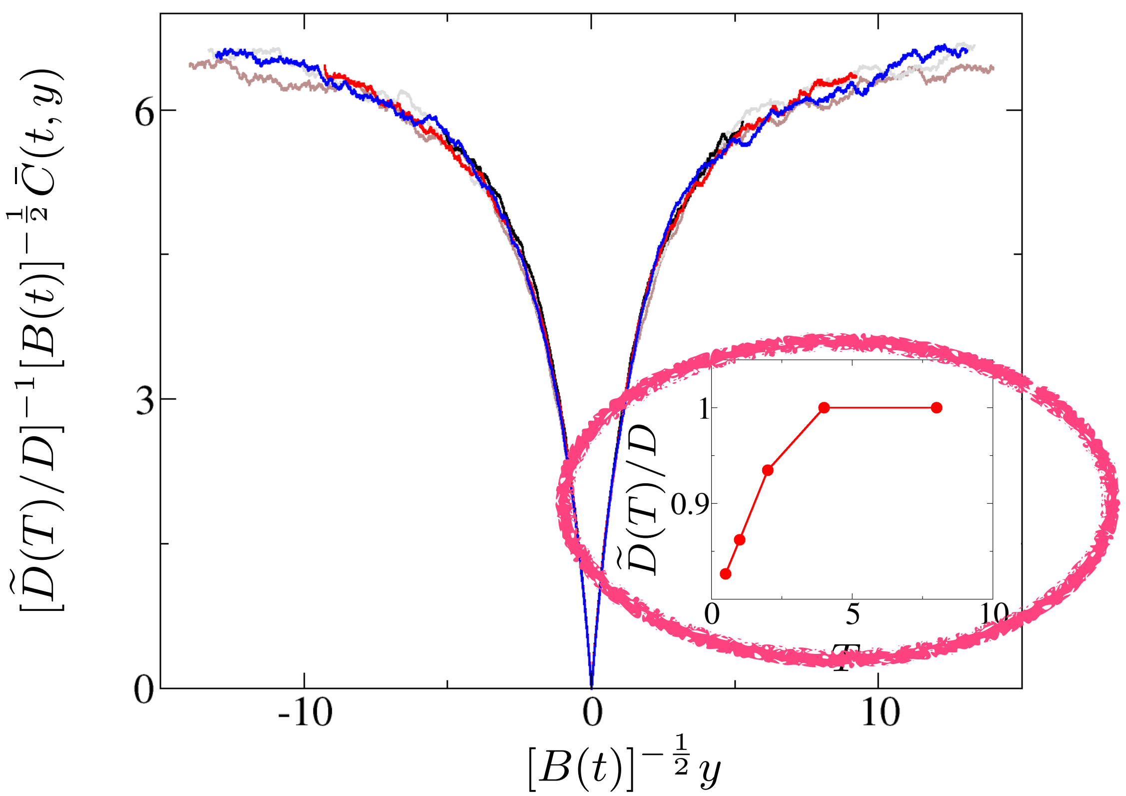
and a fitting parameter $\widetilde{D}$

E. Agoritsas, S. Bustingorry, V. Lecomte, G. Schehr & T. Giamarchi, *Phys. Rev. E* 86, 031144 (2012).

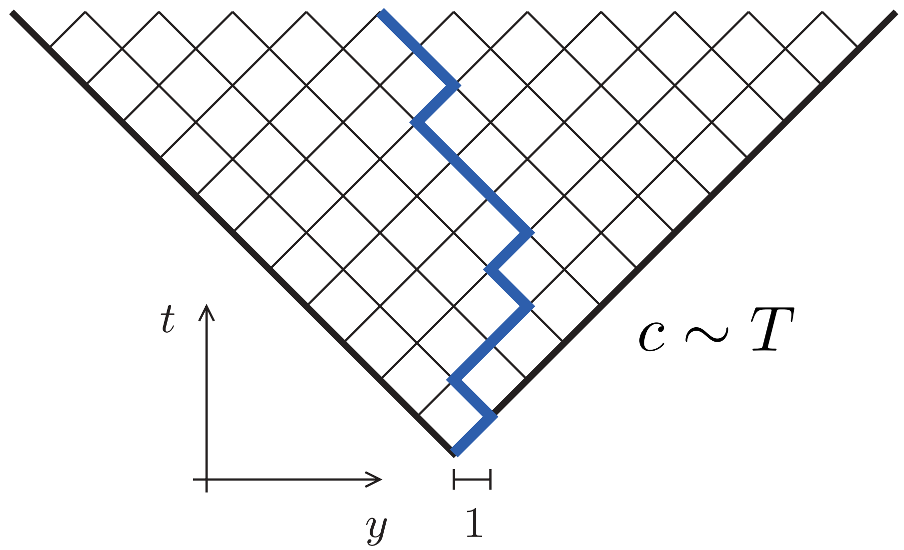
Numerics: continuous vs discrete Directed Polymer

- Airy-like 2-pt correlators at large times: $t = 16384$





⚠ Cf. Appendix E in Ref. below,
for translation from the discrete to the continuous DP



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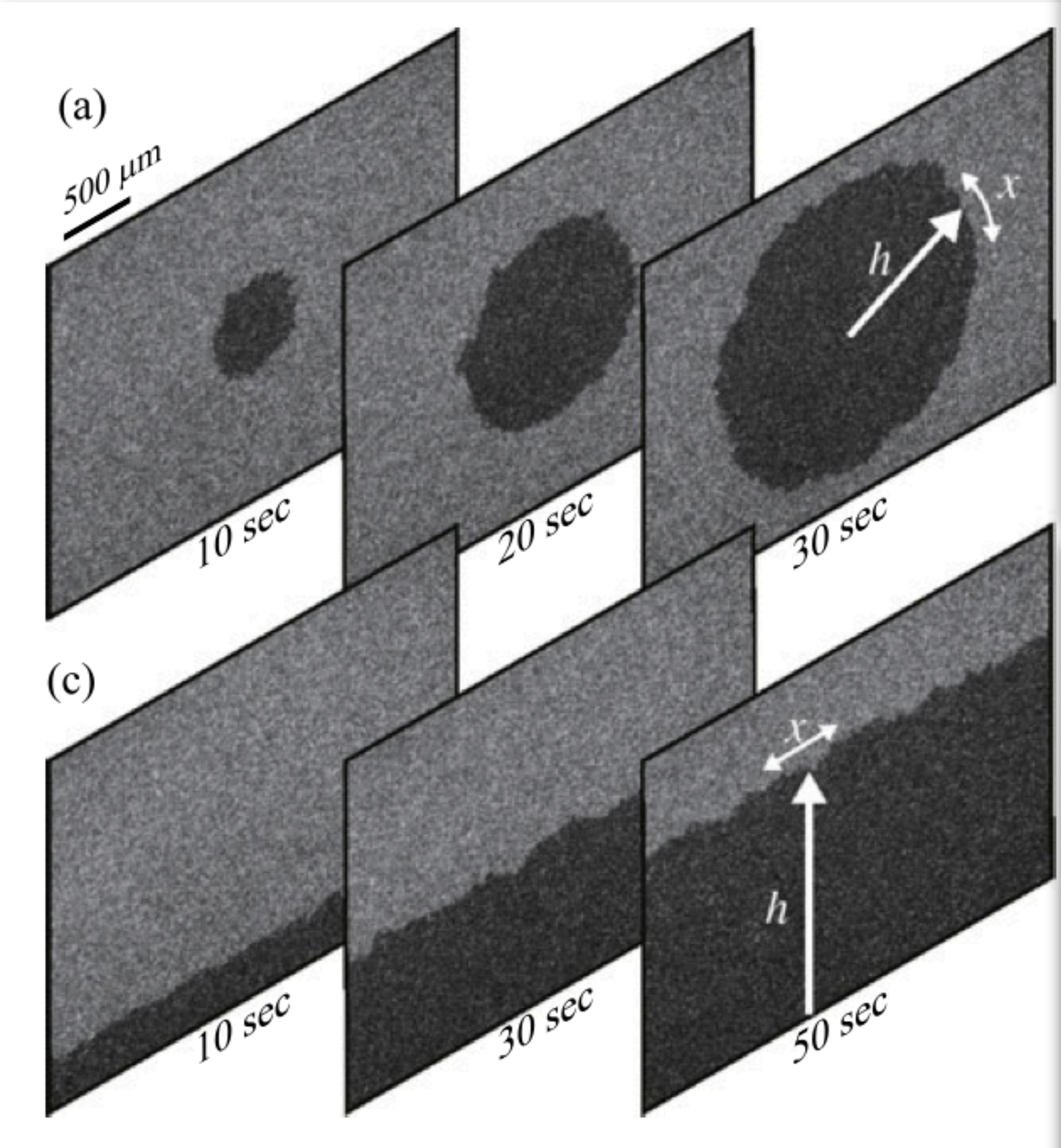
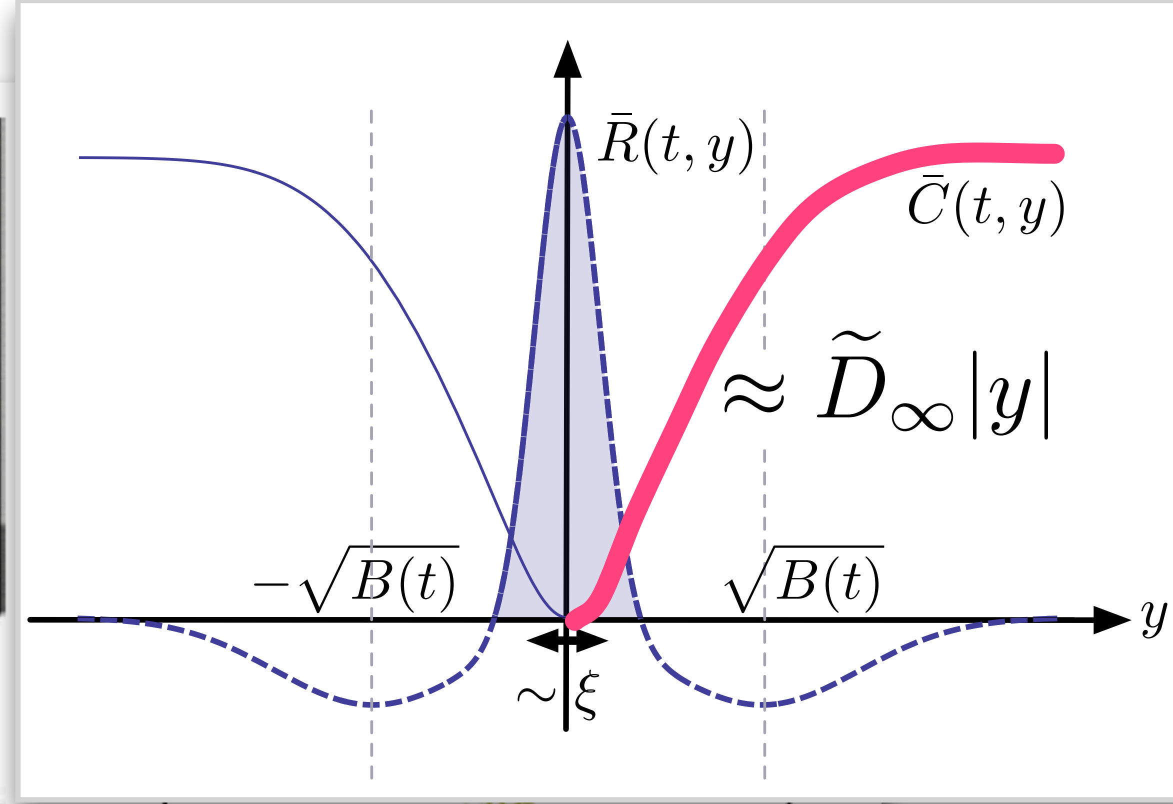
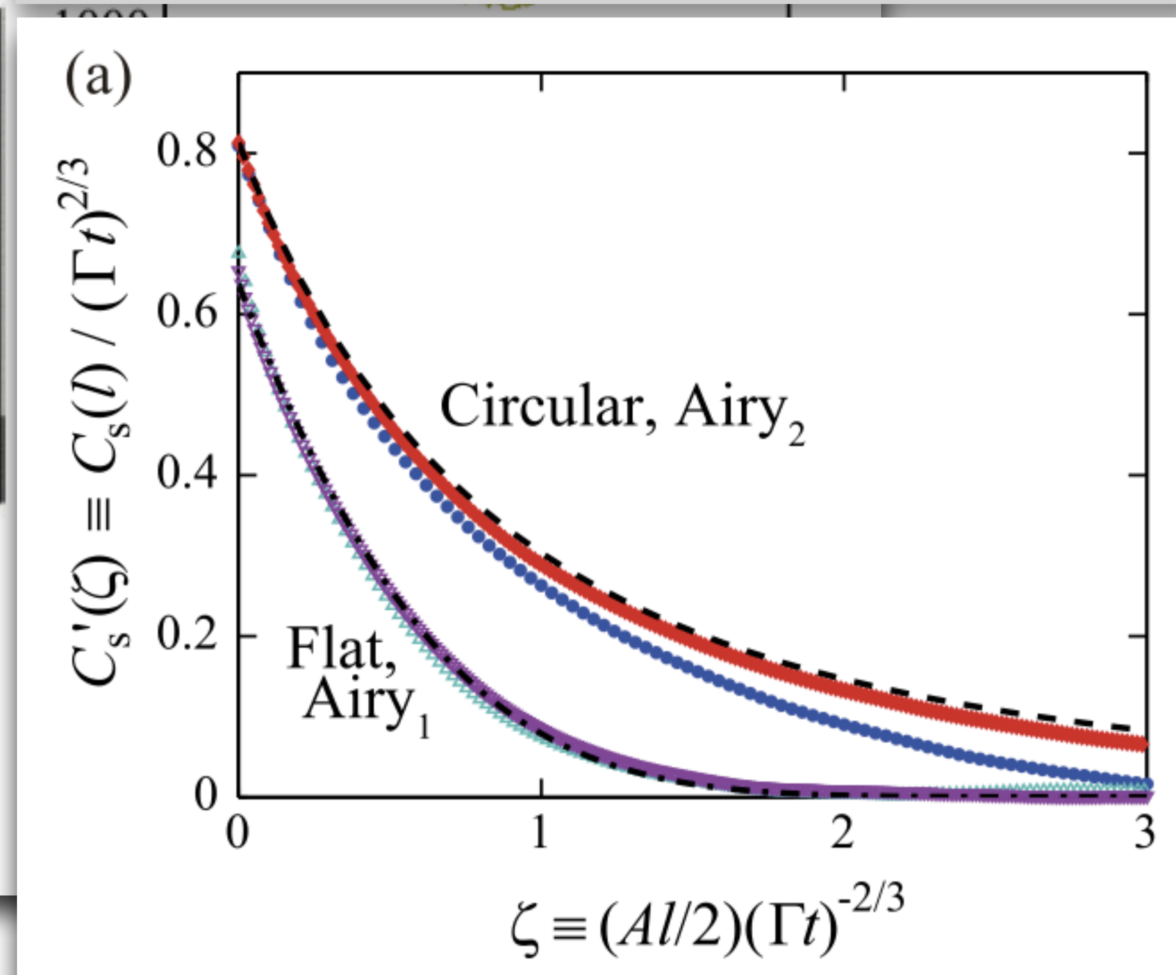
Experimental 1D KPZ interfaces: nematic liquid crystals





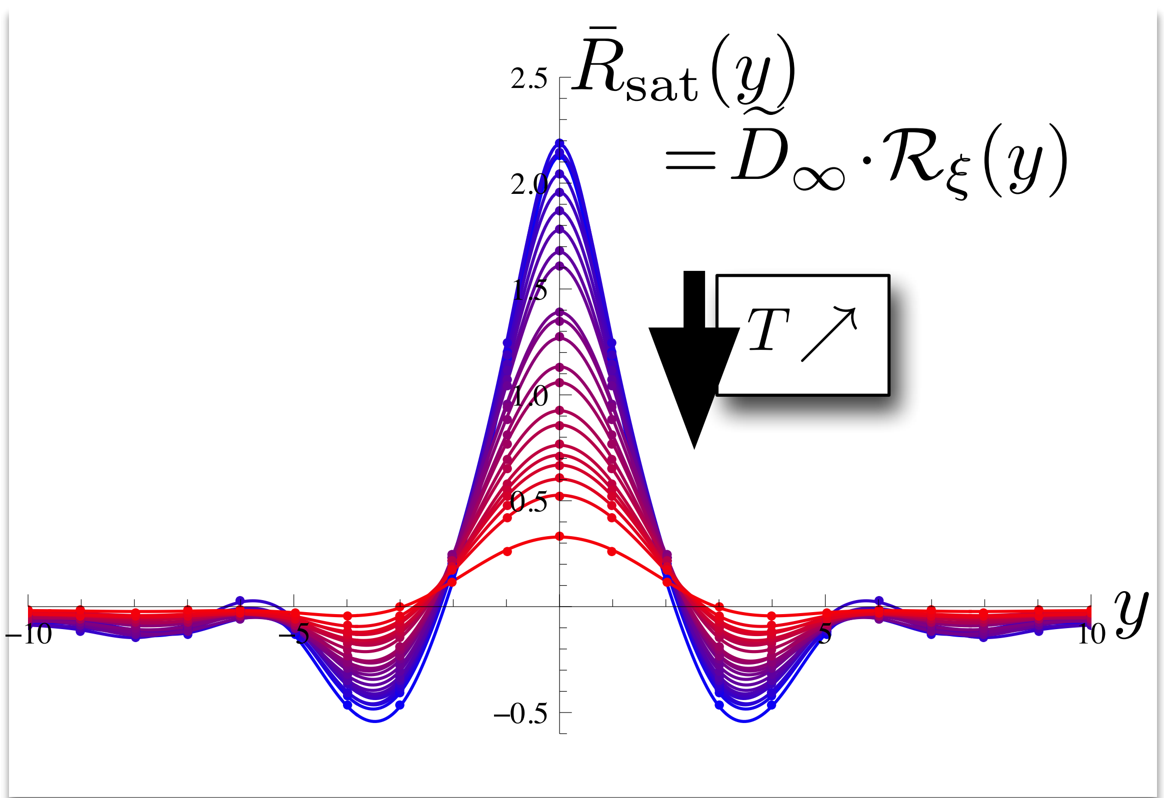


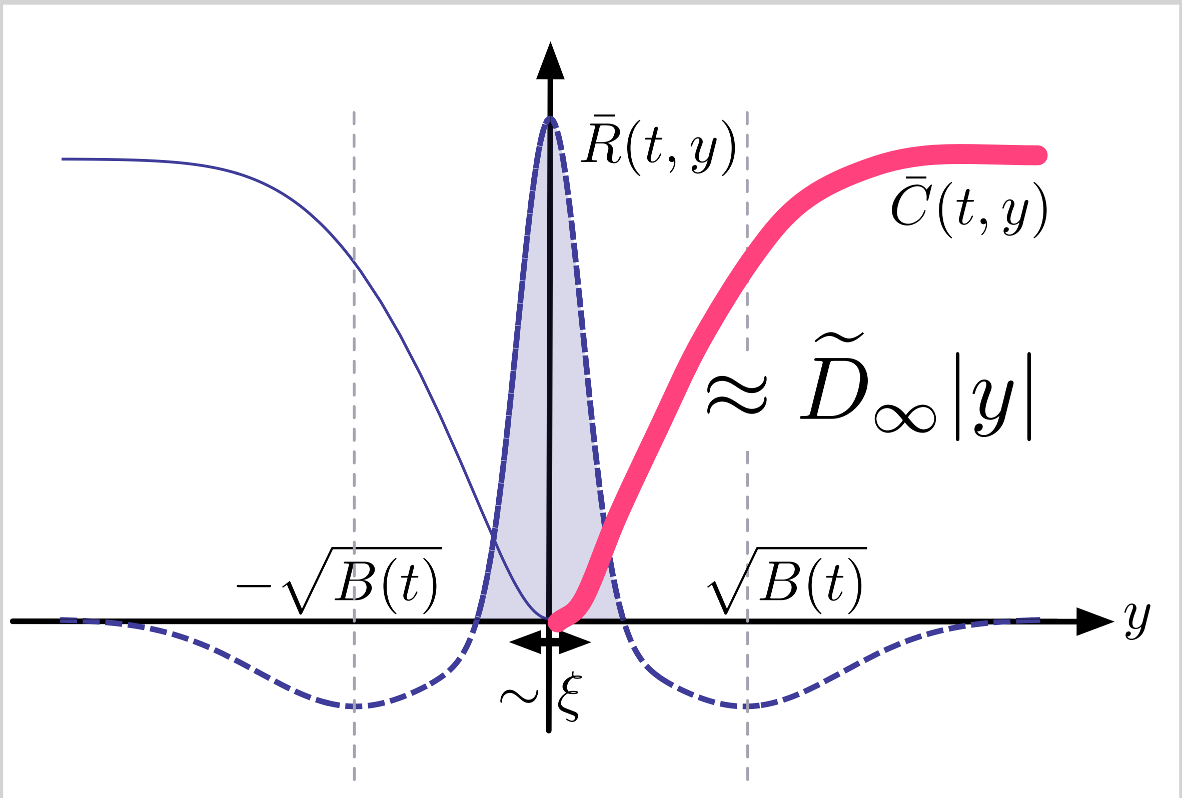
K. Takeuchi & M. Sano, *J. Stat. Phys.* 147, 853 (2012).

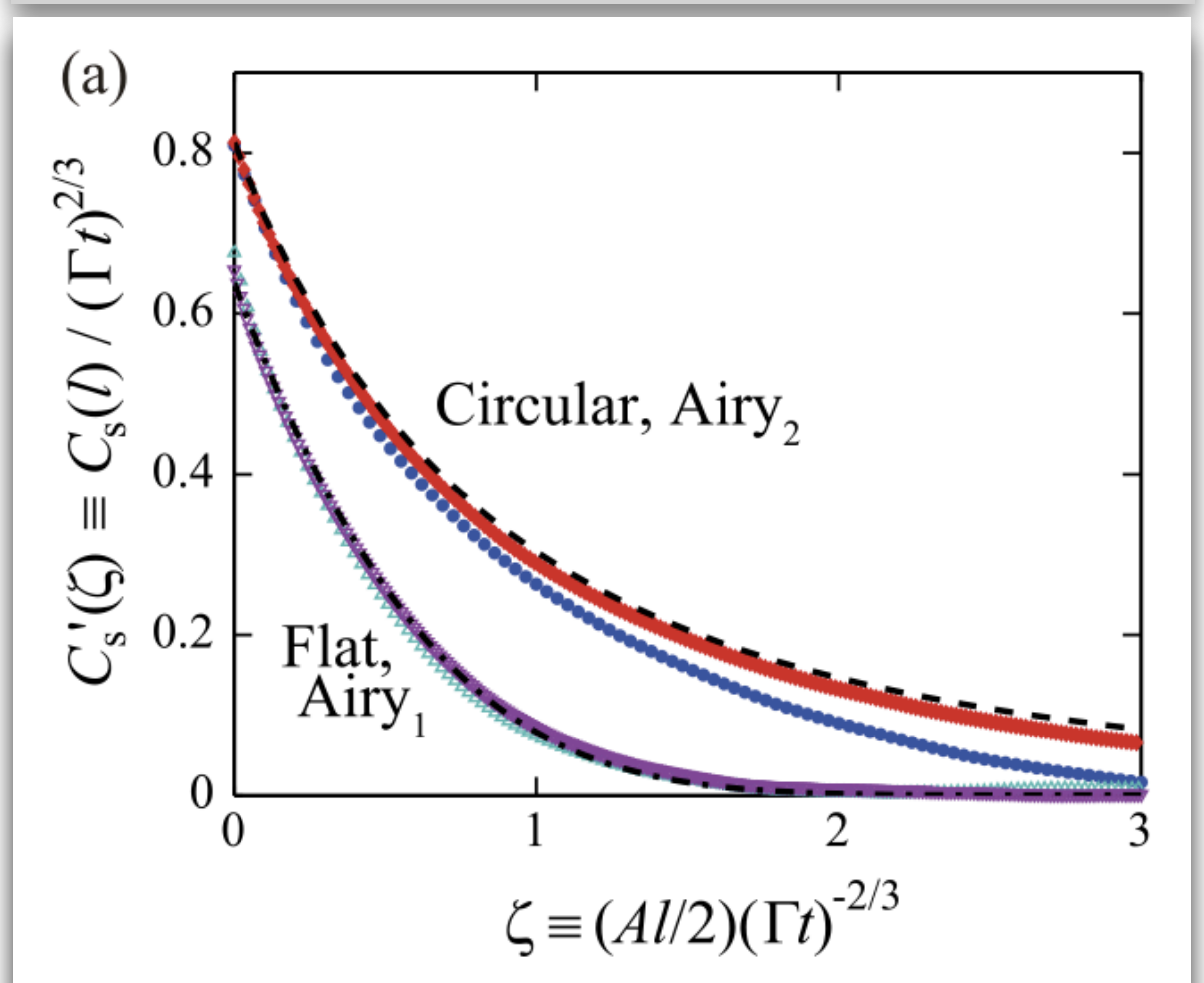
Experimental 1D KPZ interfaces: nematic liquid crystals

- **Possible to observe the signature of a finite disorder correlation length?**

- Shape of the asymptotic correlator accessible experimentally?





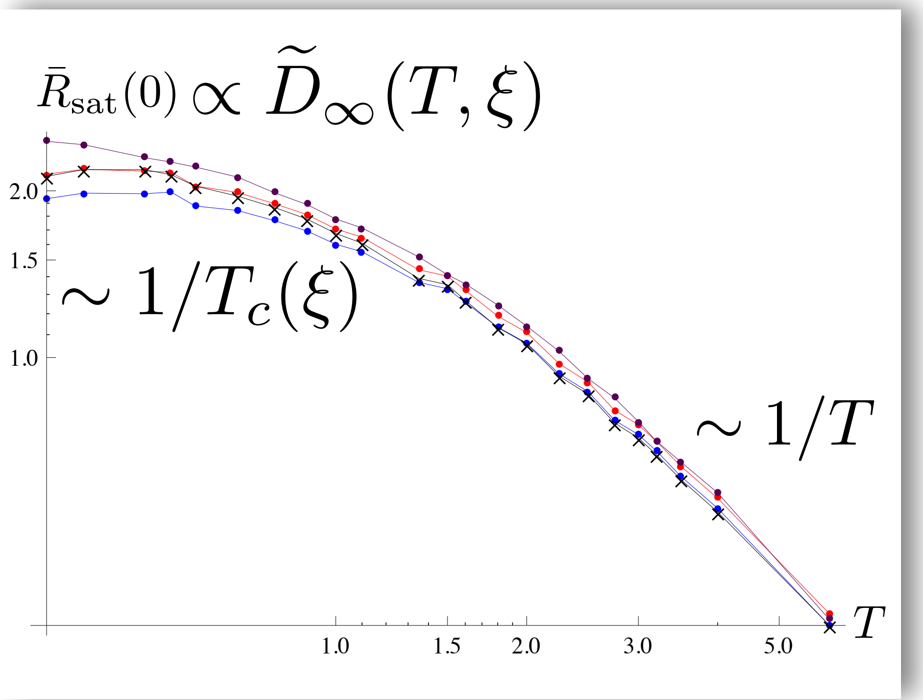


K. Takeuchi & M. Sano, *J. Stat. Phys.* 147, 853 (2012).

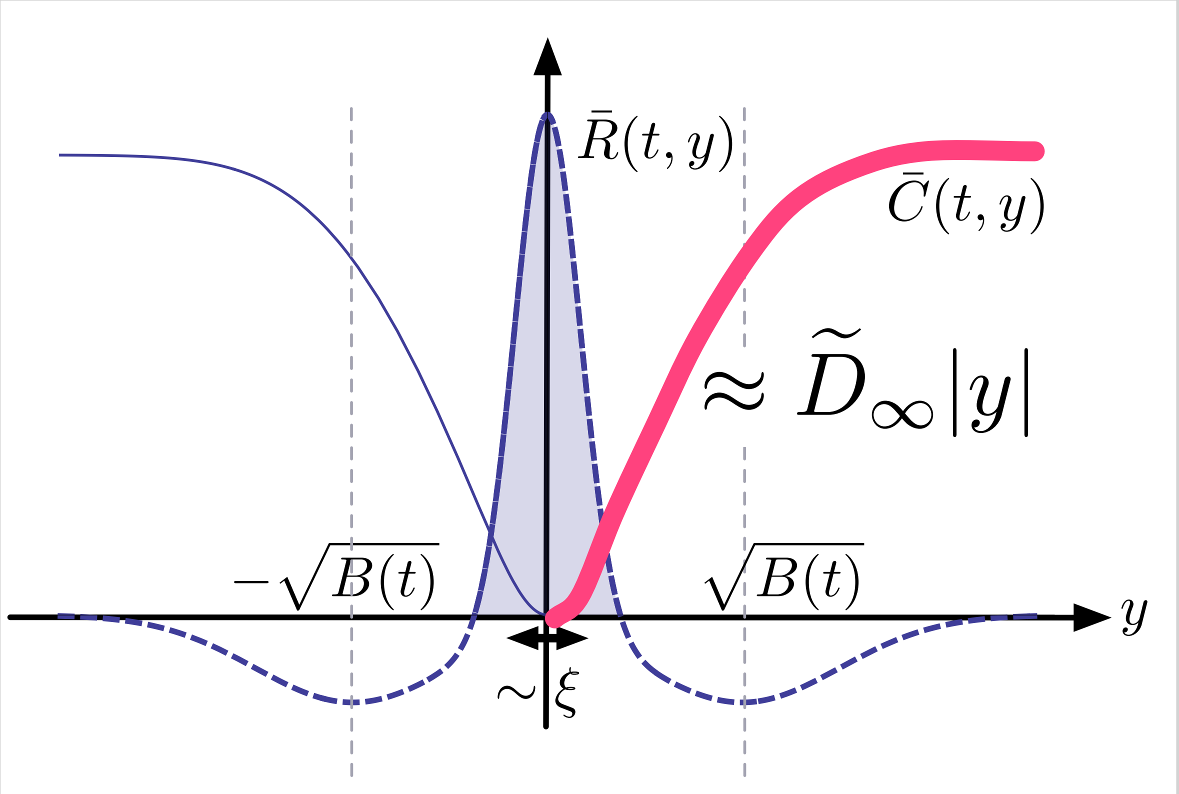
Experimental 1D KPZ interfaces: nematic liquid crystals

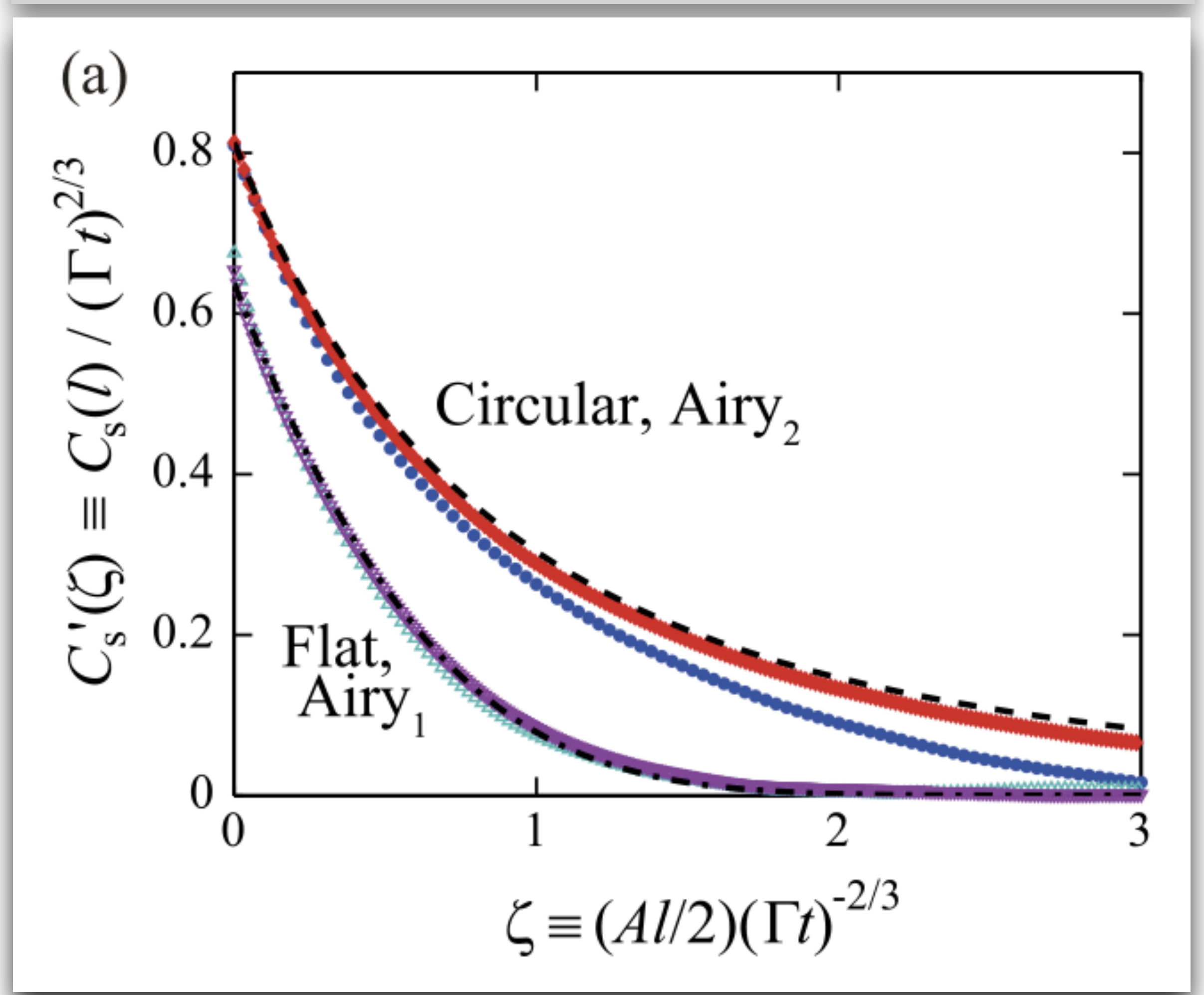
- **Possible to observe the signature of a finite disorder correlation length?**

- Is there a crossover of the amplitude of the asymptotic correlator?



Low-temperature regime in our model
$\overset{?}{\leftrightarrow}$ **High-velocity for the liquid crystals**





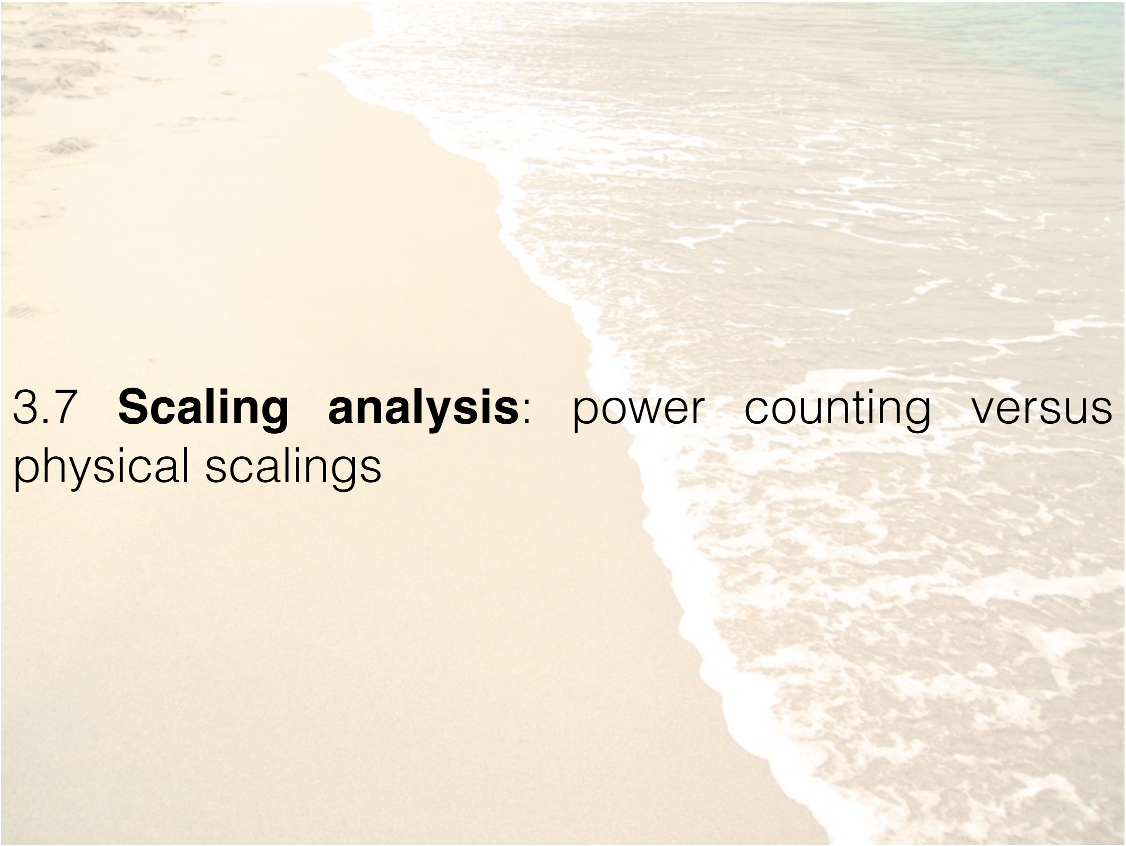
K. Takeuchi & M. Sano, *J. Stat. Phys.* 147, 853 (2012).

Doctoral Training School *Kardar-Parisi-Zhang equation: new trends in theories and experiments*
April 15-26, 2024 — Ecole de Physique des Houches (France)

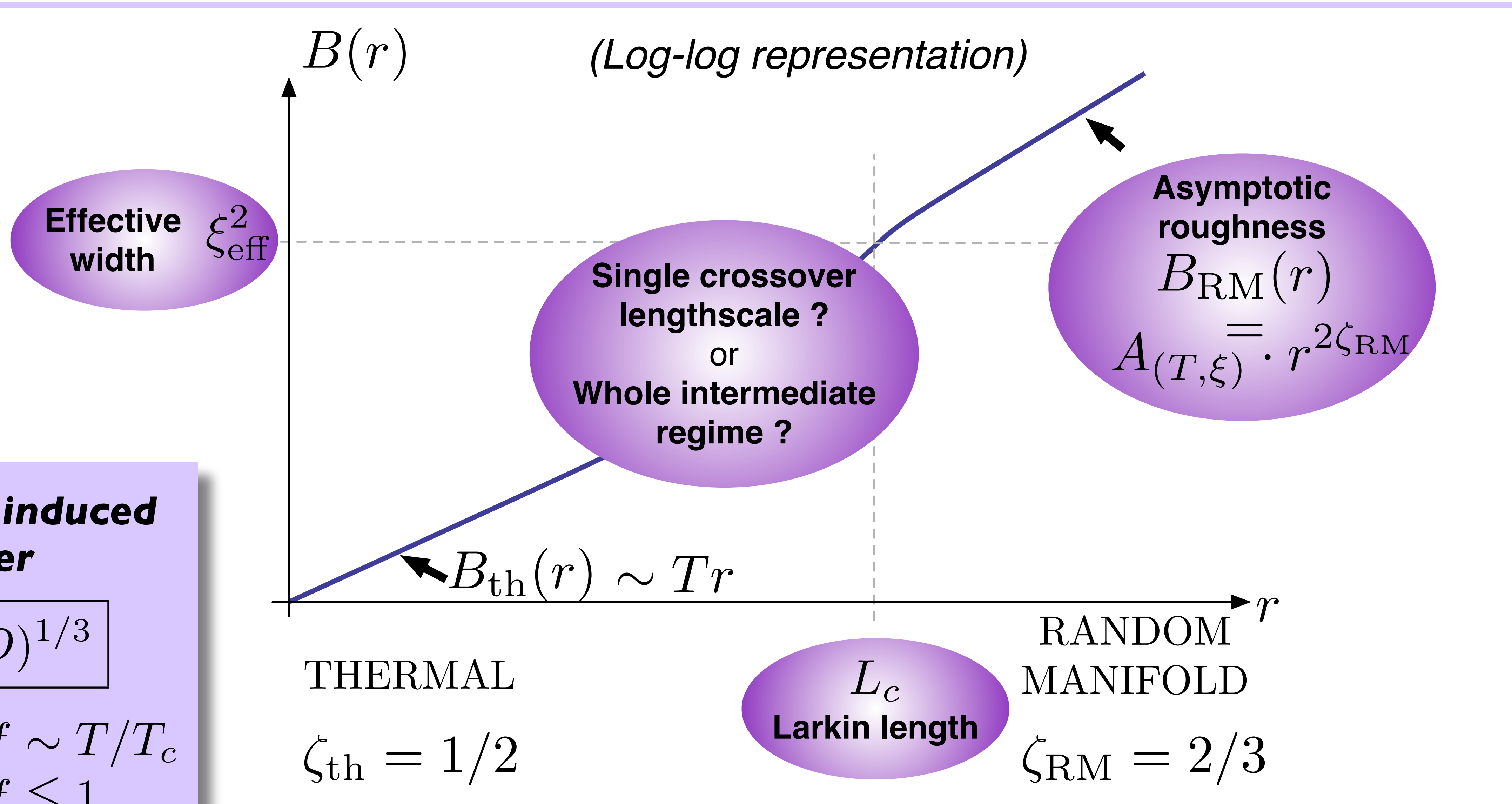
Interfaces in disordered systems and directed polymer

Elisabeth Agoritsas

- 1. Introduction
- 2. Disordered elastic systems: Recipe
- 3. Disordered elastic systems: Statics
- 4. Disordered elastic systems: Dynamics
- 5. Concluding remarks



Roughness regimes & characteristic crossover scales



Temperature-induced crossover

$$\boxed{T_c = (\xi c D)^{1/3}}$$

$$\begin{cases} T \ll T_c : & f \sim T/T_c \\ T \gg T_c : & f \lesssim 1 \end{cases}$$

Universal scaling

KPZ scaling for the asymptotic roughness $\quad B(r) \overset{(r\to\infty)}{\sim} \left(\frac{D}{cT/f}\right)^{2/3} r^{4/3}$

Non-universal features

Roughness amplitude: $\quad A_{(T,\xi)} \sim \left(\frac{D}{cT/f}\right)^{2/3}$

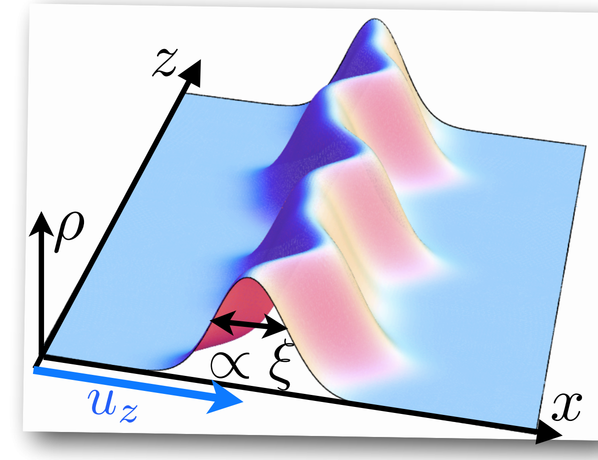
Larkin length: $\quad L_c = \frac{(T/f)^5}{cD^2}$

Interlude: 'Standard' Flory/Imry-Ma scaling argument

- Short-range elasticity & Elastic limit / Quenched random-bond weak disorder

$$\boxed{\mathcal{H}_{\mathrm{DES}} = \mathcal{H}_{\mathrm{el}} + \mathcal{H}_{\mathrm{dis}}}$$

$$\mathcal{H}[u,\widetilde{V}] = \int_{\mathbb{R}} dz \cdot \left[\frac{c}{2}(\nabla_z u(z))^2 + \int_{\mathbb{R}} dx \cdot \rho_\xi(x-u(z))\,\widetilde{V}(z,x)\right]$$



- Dimensional analysis / power counting:

  - $\mathcal{H}_{\mathrm{el}}[u] = \frac{c}{2}\int d^dz \cdot (\nabla u(z))^2 \sim L^d \cdot \left(\frac{u}{L}\right)^2 = L^{d-2}u^2;$
  - $\overline{V(x,z)V(x',z')} = D\cdot\delta^{(m)}(x-x')\delta^{(d)}(z-z') \Longrightarrow V^2 \sim \left(\frac{1}{x}\right)^m\left(\frac{1}{z}\right)^d \sim u^{-m}\cdot L^{-d};$
  - $\rho(x,z) = \frac{1}{(2\pi\xi^2)^{m/2}} e^{-\frac{(x-u(z))^2}{2\xi^2}} \Longrightarrow \rho \sim \xi^{-m} \sim u^{-m};$
  - $\mathcal{H}_{\mathrm{dis}}[u,V] = \int d^mx\, d^dz \cdot V(x,z)\rho(x,z) \sim u^m \cdot L^d \cdot u^{-m/2}\cdot L^{-d/2}\cdot u^{-m} = L^{d/2}u^{-m/2};$
  - imposing $\mathcal{H}_{\mathrm{el}} \sim \mathcal{H}_{\mathrm{dis}}$ on the thermal and disorder energetic contributions at the lengthscale L, we thus have:

$$L^{d-2}u^2 \sim L^{d/2}u^{-m/2} \Longleftrightarrow u^{\frac{4+m}{2}} \sim L^{\frac{4-d}{2}} \Longleftrightarrow \boxed{u(L) \sim L^{\frac{4-d}{4+m}} \equiv L^{\zeta_F}}$$

  - and eventually $B(r) \equiv \overline{\left\langle [u(r)-u(0)]^2\right\rangle} \sim u(r)^2 \sim r^{2\zeta_F}$ with

$$\zeta_F = \frac{4-d}{4+m}$$

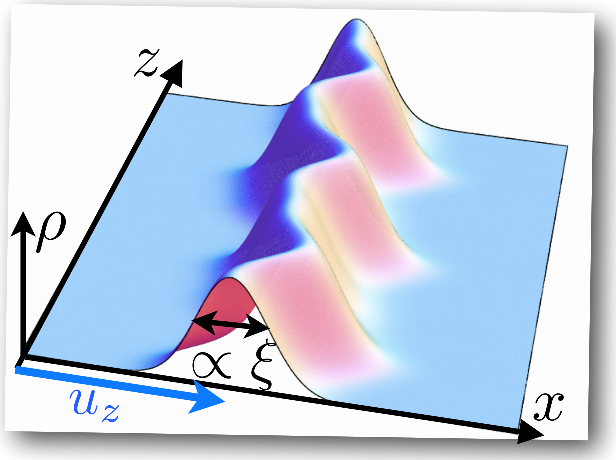
$$d = m = 1$$
$$\Rightarrow \zeta_F = 3/5$$

What is wrong with the 'standard' Flory/Imry-Ma scaling argument?

- Short-range elasticity & Elastic limit / Quenched random-bond weak disorder

$$\boxed{\mathcal{H}_{\mathrm{DES}} = \mathcal{H}_{\mathrm{el}} + \mathcal{H}_{\mathrm{dis}}}$$

$$\mathcal{H}[u, \widetilde{V}] = \int_{\mathbb{R}} dz \cdot \left[\frac{c}{2} (\nabla_z u(z))^2 + \int_{\mathbb{R}} dx \cdot \rho_\xi(x - u(z))\, \widetilde{V}(z, x) \right]$$



- Dimensional analysis / power counting on the spatial dimension only!

$$\left.\begin{array}{c} \mathcal{H}_{\mathrm{el}} \sim L^{d-2} u^2 \\ \mathcal{H}_{\mathrm{dis}} \sim L^{d/2} u^{-m/2} \end{array}\right\} \quad \mathcal{H}_{\mathrm{el}} \sim \mathcal{H}_{\mathrm{dis}} \Rightarrow u(L) \sim L^{\frac{4-d}{4+m}} \equiv L^{\zeta_F}$$

- Much more freedom if we allow to rescale also all the other parameters $\{c, D, T, \xi, L\}$ of the DES model!
  - Allowing to rescale all these parameters (here 5 of them), while keeping the Gibbs-Boltzmann weight invariant (2 conditions), leaves 3 parameters free to be fixed (potentially dependent on the lengthscale)
  - Reasoning on the Hamiltonian yields a list of possible candidates for crossover scales.
  - Reasoning on the replicated Hamiltonian (although strictly equivalent to the non-replicated one) turns out to yield the asymptotic KPZ exact scalings.
  - Reasoning on the DP free-energy, additional input at fixed time ↔ our lengthscale (e.g. Brownian scaling)

E. Agoritsas & V. Lecomte, *J. Phys. A* 50, 104001 (2017), "Power countings versus physical scalings in disordered elastic systems - Case study of the one-dimensional interface"

- **Scaling arguments:**
  - 'back-of-the-envelope' shortcuts for theoretical computations
  - reasoning on the interplay between a model's relevant 'typical' scales
  - sometimes untrustworthy beforehand; rather a posteriori explanation

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- Flory or 'Imry-Ma' construction: power countings on the Hamiltonian or free energy; which are the possible candidates for the Flory roughness exponents?

$$\{t = b\hat{t}, y = a\hat{y}\} \quad \& \quad a \sim b^{\zeta_{\mathrm{F}}}$$

$$B(t; c, D, T, \xi, t_{\mathrm{f}}) = a^2\, \bar{B}(t/b; c', D', T', \xi', t_{\mathrm{f}}/b)$$

$$\{c' = 1,\ D' = \frac{D}{D_0},\ T' = \frac{T}{\widetilde{E}},\ \xi' = \frac{\xi}{a}\}$$

Requirement: Boltzmann weight invariant under rescaling $\propto e^{-\frac{1}{T}\mathcal{H}[y(t),V;t_{\mathrm{f}}]}$

Starting point	*Power counting*	ζ_{F}	$\widetilde{E} = ca^2/b$
$\mathcal{H}\,[y(t),V;t_{\mathrm{f}}]$	$a = \left(D_0^{1/3}c^{-2/3}b\right)^{3/5},$	$3/5$	$\left(cD_0^2 b\right)^{1/5} = (cD_0 a)^{1/3}$
$\widetilde{\mathcal{H}}\,[y_1(t),\dots,y_n(t);t_{\mathrm{f}}]$	$a = \left(\frac{D_0}{c\widetilde{E}}\right)^{1/3} b^{2/3},$	$2/3$	$\left(cD_0^2 b\right)^{1/5} = (cD_0 a)^{1/3}$
$F_V(t_{\mathrm{f}},y)$	$a = \left(\widetilde{D}_0/c^2\right)^{1/3} b^{2/3},$	$2/3$	$\left(\widetilde{D}_0^2 b/c\right)^{1/3} = (\widetilde{D}_0 a)^{1/2}$
$\widetilde{F}(t_{\mathrm{f}},y_1,\dots,y_n)$	$a = \frac{\widetilde{D}_0}{c\widetilde{E}} b,$	1	$\left(\widetilde{D}_0^2 b/c\right)^{1/3} = (\widetilde{D}_0 a)^{1/2}$

E. Agoritsas & V. Lecomte, *J. Phys. A* 50, 104001 (2017)

Revisiting the Flory construction

- **Scaling arguments:**
  - 'back-of-the-envelope' shortcuts for theoretical computations
  - reasoning on the interplay between a model's relevant 'typical' scales
  - sometimes untrustworthy beforehand; rather a posteriori explanation

- **Flory or 'Imry-Ma' construction: power countings on the Hamiltonian or free energy; which are the possible candidates for the Flory roughness exponents?**

$$\{t = b\hat{t}, y = a\hat{y}\} \quad \& \quad a \sim b^{\zeta_{\mathrm{F}}}$$

$$B(t; c, D, T, \xi, t_{\mathrm{f}}) = a^2\, \bar{B}(t/b; c', D', T', \xi', t_{\mathrm{f}}/b)$$

$$\{c' = 1,\ D' = \frac{D}{D_0},\ T' = \frac{T}{\widetilde{E}},\ \xi' = \frac{\xi}{a}\}$$

Requirement: Boltzmann weight invariant under rescaling $\propto e^{-\frac{1}{T}\mathcal{H}[y(t),V;t_{\mathrm{f}}]}$

Constraint: $D' = 1$	b	$\widetilde{E}$	D_0	a	*Possibly relevant for*
$T' = 1$	$\frac{T^5}{cD^2}$	T	D	$\frac{T^3}{cD}$	$\xi \to 0$
$\xi' = 1$	$\frac{\xi^{1/3}}{D^{5/3}c^{2/3}} = \frac{T_c^5}{cD^2}$	$(\xi c D)^{1/3} \equiv T_c$	D	ξ	$T \to 0$
$\hat{t}_{\mathrm{f}} = 1$	t_{f}	$(cD^2 t_{\mathrm{f}})^{1/5}$	D	$\left(\frac{D^{1/3}t_{\mathrm{f}}}{c^{2/3}}\right)^{3/5}$	$t_{\mathrm{f}} \to \infty$
$T' = T/f$	$\frac{(T/f)^5}{cD^2}$	f	D	$\frac{(T/f)^3}{cD}$	Temperature crossover

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$$\boxed{B(t; c, D, T, \xi, t_{\mathrm{f}}) = a^2\, \bar{B}(t/b; c', D', T', \xi', t_{\mathrm{f}}/b)}$$

$$\{c' = 1,\ D' = \frac{D}{D_0},\ T' = \frac{T}{\widetilde{E}},\ \xi' = \frac{\xi}{a}\}$$

Requirement: Boltzmann weight invariant under rescaling $\propto e^{-\frac{1}{T}\mathcal{H}[y(t),V;t_{\mathrm{f}}]}$

- Alternative choice which happens to yield the KPZ & Brownian scalings:

$$\boxed{\frac{D'}{T'} = \frac{D/D_0}{T/\widetilde{E}} = \frac{1}{f}}$$

$$b = t_{\mathrm{f}}, \quad a = \left(\frac{D}{cT/f}\right)^{1/3} t_{\mathrm{f}}^{2/3}, \quad \widetilde{E} = \left[\left(\frac{D}{T/f}\right)^2 c t_{\mathrm{f}}\right]^{1/3}$$

Natural choice for the replicated Hamiltonian!

We can recover the exact KPZ scalings :-)

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Rescaling options in path-integral representations

- How do these power countings translate in terms of path integrals?
  Existence (or not) of well-defined optimal trajectories / path-integral saddle points,
  in large-size or low-temperature asymptotics?

- Large-size asymptotic, option I: Flory on the Hamiltonian

$$t = t_{\mathrm{f}}\,\hat{t}, \qquad y = t_{\mathrm{f}}^{\zeta_{\mathrm{F}}} \left(\tfrac{D}{c^2}\right)^{\frac{1}{5}} \hat{y}, \qquad \zeta_{\mathrm{F}} = \frac{3}{5}$$

$$B(t_{\mathrm{f}}) \;=\; \left[\frac{D}{c^2}\right]^{\frac{2}{5}} t_{\mathrm{f}}^{2\zeta_{\mathrm{F}}}\, \overline{\frac{\displaystyle\int_{\hat{y}(0)=0} \mathcal{D}\hat{y}(\hat{t})\; \hat{y}(1)^2 \exp\left\{-\frac{(cD^2)^{\frac{1}{5}}}{T} t_{\mathrm{f}}^{\frac{1}{5}} \int_0^1 d\hat{t}\left[\tfrac{1}{2}(\partial_{\hat{t}}\hat{y})^2 + \hat{V}_{\hat{\xi}(t_{\mathrm{f}})}(\hat{t},\hat{y}(\hat{t}))\right]\right\}}{\displaystyle\int_{\hat{y}(0)=0} \mathcal{D}\hat{y}(\hat{t}) \exp\left\{-\frac{(cD^2)^{\frac{1}{5}}}{T} t_{\mathrm{f}}^{\frac{1}{5}} \int_0^1 d\hat{t}\left[\tfrac{1}{2}(\partial_{\hat{t}}\hat{y})^2 + \hat{V}_{\hat{\xi}(t_{\mathrm{f}})}(\hat{t},\hat{y}(\hat{t}))\right]\right\}}}$$

- Large-size asymptotic, option II: Flory on the free-energy (hyp. Brownian scaling)

$$t = t_{\mathrm{f}}\hat{t}, \quad y = (\widetilde{D}/c^2)^{\frac{1}{3}}\, t_{\mathrm{f}}^{\frac{2}{3}}\, \hat{y}, \quad \bar{F}_V(t,y) \overset{(d)}{=} (\widetilde{D}^2 t_{\mathrm{f}}/c)^{1/3} \hat{F}(\hat{t},\hat{y})$$

$$B(t_{\mathrm{f}}) \underset{t_{\mathrm{f}}\to\infty}{\sim} \left[\frac{\widetilde{D}}{c^2}\right]^{\frac{2}{3}} t_{\mathrm{f}}^{\frac{4}{3}}\, \overline{\frac{\displaystyle\int_{\mathbb{R}} d\hat{y}\; \hat{y}^2 \exp\left\{-\tfrac{1}{T}\left(\tfrac{\widetilde{D}^2}{c} t_{\mathrm{f}}\right)^{\frac{1}{3}}\left[\tfrac{\hat{y}^2}{2} + \hat{F}(\hat{t},\hat{y})\right]\right\}}{\displaystyle\int_{\mathbb{R}} d\hat{y}\; \exp\left\{-\tfrac{1}{T}\left(\tfrac{\widetilde{D}^2}{c} t_{\mathrm{f}}\right)^{\frac{1}{3}}\left[\tfrac{\hat{y}^2}{2} + \hat{F}(\hat{t},\hat{y})\right]\right\}}} \underset{t_{\mathrm{f}}\to\infty}{\sim} \overline{(\hat{y}^\star[\hat{F}])^2}\; (\widetilde{D}/c^2)^{\frac{2}{3}}\, t_{\mathrm{f}}^{\frac{4}{3}}$$

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Rescaling options in path-integral representations

4.1. Saddle point on the free energy at large lengthscale t_{f}

$$t = t_{\mathrm{f}}\hat{t}, \qquad y = (\widetilde{D}/c^2)^{\frac{1}{3}}\, t_{\mathrm{f}}^{\frac{2}{3}}\, \hat{y}, \qquad \bar{F}_V(t,y) \stackrel{(d)}{=} (\widetilde{D}^2 t_{\mathrm{f}}/c)^{1/3} \hat{F}(\hat{t},\hat{y})$$

$$B(t_{\mathrm{f}}) \underset{t_{\mathrm{f}}\to\infty}{\sim} \left[\frac{\widetilde{D}}{c^2}\right]^{\frac{2}{3}} t_{\mathrm{f}}^{\frac{4}{3}}\; b_1(t_{\mathrm{f}})$$

$$B(t_{\mathrm{f}}) \underset{t_{\mathrm{f}}\to\infty}{\sim} \overline{(\hat{y}^\star[\hat{F}])^2}\, (\widetilde{D}/c^2)^{\frac{2}{3}}\, t_{\mathrm{f}}^{\frac{4}{3}}$$

Rely on the assumption that there exists an optimal trajectory!

$$b_1(t_{\mathrm{f}}) = \overline{\frac{\displaystyle\int_{\mathbb{R}} d\hat{y}\; \hat{y}^2 \exp\left\{-\tfrac{1}{T}\left(\tfrac{\widetilde{D}^2}{c}t_{\mathrm{f}}\right)^{\frac{1}{3}}\left[\tfrac{\hat{y}^2}{2}+\hat{F}(\hat{t},\hat{y})\right]\right\}}{\displaystyle\int_{\mathbb{R}} d\hat{y}\; \exp\left\{-\tfrac{1}{T}\left(\tfrac{\widetilde{D}^2}{c}t_{\mathrm{f}}\right)^{\frac{1}{3}}\left[\tfrac{\hat{y}^2}{2}+\hat{F}(\hat{t},\hat{y})\right]\right\}}}$$

4.2. Saddle point on the Hamiltonian with the Flory scaling

$$t = t_{\mathrm{f}}\,\hat{t}, \qquad y = t_{\mathrm{f}}^{\zeta_{\mathrm{F}}}\left(\tfrac{D}{c^2}\right)^{\frac{1}{5}}\hat{y}, \qquad \zeta_{\mathrm{F}} = \frac{3}{5}$$

$$B(t_{\mathrm{f}}) \overset{\text{wrong!}}{\underset{t_{\mathrm{f}}\to\infty}{\sim}} \overline{(\hat{y}^\star_{\hat{V}}(1))^2}\;\left(\tfrac{D}{c^2}\right)^{\frac{2}{5}} t_{\mathrm{f}}^{\frac{6}{5}}$$

$$B(t_{\mathrm{f}}) \overset{\text{true!}}{\underset{t_{\mathrm{f}}\to\infty}{\sim}} \overline{(\hat{y}^\star_{\hat{V}}(1;t_{\mathrm{f}}))^2}\;\left(\tfrac{D}{c^2}\right)^{\frac{2}{5}} t_{\mathrm{f}}^{\frac{6}{5}}$$

$$B(t_{\mathrm{f}}) = \left[\frac{D}{c^2}\right]^{\frac{2}{5}} t_{\mathrm{f}}^{2\zeta_{\mathrm{F}}}\; b_2(t_{\mathrm{f}})$$

$$\overline{(\hat{y}^\star_{\hat{V}}(1;t_{\mathrm{f}}))^2} \sim t_{\mathrm{f}}^{2/15}$$

$$b_2(t_{\mathrm{f}}) = \overline{\frac{\displaystyle\int_{\hat{y}(0)=0}\mathcal{D}\hat{y}(\hat{t})\; \hat{y}(1)^2 \exp\left\{-\tfrac{(cD^2)^{\frac{1}{5}}}{T}t_{\mathrm{f}}^{\frac{1}{5}}\int_0^1 d\hat{t}\left[\tfrac{1}{2}(\partial_{\hat{t}}\hat{y})^2+\hat{V}_{\hat{\xi}(t_{\mathrm{f}})}(\hat{t},\hat{y}(\hat{t}))\right]\right\}}{\displaystyle\int_{\hat{y}(0)=0}\mathcal{D}\hat{y}(\hat{t})\; \exp\left\{-\tfrac{(cD^2)^{\frac{1}{5}}}{T}t_{\mathrm{f}}^{\frac{1}{5}}\int_0^1 d\hat{t}\left[\tfrac{1}{2}(\partial_{\hat{t}}\hat{y})^2+\hat{V}_{\hat{\xi}(t_{\mathrm{f}})}(\hat{t},\hat{y}(\hat{t}))\right]\right\}}}$$

E. Agoritsas & V. Lecomte, *J. Phys. A* 50, 104001 (2017)