Non-equilibrium steady-state of the open KPZ equation

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Les Houches

One dimensional KPZ equation

The [Kardar-Parisi-Zhang 1986] equation is a nonlinear stochastic PDE describing the time evolution of a height function $h(t, x)$. In one spatial dimension,

$$\partial_t h(x,t) = \tfrac{1}{2}\partial_{xx} h(x,t) + \tfrac{1}{2}\big(\partial_x h(x,t)\big)^2 + \xi(x,t),$$

where ξ is a space-time white noise.

The KPZ equation is one toy model in a much wider class of models of stochastic growth, directed polymers, out of equilibrium interacting particle systems... Over the years, the subject has grown to include more and more (sometimes unexpected) topics.

Mathematical definition

$$\partial_t h(x,t) = \tfrac{1}{2}\partial_{xx} h(x,t) + \tfrac{1}{2}\big(\partial_x h(x,t)\big)^2 + \xi(x,t)$$

When ξ is a space-time white noise, $\partial_x h(x,t)$ is not a function, it can only be understood as a distribution, thus $\big(\partial_x h(x,t)\big)^2$ is ill-defined.

Stochastic PDE approach

Usually in SPDE theory, one mollifies the noise as $\xi^\epsilon = \xi * p_\varepsilon$ with some smooth $p_\varepsilon \xrightarrow[\varepsilon\to 0]{} \delta_0$, consider the solution h^ε of

$$\partial_t h^\varepsilon = \tfrac{1}{2}\partial_{xx} h^\varepsilon + \tfrac{1}{2}\big(\partial_x h^\varepsilon\big)^2 + \xi^\varepsilon,$$

and show that the solution h^ϵ converges as $\epsilon \to 0$. This is particularly difficult for the KPZ equation and was achieved by [Bertini-Cancrini 1995] and more generally by [Hairer 2011] [Gubinelli-Imkeller-Perkowski 2012]

For the KPZ equation, we define solutions as $h(x,t) = \log Z(x,t)$ where

$$\partial_t Z(x,t) = \tfrac{1}{2}\partial_{xx} Z(x,t) + Z(x,t)\xi(x,t),$$

Directed polymer interpretation

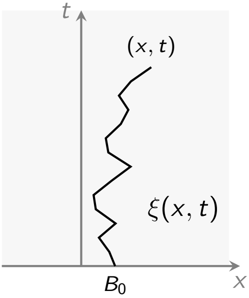
By the Feynman-Kac formula, a solution of

$$\partial_t Z(x,t) = \tfrac{1}{2}\partial_{xx} Z(x,t) + Z(x,t)\xi(x,t),$$

can be viewed as a path integral

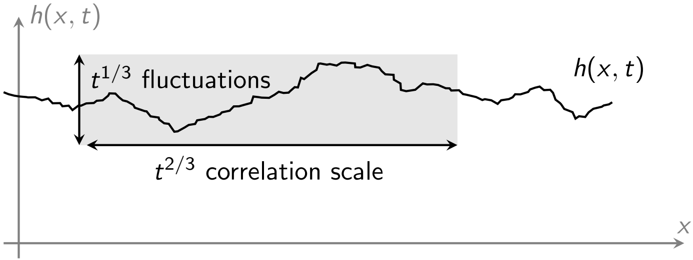
$$Z(x,t) = \left\langle Z_0(B_0) :\exp: \left(\int_0^t \xi(B_s,s)ds \right) \right\rangle$$

where Z_0 is the initial condition at $t=0$ and the expectation $\langle\cdot\rangle$ is taken over a Brownian motion B starting from a free point B_0 and ending at $B_t = x$.



Universality

Large scale fluctuations are characterized by universal exponents $(1/3, 2/3)$ (roughness $\chi = 1/2$, dynamical exponent $z = 3/2$) [Nelson-Forster-Stephen 1977], [Kardar-Parisi-Zhang]



For every model in the KPZ class described by a height function $h(t, x)$, we expect that

$$\lim_{L\to+\infty} \left\{ \frac{1}{L} h(L^2 x, L^3 t) \right\} \xrightarrow{L\to\infty} \mathfrak{h}(t, x),$$

(up to some constants) where $\mathfrak{h}(t, x)$ is called the KPZ fixed point [Matetski, Quastel, Remenik 2017] and there is much activity in understanding its properties.

One point fluctuations

For the KPZ equation, starting from $h(x, t = 0) = \log(\delta_0)$, we have

$$h(x,t) \underset{t\to\infty}{\sim} -t/24$$

and

$$\mathbb{P}\left(\frac{h(0,t)+t/24}{(t/2)^{1/3}} \leqslant s\right) \xrightarrow[n\to\infty]{} F_2(s),$$

where F_2 is the Tracy-Widom cumulative distribution function.

Proof ($\approx$ 2010):

- Using directed polymers: [Calabrese-Le Doussal-Rosso][Dotsenko] via Replica method + Bethe ansatz .
- Using ASEP: [Amir-Corwin-Quastel][Sasamoto-Spohn] via [Tracy-Widom] Bethe ansatz solution of ASEP.

Today, one knows much more about fluctuations for other initial conditions, multipoint correlations, large deviations (cf Gregory Schehr's talk), etc.

In this talk

In this talk, I will discuss

- **A more complicated setup:** the KPZ equation on a bounded domain $[0, L]$ instead of $\mathbb{R}$;
- **A simpler question:** finding the non-equilibrium steady-state.

Vocabulary

There are some differences of vocabulary between statistical Physics and Markov processes literature.

$$\text{steady-state (equilibrium)} = \text{reversible stationary measure}$$

where *reversible* means *satisfying detailed balance* $\pi_i P_{i\to j} = \pi_j P_{j\to i}$.

When detailed balance is not satisfied, as in out-of-equilibrium systems, there may still be a

$$\text{non-equilibrium steady state} = \text{non reversible stationary measure.}$$

Stationary measures of the KPZ equation

The KPZ equation is such that $h(0,t) \sim \frac{-t}{24}$, which clearly diverges, so it does not have a true stationary measure. But it has non-equilibrium steady states in the following sense:

Definition (Non-equilibrium steady-state)

We say that the law of a process $h^{\mathrm{stat}}(x)$ is stationary for the KPZ equation when the following holds:

If $h(x,0) = h^{\mathrm{stat}}(x)$, then for all $t > 0$,

$$h(t,x) - h(t,0) \overset{(d)}{=} h^{\mathrm{stat}}(x) - h^{\mathrm{stat}}(0).$$

For the KPZ equation on $\mathbb{R}$, the Brownian motion with drift μ ($\mu \in \mathbb{R}$ can be arbitrary) is stationary for the KPZ equation [Forster-Nelson-Stephen 1977, Bertini-Giacomin 1997].

Stationary measures of stochastic PDEs

This stationarity of the Brownian motion is far from obvious!

- (Linear case) For stochastic PDEs of the form

$$\partial_t u = Lu + \xi$$

  where L is a linear differential operator, stationary measures are Gaussian and there exists a general theory.
- (Equilibrium case) The path integral measure

$$e^{-S[\varrho]}\mathcal{D}\varrho$$

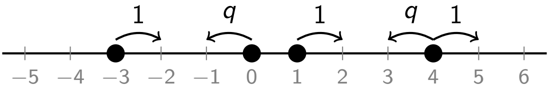
  is the stationary measure for the equation

$$\partial_t u = -\frac{\delta S[u]}{\delta u} + \sqrt{2}\xi.$$

  [Nelson 1966, Parisi-Wu 1981]
- The KPZ equation is non linear and out of equilibrium.

Bertini-Giacomin's proof via ASEP

ASEP (asymmetric simple exclusion process) is a continuous Markov process on $\{0,1\}^{\mathbb{Z}}$, whose transition rates depend on an asymmetry parameter $q < 1$.



- For any $\varrho \in [0,1]$, i.i.d. Bernoulli(ϱ) is a stationary measure.
- Define a height function $H(x,t)$ so that

$$H(x,t) - H(x-1,t) = \begin{cases} 1 & \text{if site } x \text{ is occupied.} \\ -1 & \text{if site } x \text{ is empty.} \end{cases}$$

and $H(0,t)$ is the number of particles which have crossed the origin.

Convergence ASEP $\to$ KPZ

Theorem ([Bertini-Giacomin 1997])

Let $\mathcal{Z}_t(x) = q^{\frac{1}{2}H(x,t)-\nu t}$*, where* $\nu = (1-\sqrt{q})^2$*. For* $q = e^{-\varepsilon}$*, when* $\varepsilon \to 0$

$$\mathcal{Z}_{\epsilon^{-4}t}(\epsilon^{-2}x) \Longrightarrow Z(x,t),$$

the solution of

$$\partial_t Z = \tfrac{1}{2}\Delta Z + Z\,\xi.$$

ASEP height function converges to a solution of KPZ equation.

When occupation variables are i.i.d. Bernoulli, ASEP's height function converges to a Brownian motion (with drift), up to a global shift.

Corollary ([Bertini-Giacomin 1997])

For any drift $\mu \in \mathbb{R}$*, the Brownian motion* $B_x + \mu x$ *is stationary*

Open KPZ equation on $[0, L]$

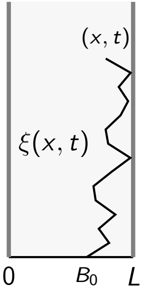
Consider the KPZ equation

$$\partial_t h(t,x) = \frac{1}{2}\partial_{xx} h(t,x) + \frac{1}{2}\big(\partial_x h(t,x)\big)^2 + \xi(t,x)$$

for $x \in [0, L]$. We may define it as $h(t,x) = \log Z(t,x)$ with

$$Z(x,t) = \left\langle Z_0(B_0) \exp\left(\int_0^t (\xi(B_s,s) - a\delta_0(B_s) - b\delta_L(B_s) ds)\right)\right\rangle,$$

where, now, B_t is a Brownian motion reflected on walls at $x = 0$ and $x = L$, and $a, b \in \mathbb{R}$ are some parameters.



Open KPZ equation

It may also be seen as the stochastic PDE

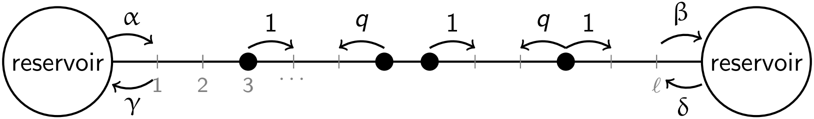
$$\partial_t h(t,x) = \tfrac{1}{2}\partial_{xx} h(t,x) + \tfrac{1}{2}\big(\partial_x h(t,x)\big)^2 + \xi(t,x)$$

with boundary conditions

$$\partial_x h(t,x)\Big|_{x=0} = u, \quad \partial_x h(t,x)\Big|_{x=L} = -v,$$

with $u = a + 1/2, v = b + 1/2$. Since $h(x,t)$ is not differentiable, some care is needed to define what the boundary conditions exactly mean.

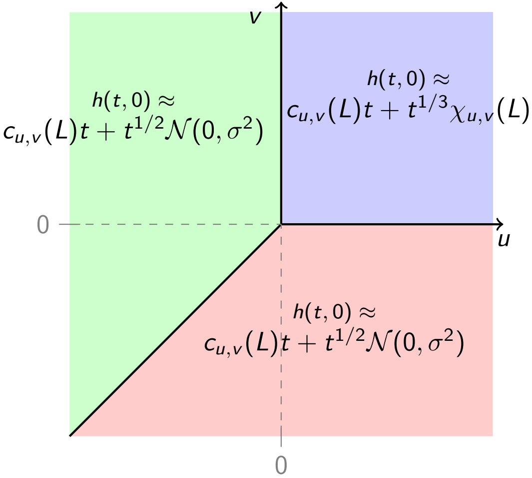
The corresponding version of ASEP is **open ASEP**



Physically, fixing $\partial_x h$ at the boundary corresponds to fixing the densities imposed by the reservoirs [Corwin-Shen 2016].

Fluctuations in the open KPZ class

When $L = O(t^{2/3})$, we expect that for all models in the KPZ class,



Finding the distribution of $\chi_{u,v}$ is open. However, one can determine the large t distribution of

$$h(x,t) - h(0,t).$$

Stationary measures on $[0, L]$

On a segment, the KPZ equation stationary measures has nontrivial **spatial correlations**.

Theorem

For any $u, v \in \mathbb{R}$, there exists a unique stationary process $h_{u,v}^{\mathrm{stat}}(x)$ with law

$$h_{u,v}^{\mathrm{stat}}(x) \overset{(d)}{=} W(x) + X(x)$$

where W is a Brownian motion on $[0, L]$ and X is a reweighted Brownian motion

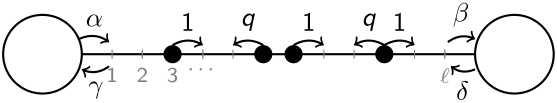
$$\mathbb{P}_{u,v}^{\mathrm{stat}}(X) = \frac{1}{\widetilde{\mathcal{Z}}_{u,v}(L)} e^{-vX(L)} \left(\int_0^L e^{-X(s)} ds \right)^{-u-v} \mathbb{P}_{X_0=0}^{Brown}(X).$$

- When $u + v = 0$, $h_{u,v}^{\mathrm{stat}}$ is a Brownian motion with drift $-v$.
- When $u, v > 0$, the profile is curved.
- Exponential functionals of the Brownian motion have been well-studied, so that one can obtain precise quantitative information about $h_{u,v}^{\mathrm{stat}}(x)$.

Proof

The initial proof of the result relied on:

- The characterization of open ASEP stationary measure via the matrix product ansatz [Derrida-Evans-Hakim-Pasquier 1993]



- A representation of the matrix product ansatz [Uchiyama-Sasamoto-Wadati 2004] and its relation to Askey-Wilson processes [Bryc-Wesolowski 2015]
- When $u + v \geqslant 0$, [Corwin-Knizel 2021] took the KPZ limit and characterized $h_{u,v}^{\mathrm{stat}}$ through Laplace transform formulas.
- [Bryc-Kuznetsov-Wang-Wesołowski 2022] and [B.- Le Doussal 2022] worked out Laplace inversion.
- [Matetski-Knizel 2023, Parekh 2023] Uniqueness of the stationary process using ideas of [Hairer-Mattingly 2015]

If time permits, we will see another, more direct, method

Brownian motion in exponential potential

When $u + v \geqslant 0$, the process $X(x)$ may be written as

$$X(x) = Y(x) - Y(0),$$

where Y is a reweighted Brownian motion

$$\mathbb{P}^{\mathrm{stat}}_{u,v}(Y) = \frac{1}{\mathcal{Z}_{u,v}(L)} \exp\left(-uY(0) - vY(L) - \int_0^L e^{-Y(s)} ds\right) \mathbb{P}^{\mathrm{Brown}}_{\mathrm{free}}(Y),$$

where

$$\mathbb{P}^{\mathrm{Brown}}_{\mathrm{free}}(Y) = \exp\left(\frac{-1}{2} \int_0^L \left(\frac{dY(s)}{ds}\right)^2 ds\right) \mathcal{D}(Y).$$

The process Y can only be defined for $u + v > 0$.

To obtain the previous description, one needs to average over $Y(0)$, and then the measure makes sense for any u, v.

Liouville quantum mechanics

$$\frac{1}{\mathcal{Z}_{u,v}(L)} \exp\left(-uY(0) - vY(L) - \int_0^L e^{-Y(s)} ds - \frac{1}{2}\int_0^L \left(\frac{dY(s)}{ds}\right)^2 ds\right) \mathcal{D}(Y)$$

The initial proof of the theorem came from recognizing eigenfunctions of

$$H = -\Delta + e^{-x}$$

in exact formulas from [Corwin-Knizel].

$Y(t)$ is a Markov process with transition probability

$$p_{s,t}(x,y) = \frac{h_t(y)}{h_s(x)} q_{t-s}(x,y), \quad q_t(x,y) = \int_{\mathbb{R}} \psi_{ik}(x)\overline{\psi_{ik}(y)} e^{-k^2 t} dk$$

where $\psi_k(x)$ form an eigenbasis of $-\Delta + e^{-x}$ with eigenvalue k^2, and

$$h_t(y) = \int_{\mathbb{R}} q_{L-t}(x,y) e^{-vy} dy.$$

This is sometimes referred to as *Liouville quantum mechanics* and used to compute exponential functionals of the Brownian motion [Comtet, Texier, Monthus, Le Doussal, ...]

Extensions, variants, universality

Analogues of these results exist for discrete models, where reweighted Brownian motions become reweighted random walks

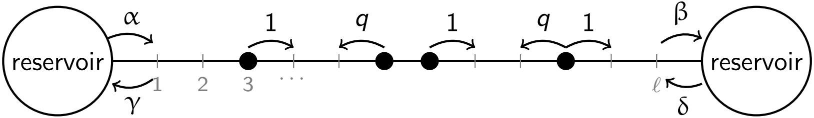
- TASEP [Derrida-Enaud-Lebowitz 2004] ASEP [B.-Le Doussal 2022], based on **Matrix Product Ansatz**.
- Last-Passage Percolation with geometric or exponential weights and the log-gamma polymer [B.-Corwin-Yang 2023], based on a different approach relying on symmetric functions and Gibbsian line ensembles.

In all cases, the stationary process converges at large scale ($L \to \infty$) to the same universal limit: a Brownian motion W plus a Brownian motion with hard wall potential.

$$V(x) = e^{-\sqrt{L}x} \xrightarrow[L\to\infty]{} \begin{cases} +\infty & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Matrix product ansatz

Consider ASEP on $\{0,1\}^\ell$ with boundary parameters $\alpha, \beta, \gamma, \delta$.



We describe the state of the system by $\eta \in \{0,1\}^\ell$. The stationary measure $\mathbb{P}$ can be written as [Derrida-Evans-Hakim-Pasquier 1993]

$$\mathbb{P}(\eta) = \frac{1}{Z_\ell} \langle w| \prod_{i=1}^{\ell} (\eta_i D + (1-\eta_i)E) \, |v\rangle$$

where

$$Z_\ell = \langle w| \, (E+D)^\ell \, |v\rangle$$

and E, D are infinite matrices, and $\langle w| \, , |v\rangle$ are row/column vectors such that

$$DE - qED = D + E$$
$$\langle w| \, (\alpha E - \gamma D) = \langle w|$$
$$(\beta D - \delta E) \, |v\rangle = |v\rangle$$

Representations of the MPA

- Finding representations, i.e. matrices E, D and explicit vectors u, v satisfying the relations, is non trivial. Special cases are worked out in [Derrida-Evans-Hakim-Pasquier 1993].
- For TASEP, $q = \gamma = \delta = 0$, we may take

$$D = \begin{pmatrix} 1 & 1 & 0 & \\ 0 & 1 & 1 & \ddots \\ 0 & 0 & 1 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}, E = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & \\ 0 & 1 & 1 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

  and easily find eigenvectors $\langle w| , |v\rangle$.
- [Sandow, 1995] proposed a representation in the most general case. The vectors $\langle w| , |v\rangle$ are complicated.
- Several families of orthogonal polynomials appear. In the most general case, [Uchiyama-Sasamoto-Wadati, 2003] found a representation using Askey-Wilson orthogonal polynomials.
- Another very simple representation was proposed in [Enaud-Derrida, 2003]

Enaud-Derrida's representation

Enaud-Derrida found a very simple representation for any parameters $q, \alpha, \beta, \gamma, \delta$. When

$$\gamma = q(1-\alpha), \quad \delta = q(1-\beta) \iff \varrho_0 = \alpha, \quad 1 - \varrho_\ell = \beta$$

it becomes :

$$D = \begin{pmatrix} [1]_q & [1]_q & 0 & 0 & 0 & \cdots \\ 0 & [2]_q & [2]_q & 0 & 0 & \cdots \\ 0 & 0 & [3]_q & [3]_q & 0 & \cdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots \end{pmatrix}, \quad E = \begin{pmatrix} [1]_q & 0 & 0 & 0 & \cdots \\ [2]_q & [2]_q & 0 & 0 & \cdots \\ 0 & [3]_q & [3]_q & 0 & \\ 0 & 0 & \ddots & \ddots & \ddots \end{pmatrix}$$

where $[n]_q = \frac{1-q^n}{1-q}$.

Denoting by $\{|n\rangle\}_{n\geqslant 1}$ the vectors of the associated basis, let

$$\langle w| = \sum_{n\geqslant 1} \left(\frac{1-\varrho_0}{\varrho_0}\right)^n \langle n|, \qquad |v\rangle = \sum_{n\geqslant 1} \left(\frac{\varrho_\ell}{1-\varrho_\ell}\right)^n [n]_q \, |n\rangle .$$

Sum over paths

Due to the bidiagonal structure, the normalization constant $Z_\ell = \langle w| (D+E)^\ell |v\rangle$ can be written as a sum over lattice paths $\vec{n} = (n_0, n_1, \dots, n_\ell) \in \mathbb{N}^\ell$ of the form

$$Z_\ell = \sum_{\vec{n}} \Omega(\vec{n})$$

where

$$\Omega(\vec{n}) = \left(\frac{1-\varrho_0}{\varrho_0}\right)^{n_0} \left(\frac{\varrho_\ell}{1-\varrho_\ell}\right)^{n_\ell} \prod_{i=1}^{\ell} v(n_{i-1}, n_i) \prod_{i=0}^{\ell} [n_i]_q,$$

with

$$v(n,n') = \begin{cases} 2 & \text{if } n = n', \\ 1 & \text{if } |n-n'| = 1 \\ 0 & \text{else.} \end{cases}$$

- This introduces a natural probability measure on random walk paths $\vec{n}$. The stationary measure $\mathbb{P}(\eta)$ can be recovered from this measure.

Open ASEP invariant measure

Following arguments similar as [Derrida-Enaud-Lebowitz 2004], one arrives at

Theorem ([B.-Le Doussal 2022])

Under the stationary measure $\mathbb{P}(\eta)$, ASEP height function $H(x) = \sum_{j=1}^{x}(2\eta_i - 1)$ is such that

$$(H(i))_{1\leqslant i\leqslant \ell} \overset{(d)}{=} (n_i - n_0 + m_i)_{1\leqslant i\leqslant \ell}\,,$$

where $(n_i, m_i)_{0\leqslant i\leqslant \ell}$ is a two dimensional random walk on $\mathbb{Z}^2$, starting from $(n_0, 0)$, distributed as

$$P(\vec{n},\vec{m}) = \frac{\mathbb{1}_{n_0>0}}{4^{-\ell} Z_\ell}\left(\frac{1-\varrho_0}{\varrho_0}\right)^{n_0}\left(\frac{\varrho_\ell}{1-\varrho_\ell}\right)^{n_\ell}\prod_{i=0}^{\ell}[n_i]_q \times P^{SSRW}_{n_0,0}(\vec{n},\vec{m}),$$

where $P^{SSRW}_{n_0,0}$ denotes the probability measure of the symmetric simple random walk (SSRW) on $\mathbb{Z}^2$ starting from $(n_0, 0)$.

Scaling limit to the KPZ equation

Under the scalings such that ASEP's height function converges to KPZ, in particular

$$q = 1 - \varepsilon, \;\; \ell = \varepsilon^{-2}, \;\; \varrho_0 = \frac{1}{2}(1 + u\varepsilon), \;\; \varrho_\ell = \frac{1}{2}(1 - v\varepsilon)$$

we find, denoting by Y_x the rescaled version of the random walk n_i

$$\prod_{i=0}^{\ell} [n_i]_q \to e^{-\int_0^L e^{-2Y_s} ds}$$

$$\left(\frac{1 - \varrho_a}{\varrho_a}\right)^{n_0} \left(\frac{\varrho_b}{1 - \varrho_b}\right)^{n_\ell} \to e^{-2uY_0 - 2vY_L}$$

so that

$$(m_i, n_i) \Longrightarrow (W_x, Y_x)$$

where W_x is a Brownian motion and Y_x is a Brownian motion reweighted by

$$\frac{1}{\mathcal{Z}_{u,v}} e^{-2uY_0 - 2vY_L} e^{-\int_0^L e^{-2Y_s} ds}.$$

Growth rate of the stationary KPZ equation

Using limits of discrete models, one can deduce that for the KPZ equation

$$\partial_t h = \tfrac{1}{2}\partial_{xx} h + \tfrac{1}{2}\big(\partial_x h\big)^2 + \xi, \quad t \geqslant 0, \quad x \in [0, L]$$

with stationary initial data $h(0, x) = h^{\mathrm{stat}}_{u,v}(x)$.

Then, for all $t > 0$,

$$\mathbb{E}[h(t, 0)] = t\ c_{u,v}(L),$$

where in the maximal current phase, $u, v > 0$,

$$\boxed{c_{u,v}(L) = \frac{-1}{24} + \frac{1}{2}\partial_L \log \mathcal{Z}_{u,v}(L)}$$

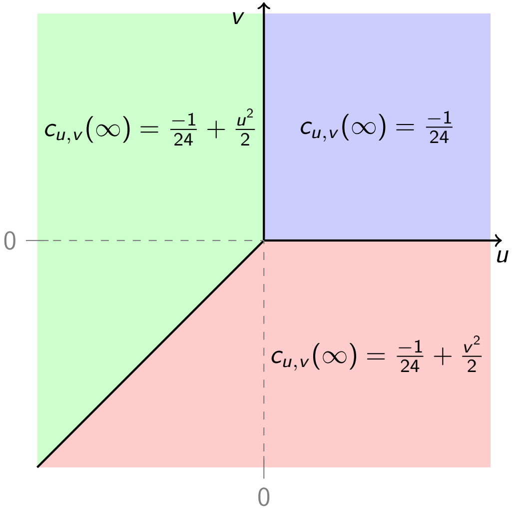
where $\mathcal{Z}_{u,v}(L)$ (the same renormalization constant as before) can be computed exactly as

$$\mathcal{Z}_{u,v}(L) = \int_{\mathbf{i}\mathbb{R}} \frac{dz}{2\mathbf{i}\pi} \left| \frac{\Gamma(u+z)\Gamma(v+z)}{\Gamma(2z)} \right|^2 \frac{e^{z^2 L}}{2}.$$

If $u < 0$ or $v < 0$, $c_{u,v}(L)$ can be obtained through analytic continuation.

Large scale limit

As L goes to infinity, $c_{u,v}(L)$ should be the same as for the KPZ equation on $\mathbb{R}$, except when boundaries play a role. Indeed, we have



Conclusion

Summary

For the KPZ equation on $[0, L]$ (and discrete analogues) the non-equilibrium steady-state is not Brownian but it can be described in terms of some path integral (a Brownian motion in an exponential potential).

Other directions

- A similar result holds for the KPZ equation on $\mathbb{R}_+$ (more is known on $\mathbb{R}_+$).
- Similar results hold for the various integrable discrete models.
- Open question: characterize the typical fluctuations of $h(x, t)$.
- What happens in higher dimensions?

Thank you for your attention