Pot of gold at the end of the KPZ rainbow

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Joint work with
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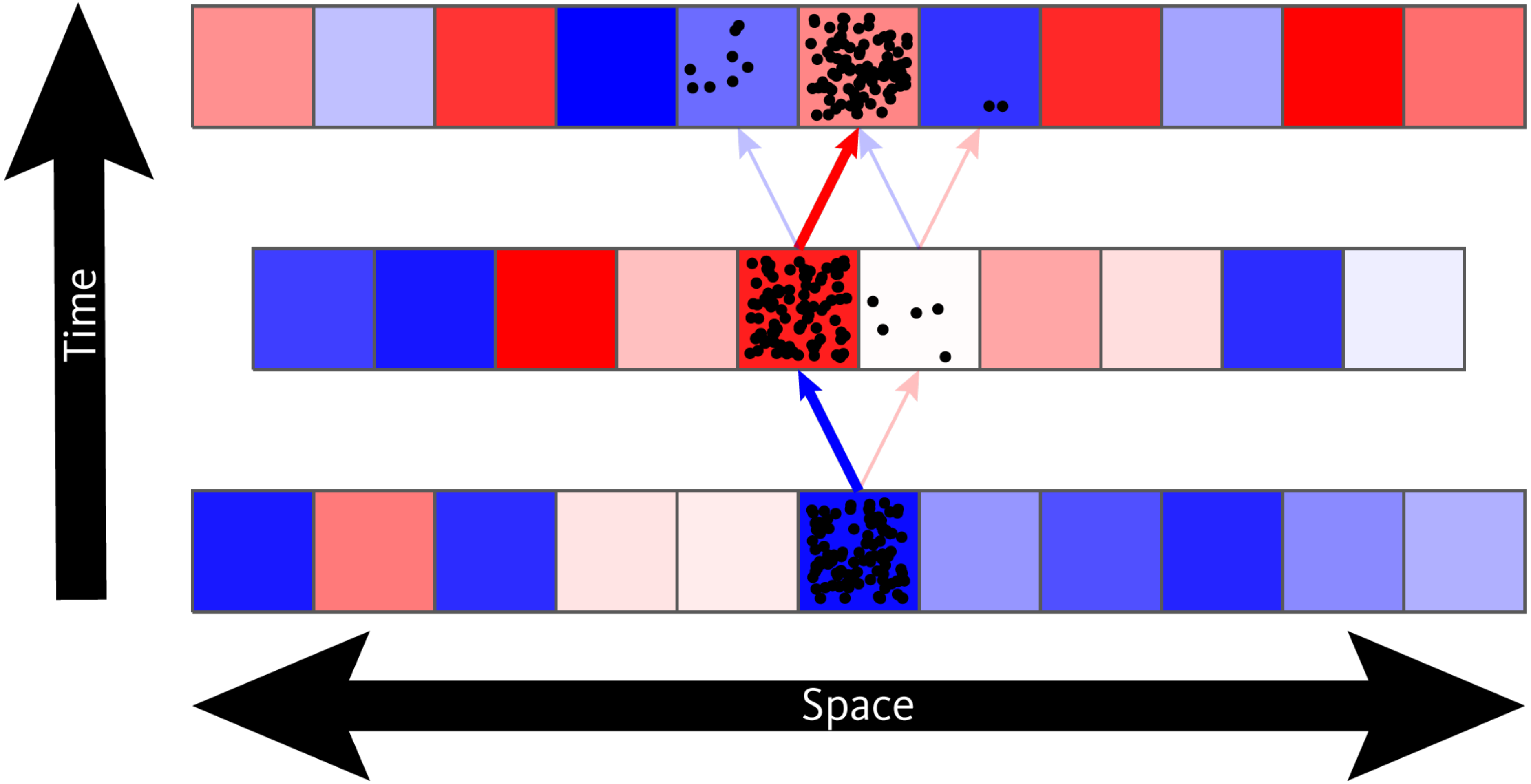
Other initial conditions and multiple species?



Asymmetric simple exclusion process (ASEP)

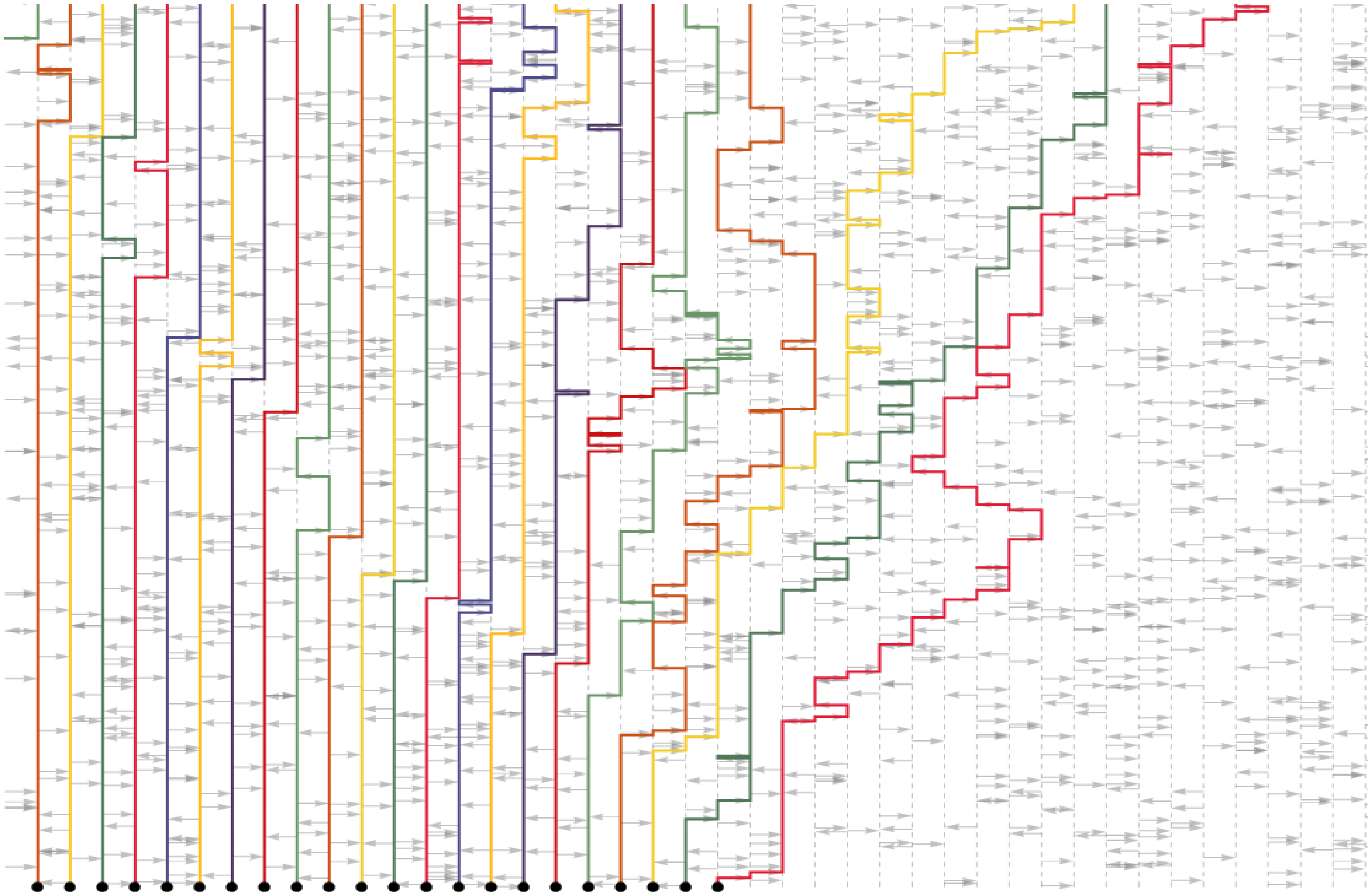
[MacDonald-Gibbs-Pipkin ‘68], [Spitzer ’70]

Environment:

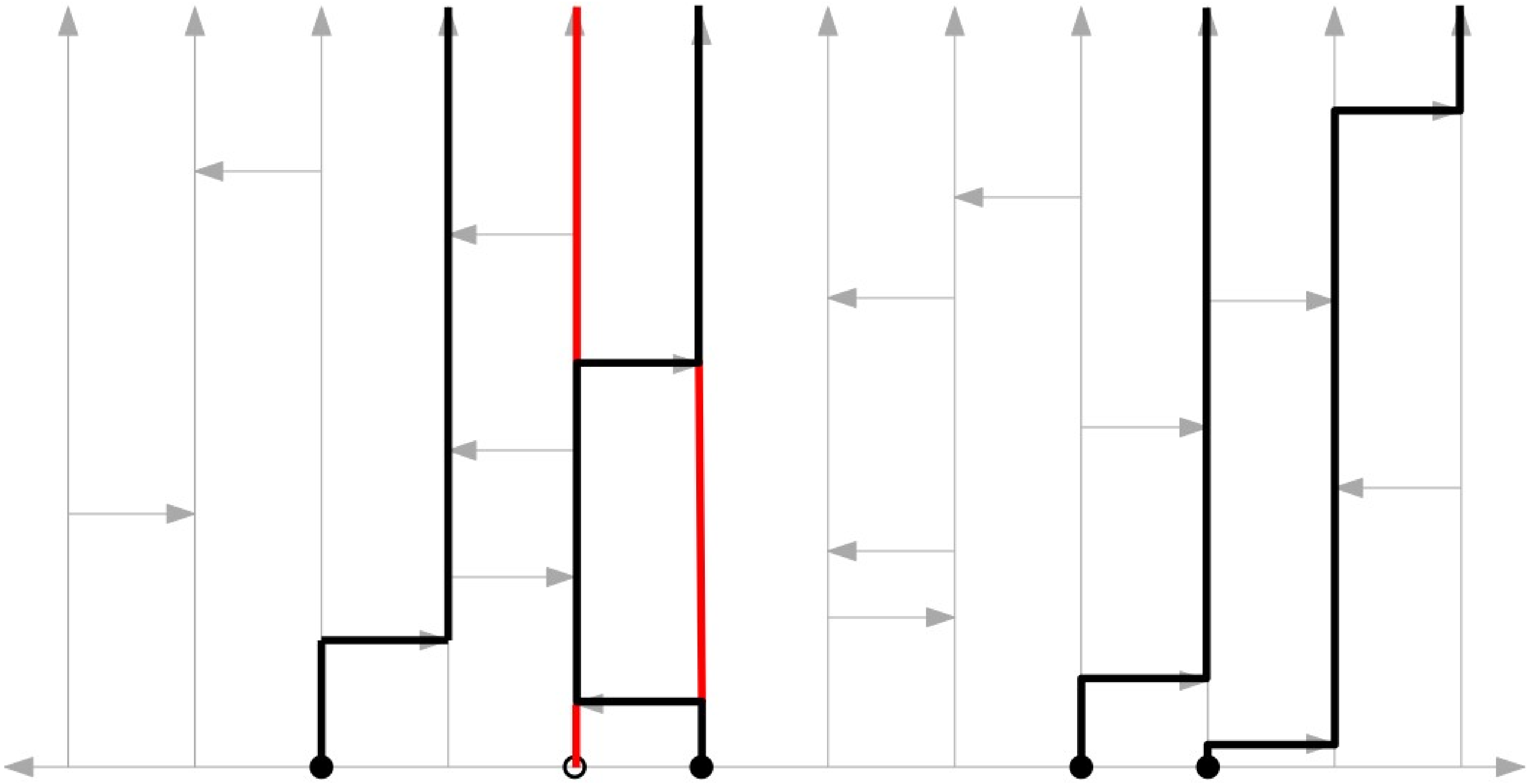
- Poisson rate 1 right arrows
- Poisson rate q<1 left arrows
- All arrows are independent

Evolution:

- Particles follow arrows if they can



Different initial data evolve as colored ASEP

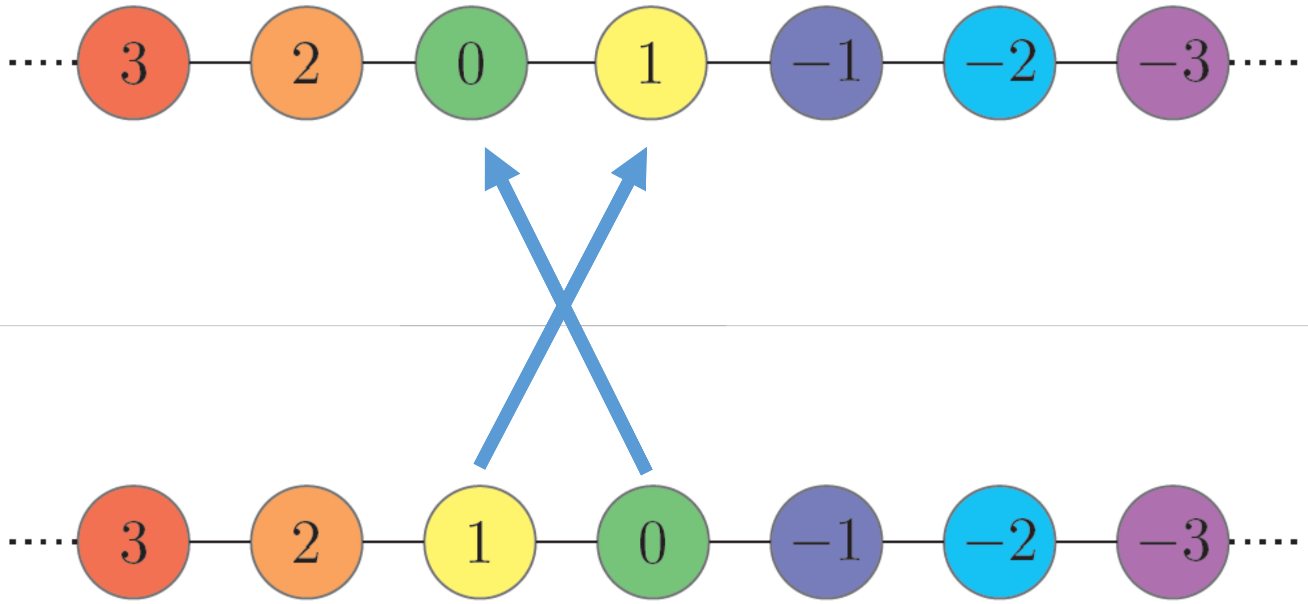


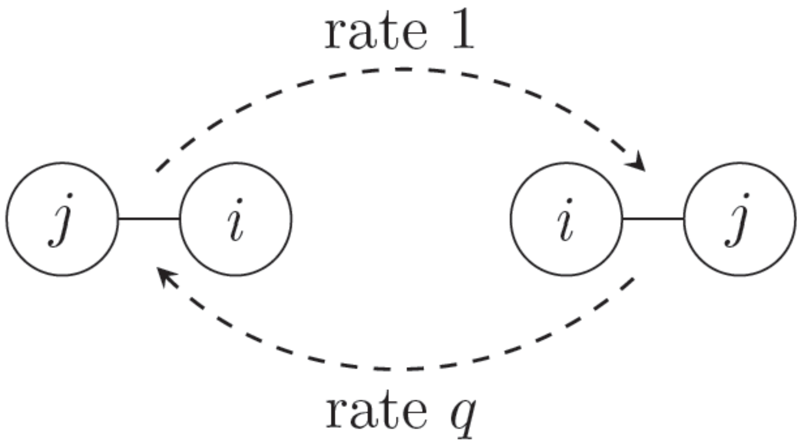
Red—**Black** becomes **Black**—**Red** at rate q

Black—**Red** becomes **Red**—**Black** at rate 1

Colored ASEP

[Liggett '76], [Harris '78]

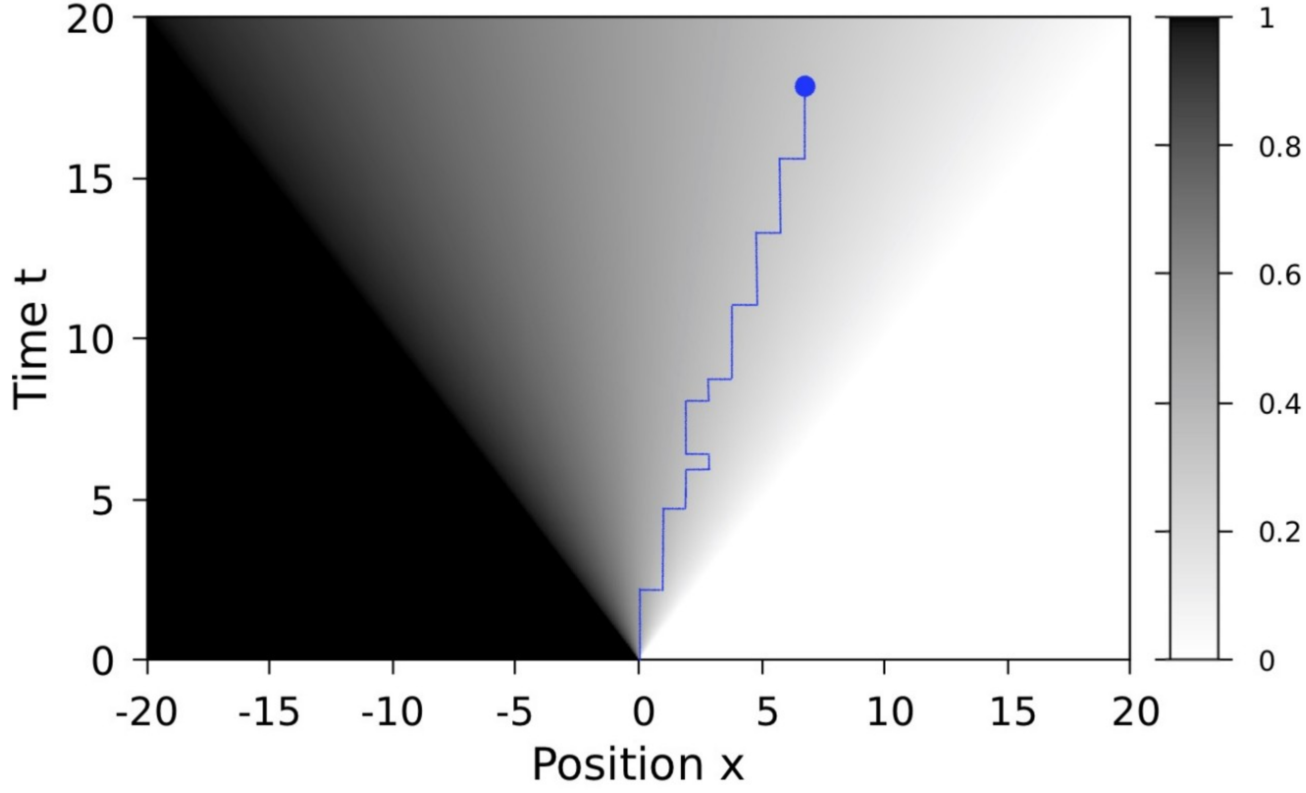


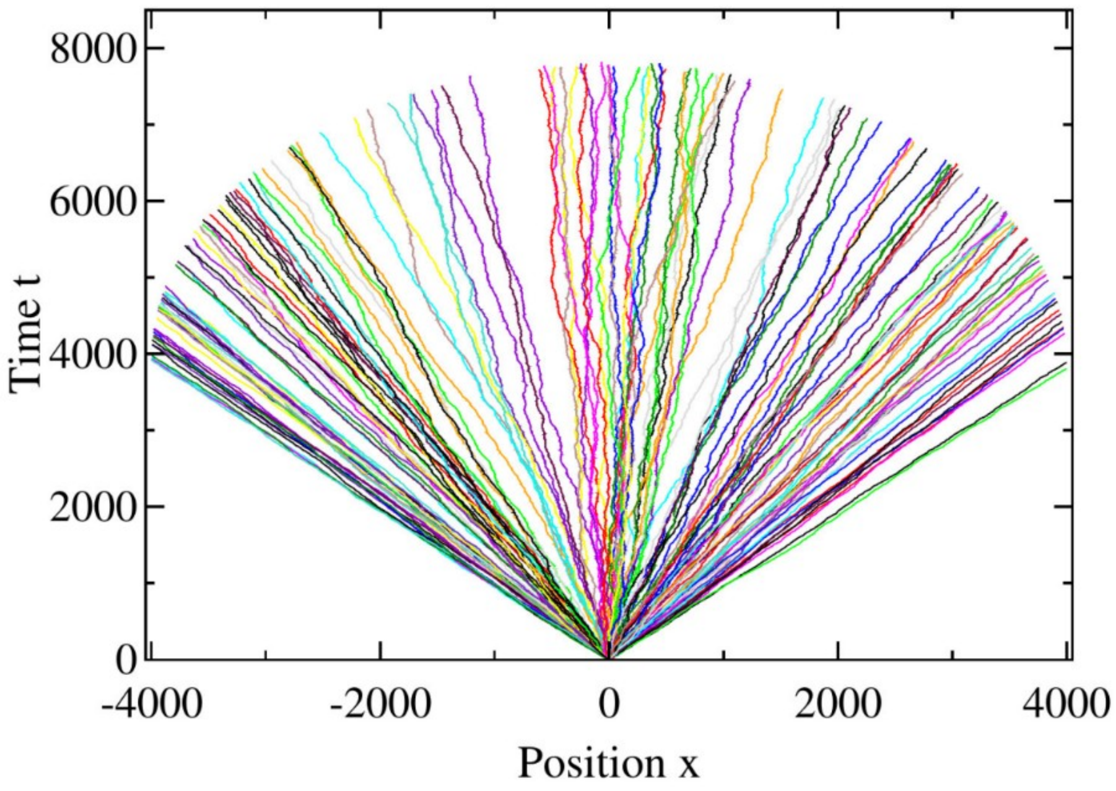


Always assume q<1 and scale time by (1-q)

Motion of individual single colors

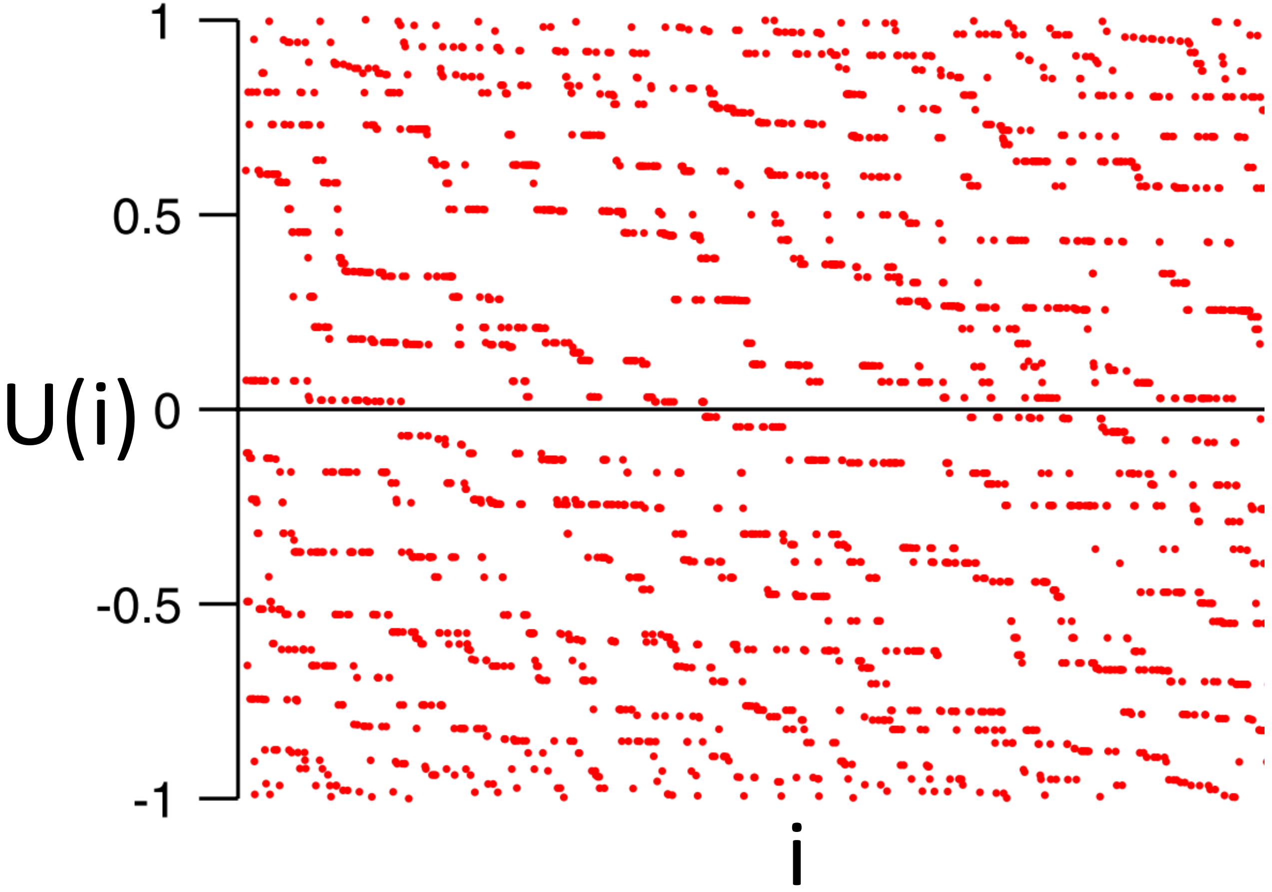
[Aggarwal-C-Ghosal '22]

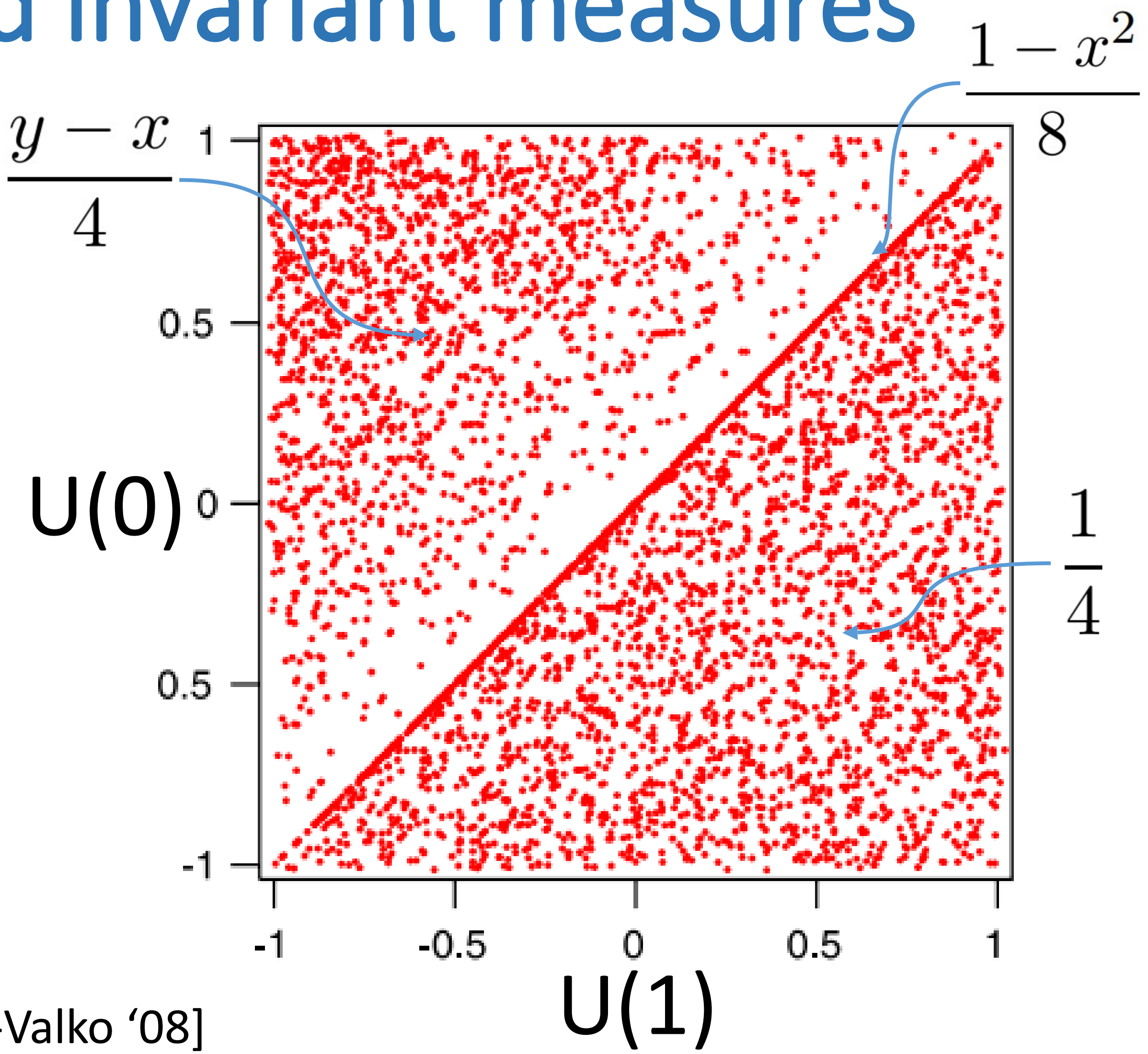




Thm: (Location of color i at time t) / t → U(i) almost surely.

ASEP speed process and invariant measures

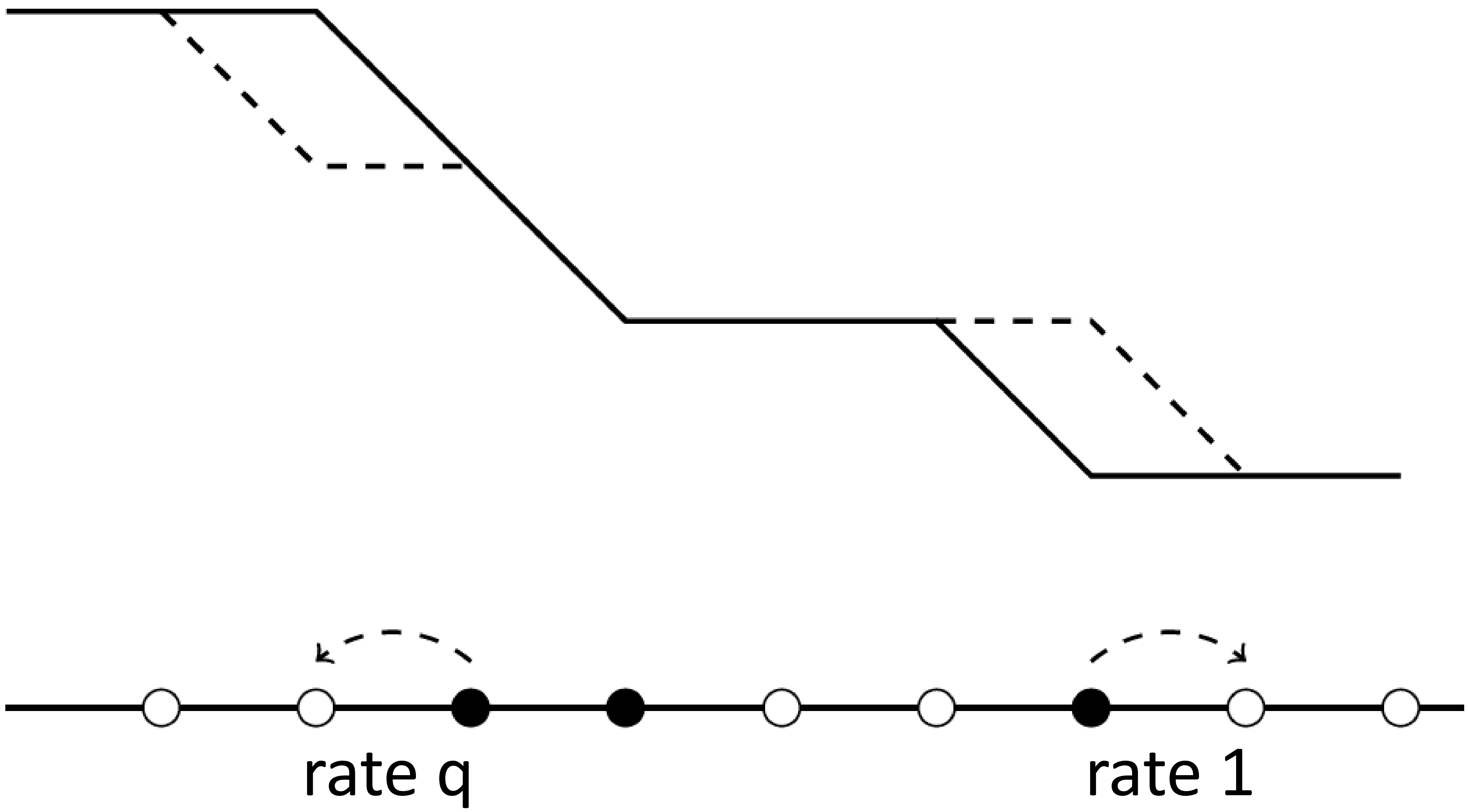




TASEP (q=0) case shown above from [Amir-Angel-Valko ‘08]

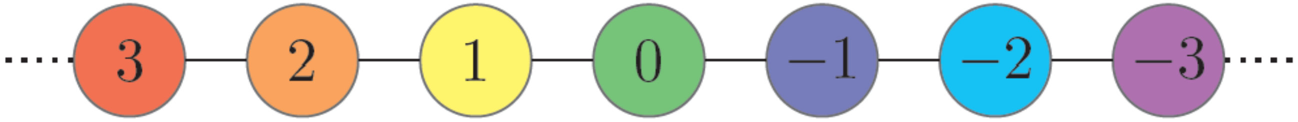
<u>Thm</u>: Colored ASEP with color U(i) at site i initialy is stationary.

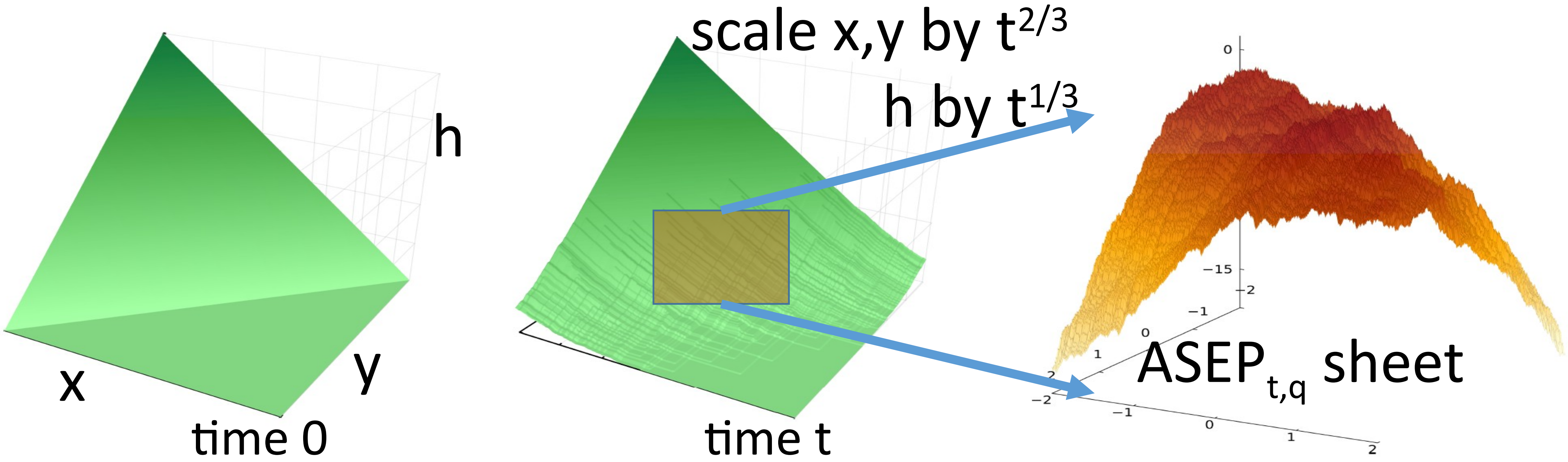
ASEP height function evolution



Mesoscopic scaling limit of colored ASEP

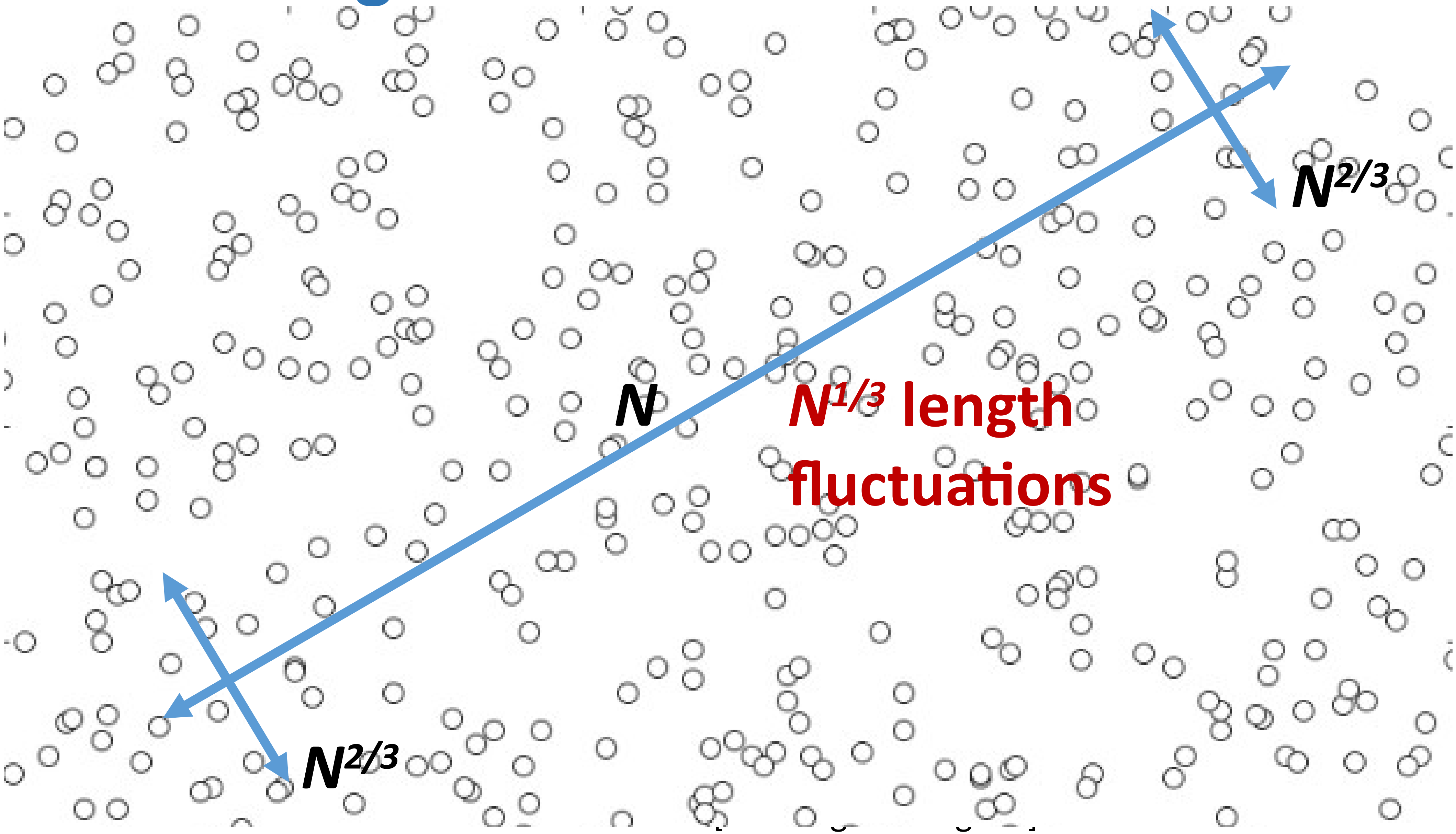
h(x;y,t) := # particles of color ≥ x to right of y at time t





<u>Thm</u>: The $\text{ASEP}_{t,q}$ sheet converges to the **Airy sheet** $\mathcal{S}(x;y)$.

Scaling limits of random 2d metric?



The directed landscape from the Airy sheet

Unique random continuous function $\mathcal{L} : \mathbb{R}^4_\uparrow \to \mathbb{R}$ that satisfies:

1. Airy sheet marginals: For any $s \in \mathbb{R}$ and $t > 0$,

$$(x, y) \mapsto \mathcal{L}(x, s; y, s+t) \stackrel{d}{=} (x, y) \mapsto t^{1/3} \mathcal{S}(xt^{-2/3}; yt^{-2/3}).$$

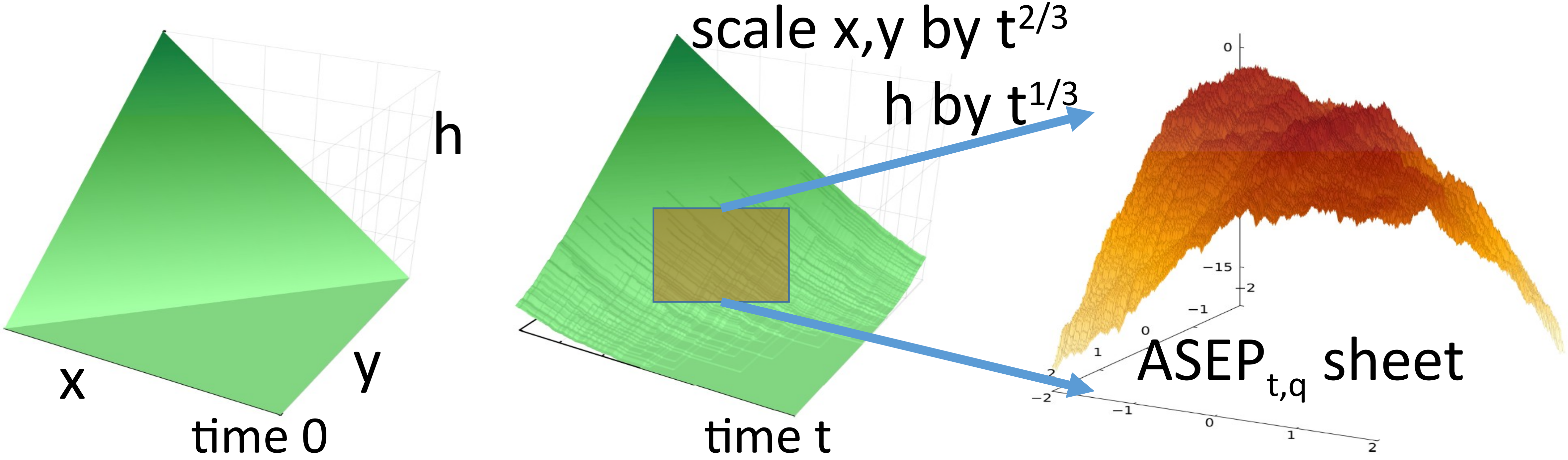
2. Independent increments: The functions $(x, y) \mapsto \mathcal{L}(x, s_i; y, t_i)$ are independent for disjoint time intervals $\{(s_i, t_i) : i \in [\![1, k]\!]\}$.

3. Composition law: For all $r < s < t$,

$$\mathcal{L}(x, r; y, t) = \max_{z \in \mathbb{R}} \Big(\mathcal{L}(x, r; z, s) + \mathcal{L}(z, s; y, t) \Big).$$

Mesoscopic scaling limit of colored ASEP

h(x;y,t) := # particles of color ≥ x to right of y at time t

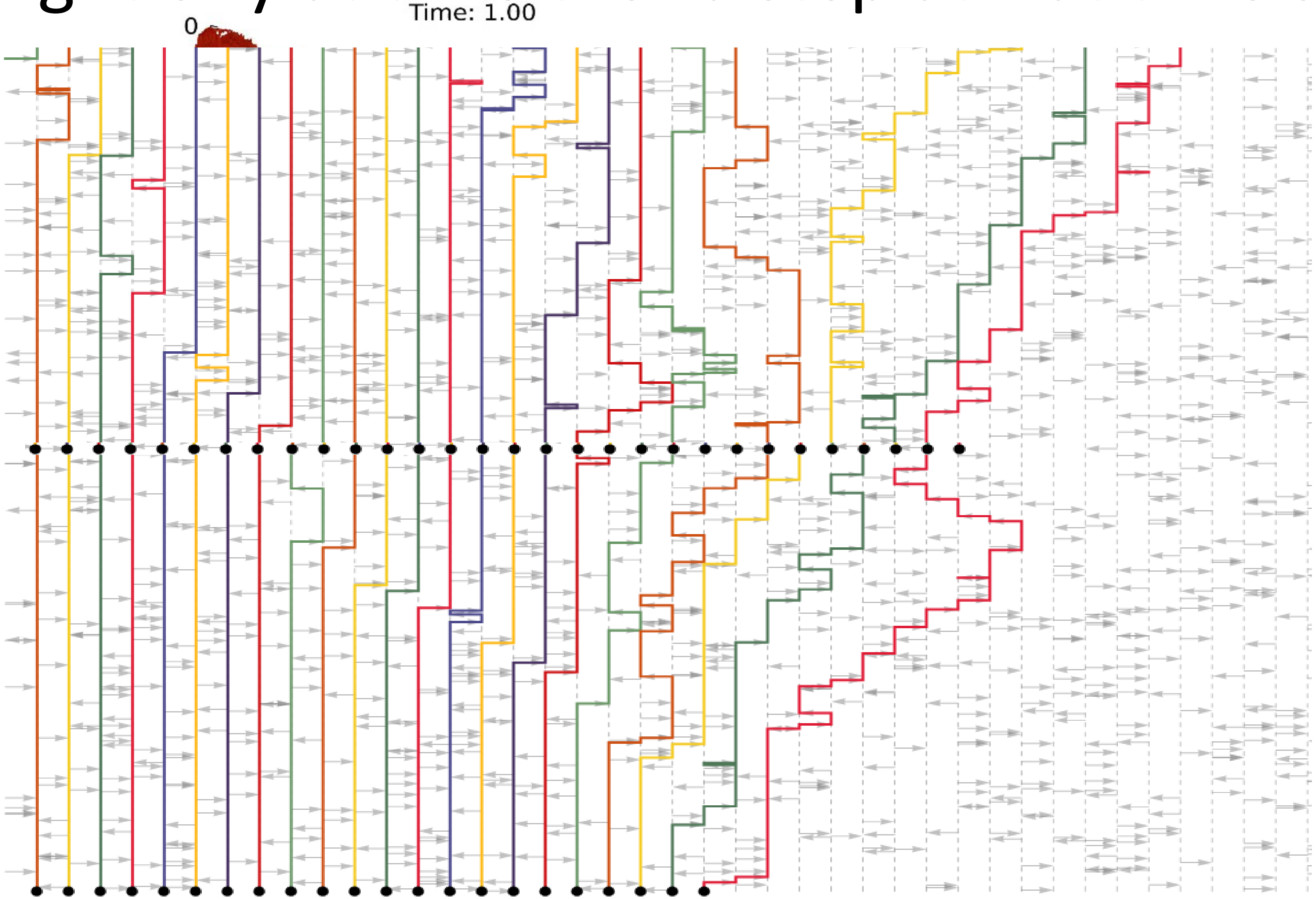


Thm: The $\mathrm{ASEP}_{t,q}$ sheet converges to the **Airy sheet** $\mathcal{S}(x;y)$.

Directed landscape from colored ASEP

h(x,s;y,t) := # particles right of y at time t for a step at x at time s

KPZ scaling/centering: Let R_ε act on h by scaling s and t by ε^{-1}, x and y by $\varepsilon^{-2/3}$, h by ε, and centering.



<u>Cor</u>: As $\varepsilon \to 0$, $R_\varepsilon h(x,s;t,y) \to$ the directed landscape.

Convergence of the random dynamic system

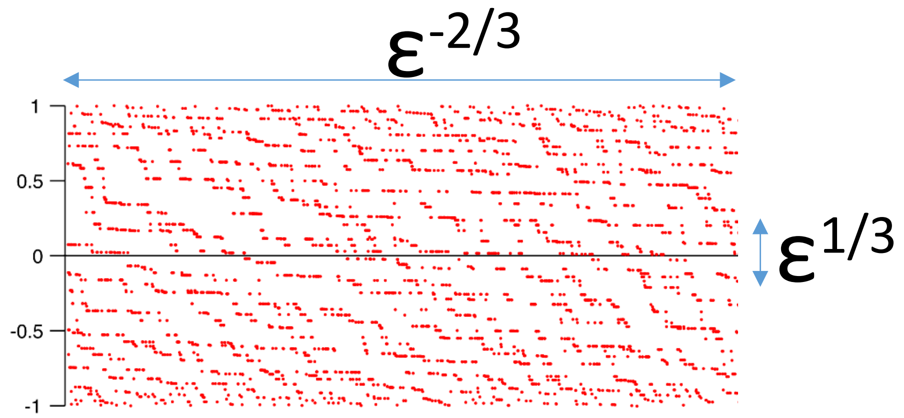
Cor: Under KPZ scaling/centering, the random dynamical system mapping height function initial data to height function evolution has a scaling limit:

$$\mathfrak{h}_0 \mapsto \mathfrak{h}(\mathfrak{h}_0; y, t) := \sup_{x \in \mathbb{R}} \Big(\mathfrak{h}_0(x) + \mathcal{L}(x, 0; y, t) \Big).$$

For a single initial data this Markov process is known as the 'KPZ fixed point' and its transition probabilities are known.

Scaling limits of coupled stationary measures

Cor: Under KPZ scaling and $\varepsilon^{1/3}$ scaling of density the ASEP speed process converges to the 'stationary horizon'.



E.g. Two coupled stationary height functions converge to

$$f_1(x), \Phi(f_1, f_2)(x)$$

where f_1, f_2 are independent two-sided Brownian motions of strictly ordered prescribed drifts and

$$\Phi(f,g)(y) = f(y) + \sup_{-\infty < x \leq y} \{g(x) - f(x)\} - \sup_{-\infty < x \leq 0} \{g(x) - f(x)\}.$$

Decoupling hypothesis in fluctuating hydrodynamics

Start colored ASEP with $\tilde{\eta}_0 \in \{0,1,2\}^{\mathbb{Z}}$ in the stationary measure with $1, 2$ density $\frac{1}{2}\beta\varepsilon^{1/3}, \frac{1}{2}$. Evolve and let

$$\eta_t^{(1)}(x) = \mathbb{1}_{\tilde{\eta}_t(x)\geq 1} \;, \qquad \eta_t^{(2)}(x) = \mathbb{1}_{\tilde{\eta}_t(x)=2} \;.$$

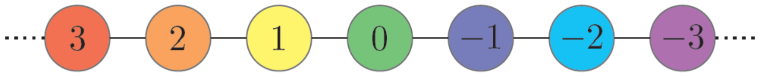
<u>Cor</u>: The two-point function (space-time covariance)

$$S_{k\ell}^{\beta,\varepsilon}(t,x) = \mathrm{Cov}(\eta_0^{(k)}(0), \eta_t^{(\ell)}(x))$$

Under KPZ scaling has a limit which, as $\beta\to\infty$, converges to zero except for $S_{kk}^{\beta,\varepsilon}$ with $k = 2$ which yields $\frac{1}{32} g''_{\mathrm{BR}}(x)$.

Summary

- ✓ Defined basic coupling and colored ASEP



- ✓ Construct ASEP speed process & coupled stationary measures
- ✓ Thm: ASEP sheets → Airy sheet
- ✓ Cor: ASEP steps under basic coupling → directed landscape
- ✓ Cor: Random dynamical system → variational problem in DL
- ✓ Cor: ASEP speed process → stationary horizon
- ✓ Cor: Verified decoupling hypothesis for colored ASEP

- Extend to other models solvable via Yang-Baxter (e.g. RWRE)
- Probe 'fine' details, mixing times, large deviations…