Back to the rainbow

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Joint work with
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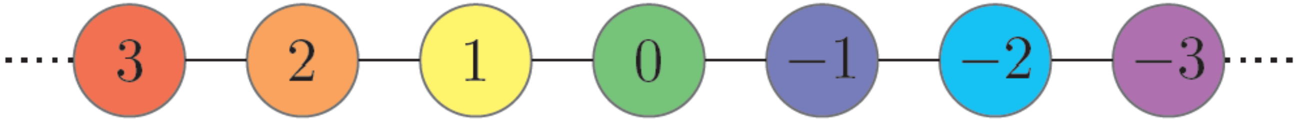
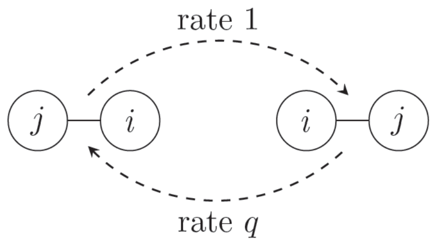
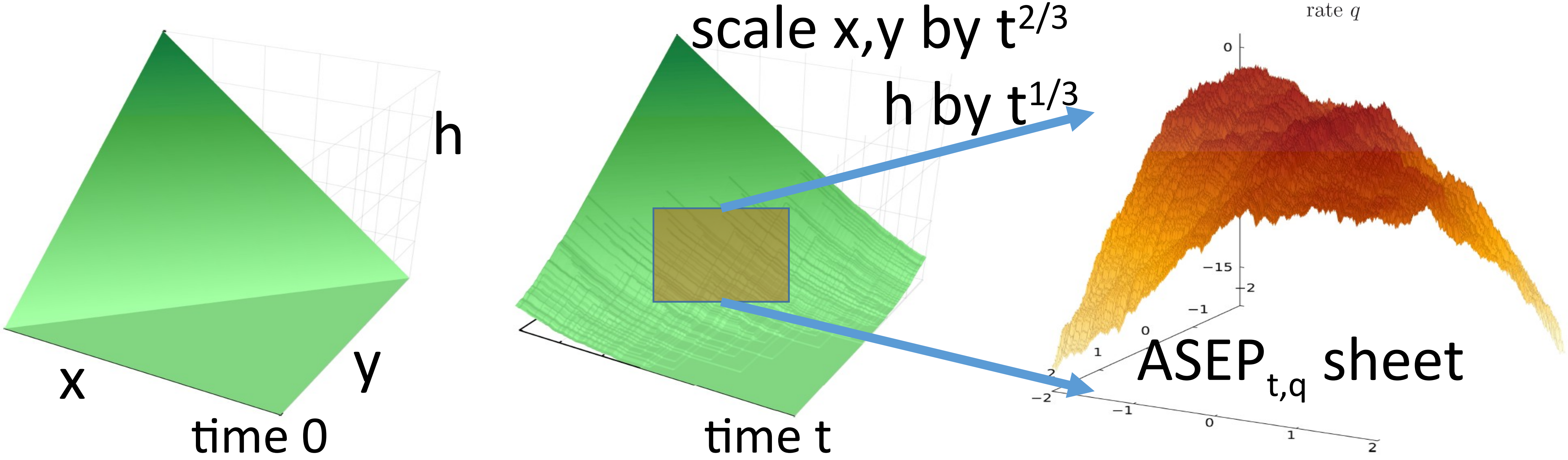
Mesoscopic scaling limit of colored ASEP

h(x;y,t) := # particles of color ≥ x to right of y at time t







<u>Thm</u>: The $\text{ASEP}_{t,q}$ sheet converges to the **Airy sheet** $\mathcal{S}(x;y)$.

Key ideas behind the theorem

- ✓ Work with colored stochastic six vertex (S6V) model
- ✓ Lecture 3 focuses on the uncolored model
  - ✓ Yang-Baxter embeds S6V height function in q-Boson model
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- Lecture 4 returns to the colored model
  - Yang-Baxter story extends to colored S6V, q-Boson models
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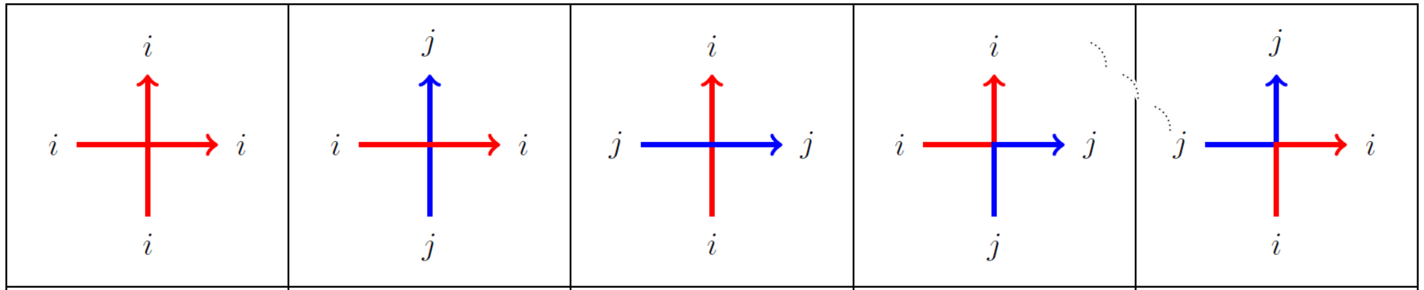
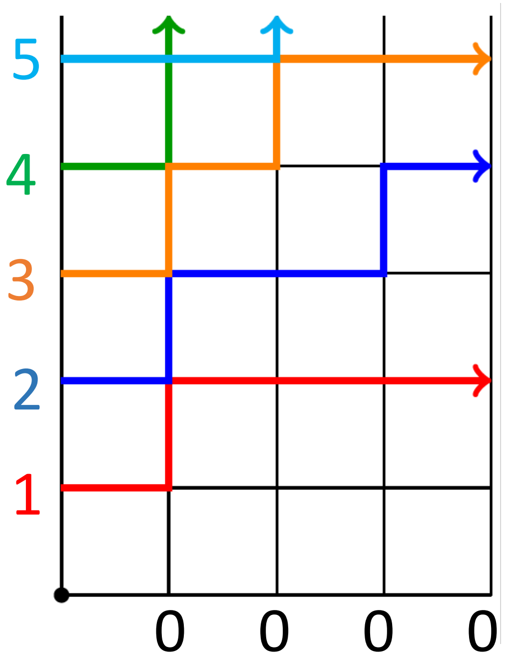
Colored stochastic six vertex (S6V) model

[Kulish-Reshetikhin-Sklyanin ’81] [Bazhanov ‘85], [Jimbo ‘86], [Kuniba-Mangazeev-Maruyama-Okado ‘16], [Borodin-Wheeler ‘18]



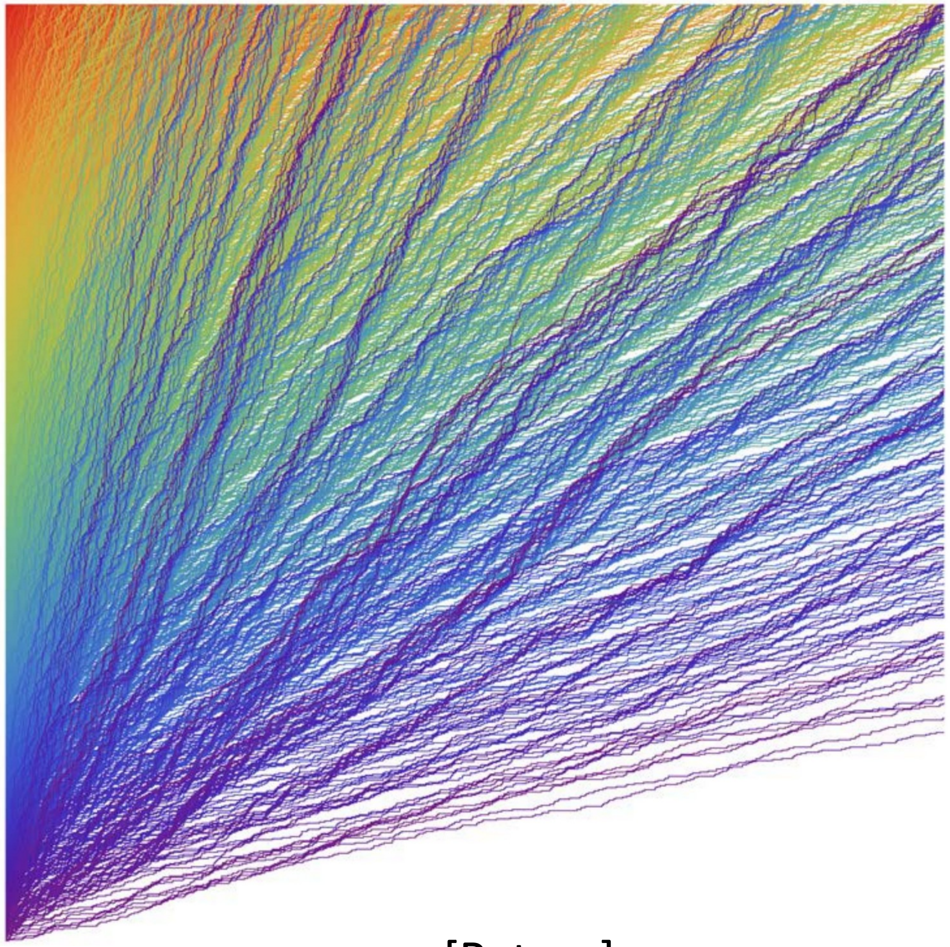
1	$\frac{q(1-z)}{1-qz}$	$\frac{1-z}{1-qz}$	$\frac{1-q}{1-qz}$	$\frac{z(1-q)}{1-qz}$



‘Quantum’ parameter q

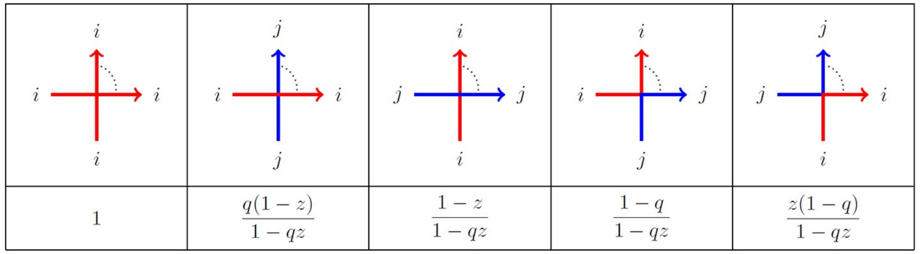
‘Spectral’ parameter z

ASEP is recovered around the diagonal as z → 1

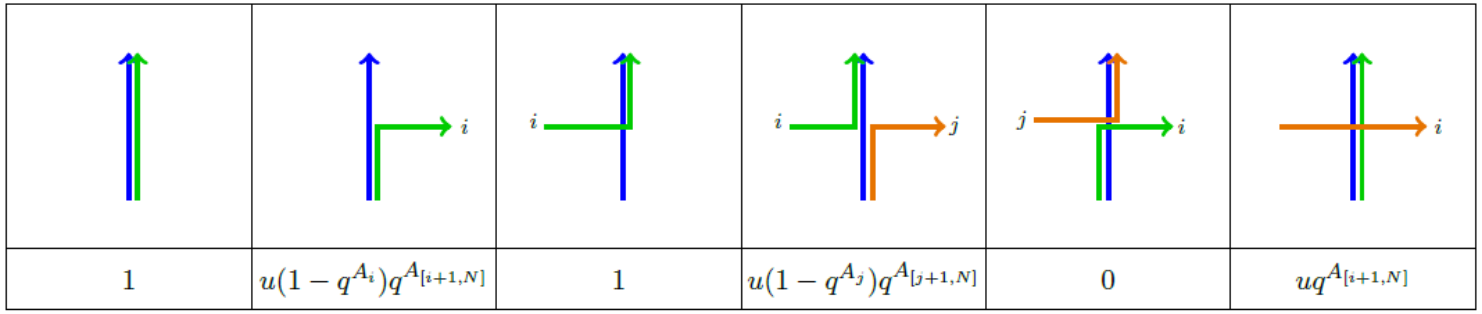


[Petrov]

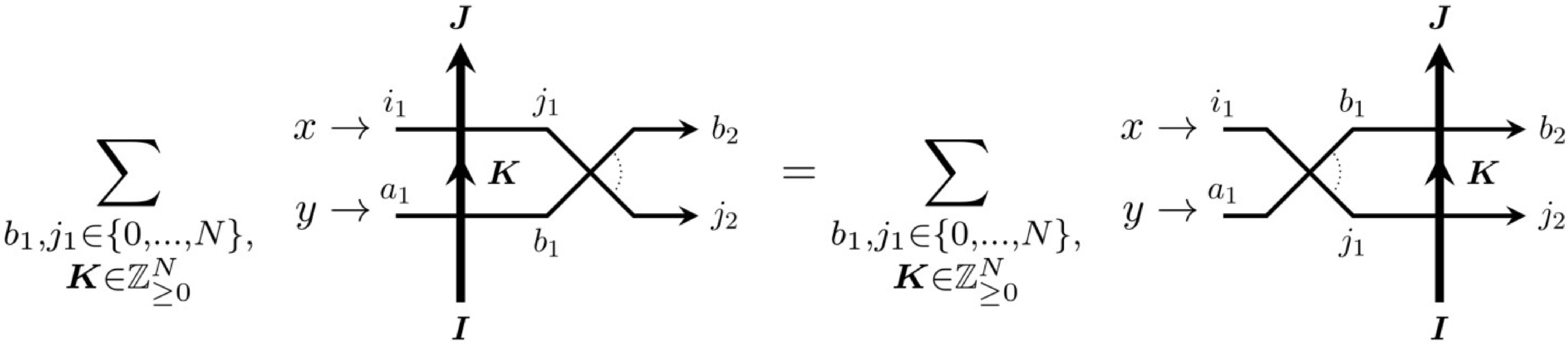
Yang-Baxter for colored S6V and q-Boson weights



S6V weights

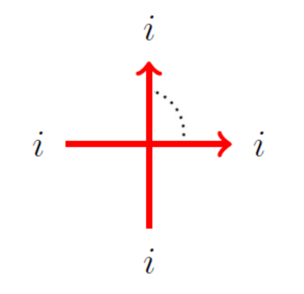
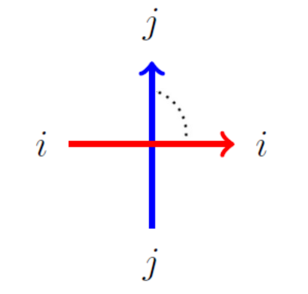
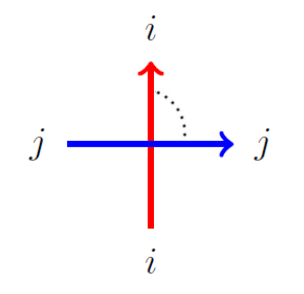
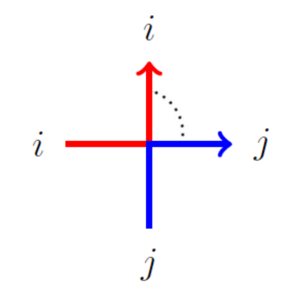
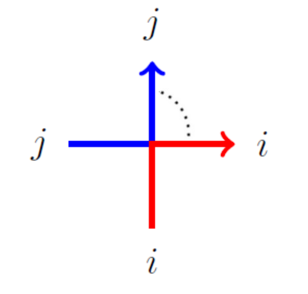


q-Boson weights

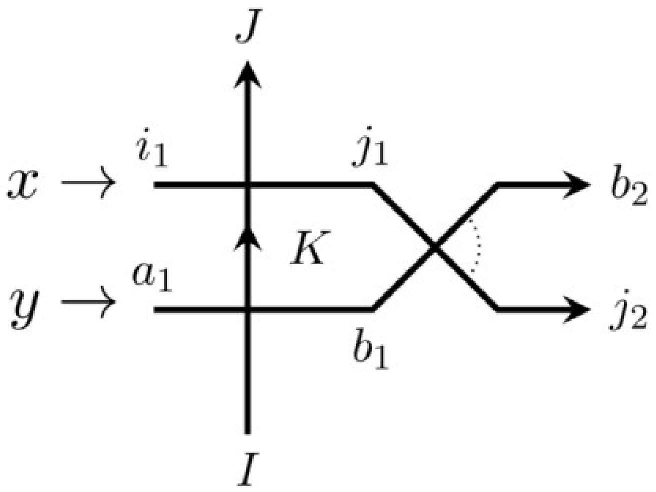


Yang-Baxter equation

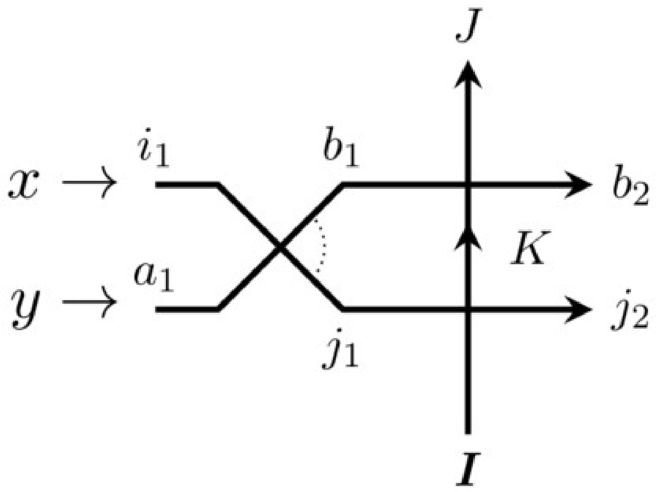
Aside: Yang-Baxter RRR relations for colored S6V

				
1	$\frac{q(1-z)}{1-qz}$	$\frac{1-z}{1-qz}$	$\frac{1-q}{1-qz}$	$\frac{z(1-q)}{1-qz}$

$$\sum_{b_1, j_1, K \in \{0,\dots,N\}}$$



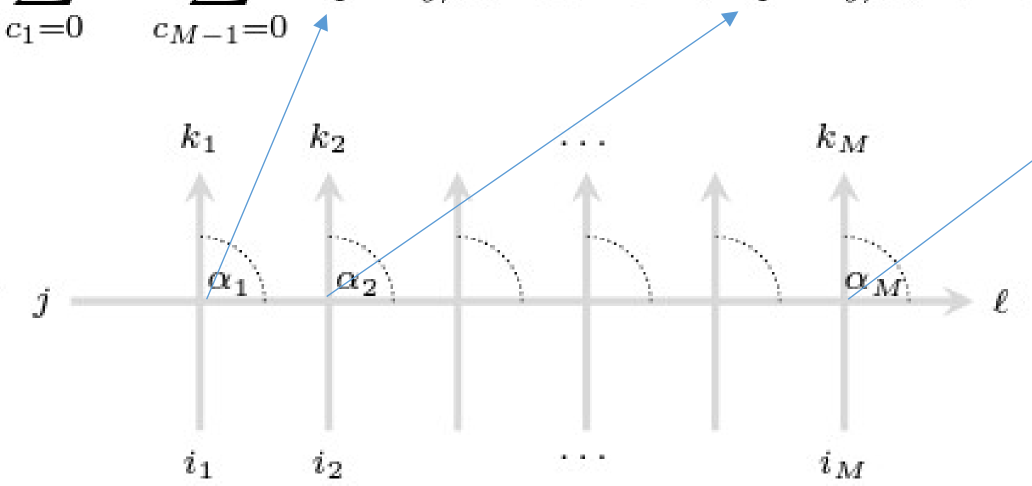
$$=\sum_{b_1, j_1, K \in \{0,\dots,N\}}$$



Aside: Fusion

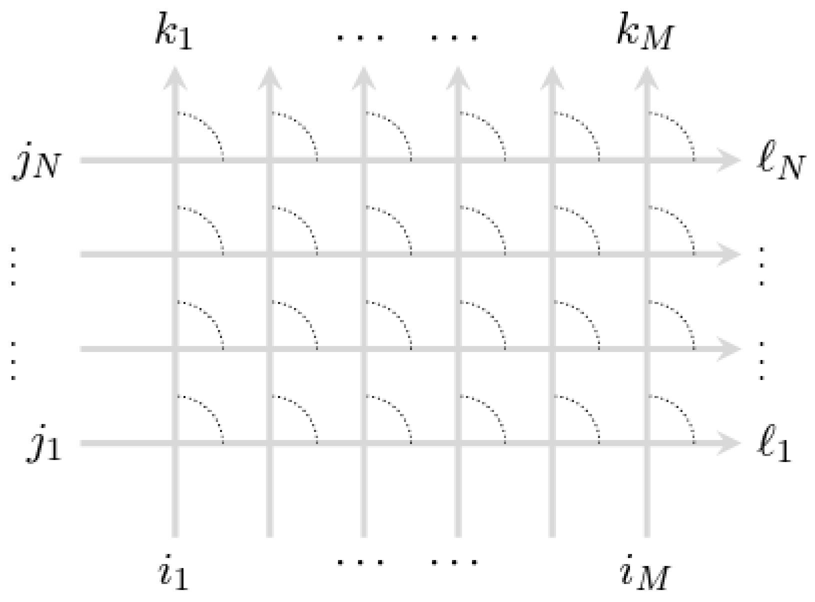
[Kulish-Reshetikhin-Sklyanin ’81]

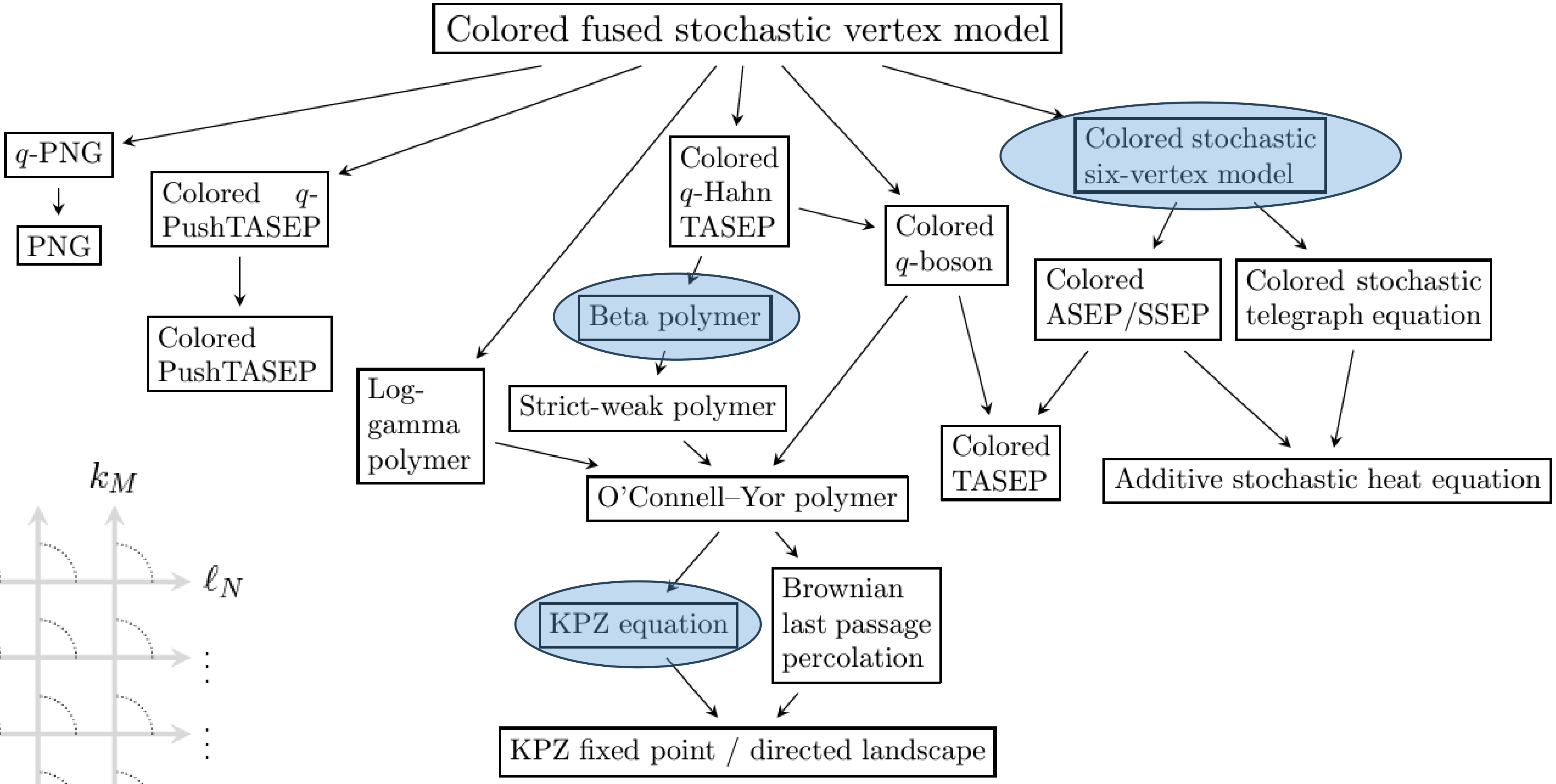
$$R_{y/x}\Big((i_1,\dots,i_M),j;(k_1,\dots,k_M),\ell\Big) := \sum_{c_1=0}^{n}\cdots\sum_{c_{M-1}=0}^{n} R_{q^{M-1}y/x}(i_1,j;k_1,c_1)R_{q^{M-2}y/x}(i_2,c_1;k_2,c_2)\dots R_{y/x}(i_M,c_{M-1};k_M,\ell)$$

$=$ 

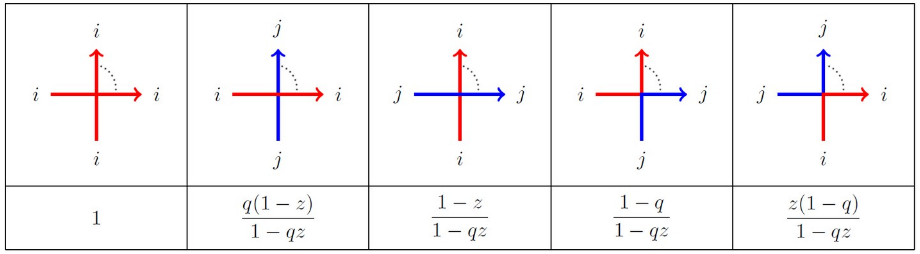
$$\mathcal{L}^{(M)}_{y/x}(\boldsymbol{I},j;\boldsymbol{K},\ell) := \frac{1}{Z_q(M;\boldsymbol{I})}\sum_{\substack{\mathcal{C}(i_1,\dots,i_M)=\boldsymbol{I}\\ \mathcal{C}(k_1,\dots,k_M)=\boldsymbol{K}}} q^{\mathrm{inv}(i_1,\dots,i_M)} R_{y/x}\Big((i_1,\dots,i_M),j;(k_1,\dots,k_M),\ell\Big)$$

Aside: Fusion

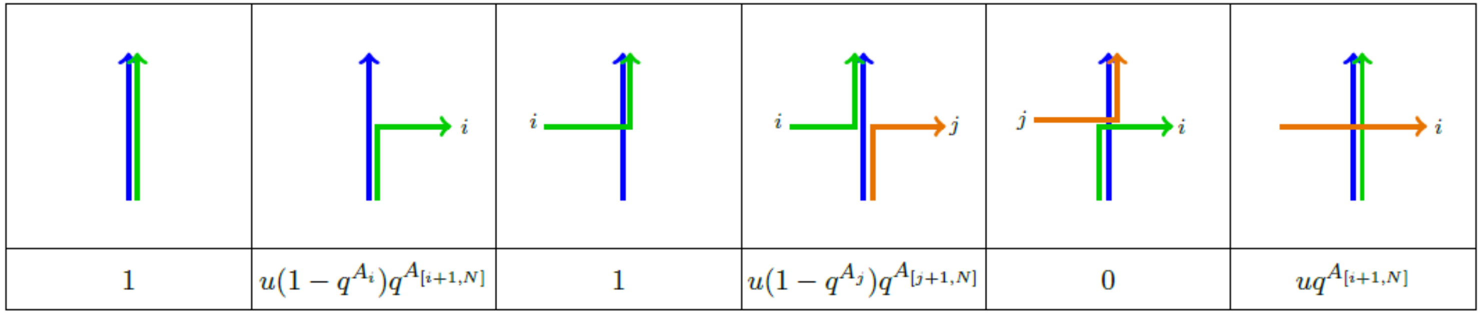




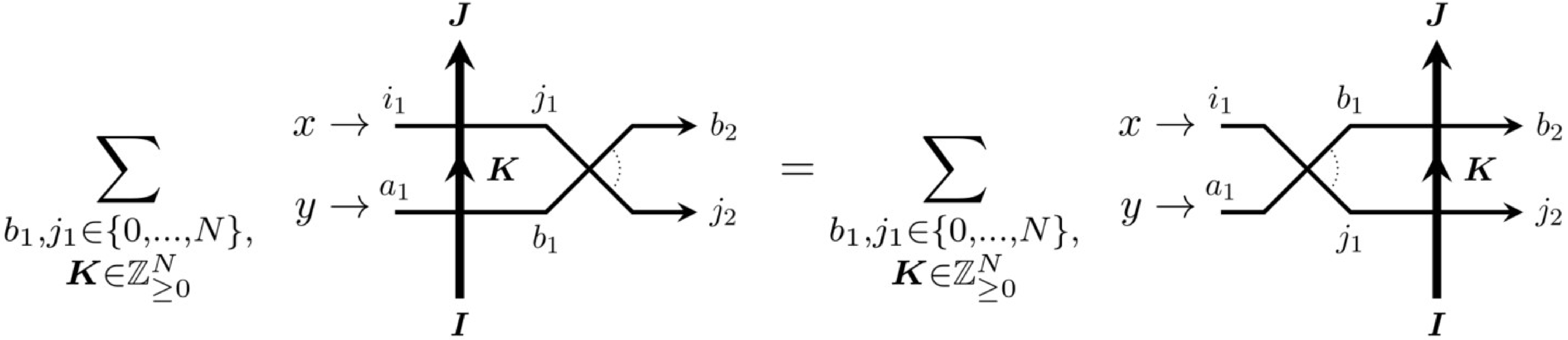
Yang-Baxter for colored S6V and q-Boson weights



S6V weights

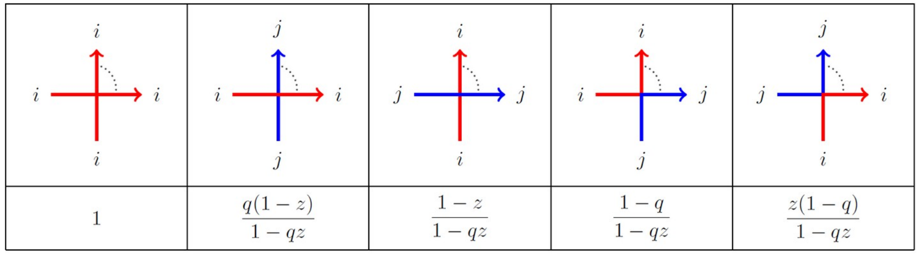


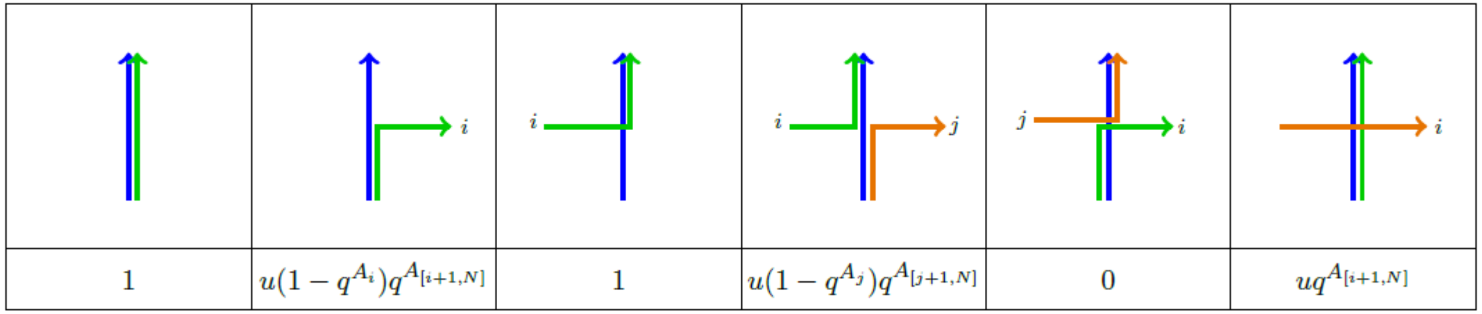
q-Boson weights



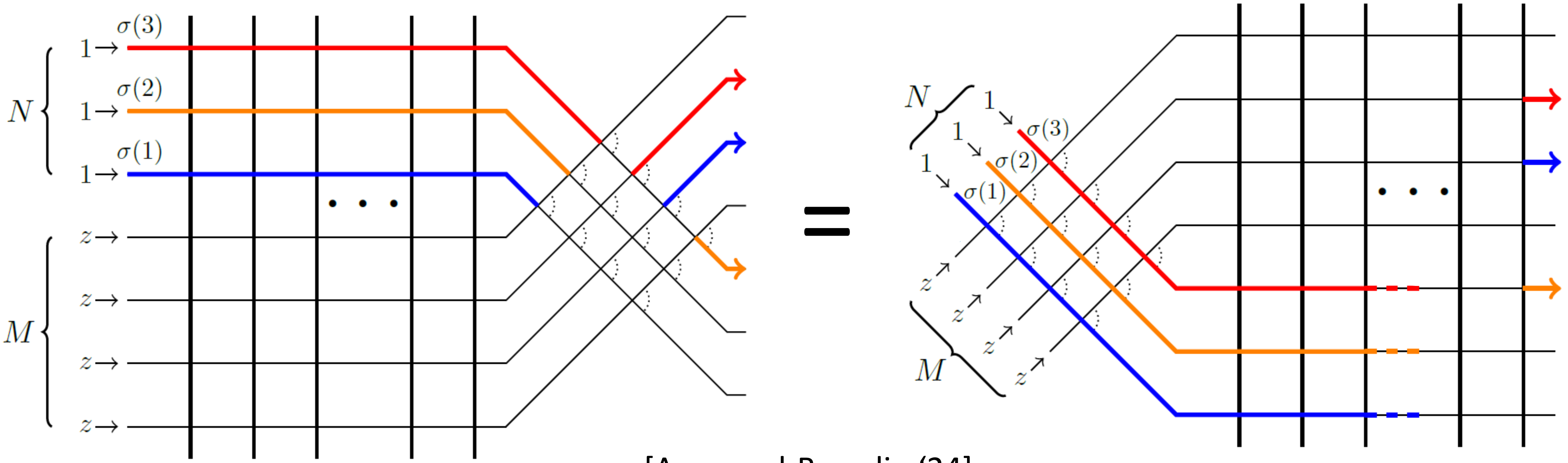
Yang-Baxter equation

Relating the colored q-Boson and S6V models



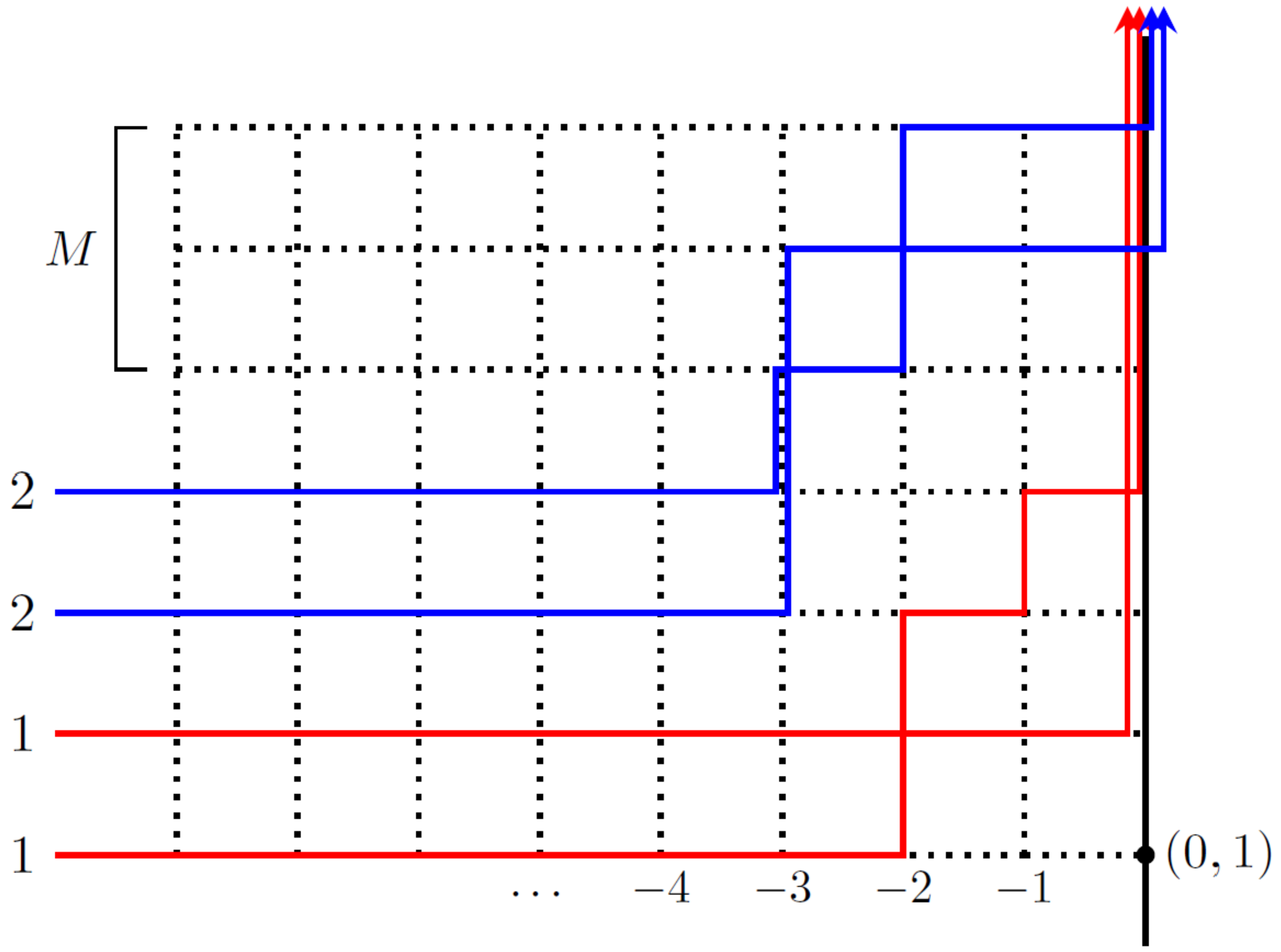


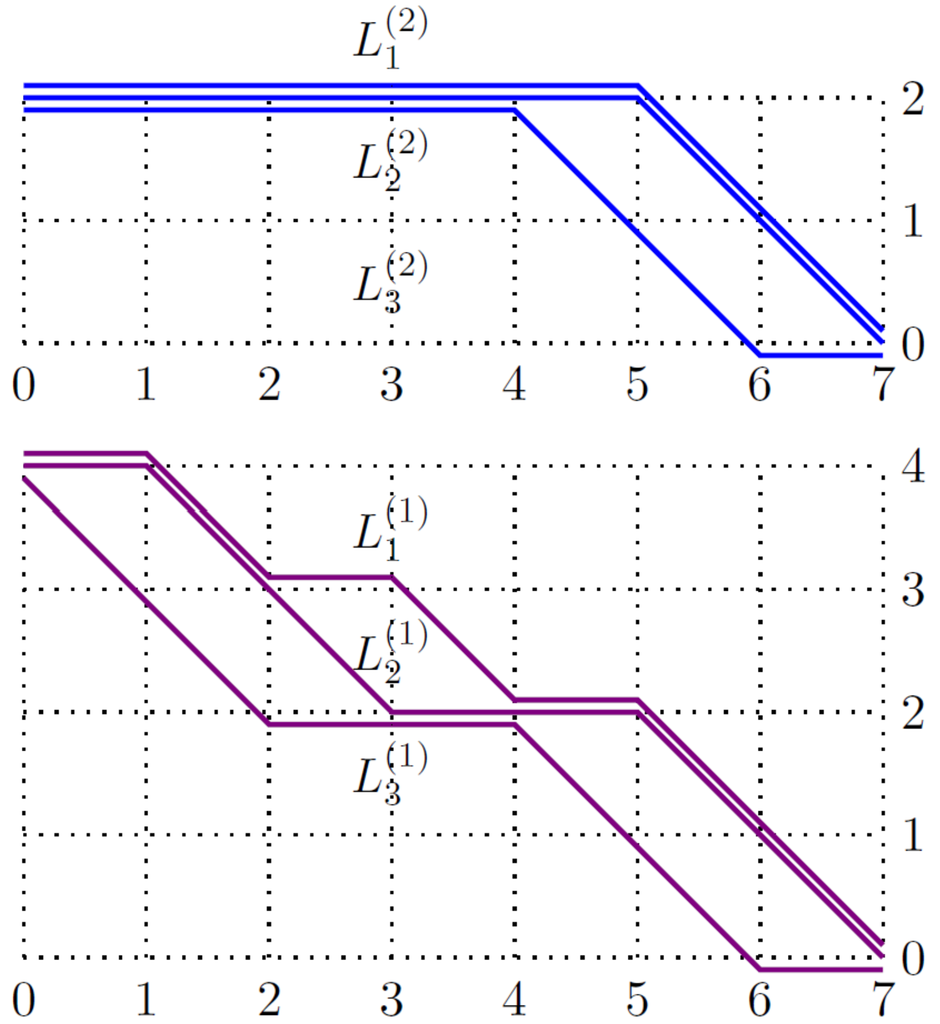
trivial weight color S6V model trivial weight colored q-Boson model



[Aggarwal-Borodin ‘24]

Colored q-Boson model as a line ensemble





Key ideas behind the theorem

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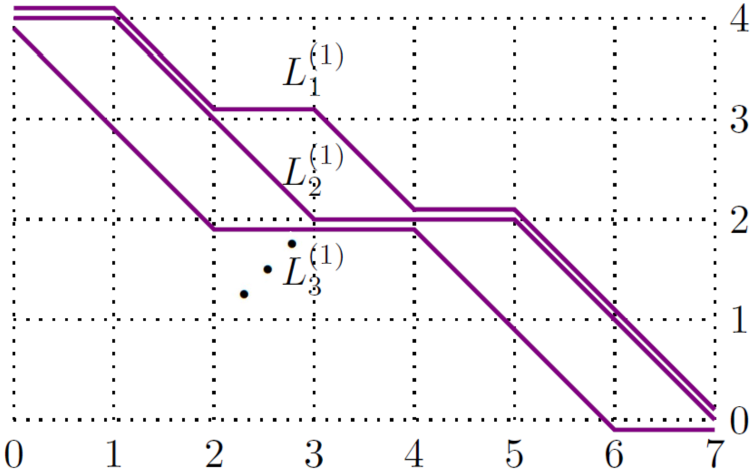
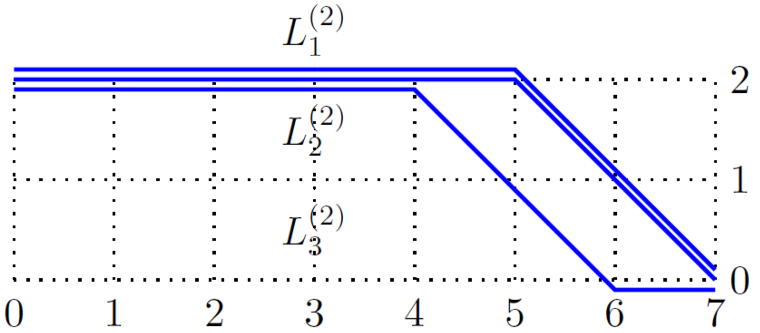
Intercolor q-Boson Gibbs property

Given $\boldsymbol{L}^{(1)}$ 

, $\boldsymbol{L}^{(2)}$ 

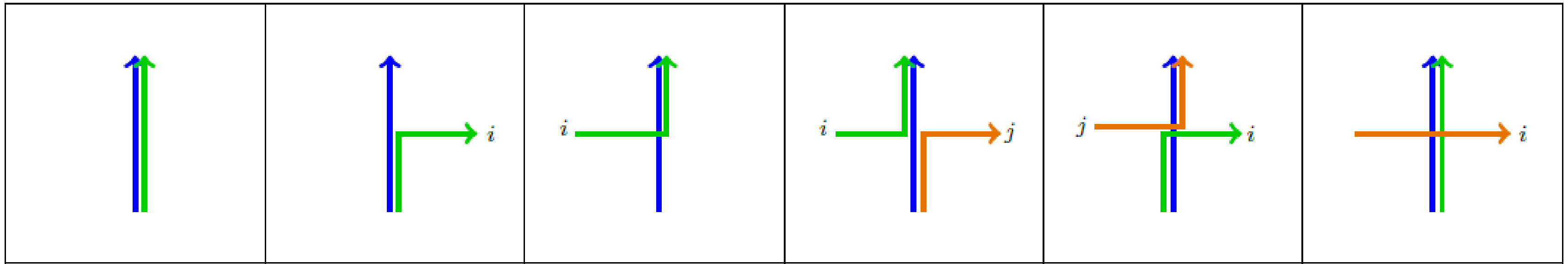
satisfies a variational formula at q=0: $L_k^{(2)} = \mathsf{PT}\left(L_k^{(1)}, L_{k+1}^{(2)}\right)$

$$\mathsf{PT}(f,g)(y) := f(y) + \max_{0\le y'\le y}\left(g(y') - f(y')\right)$$

and an approximate variational formula at q>0

$$\mathbb{P}\left(\max_{y\in[\![0,N+M]\!]}\left(L_k^{(2)}(y) - \mathsf{PT}(L_k^{(1)}, L_{k+1}^{(2)})(y)\right) \ge m\right) \le q^{cm^2}$$

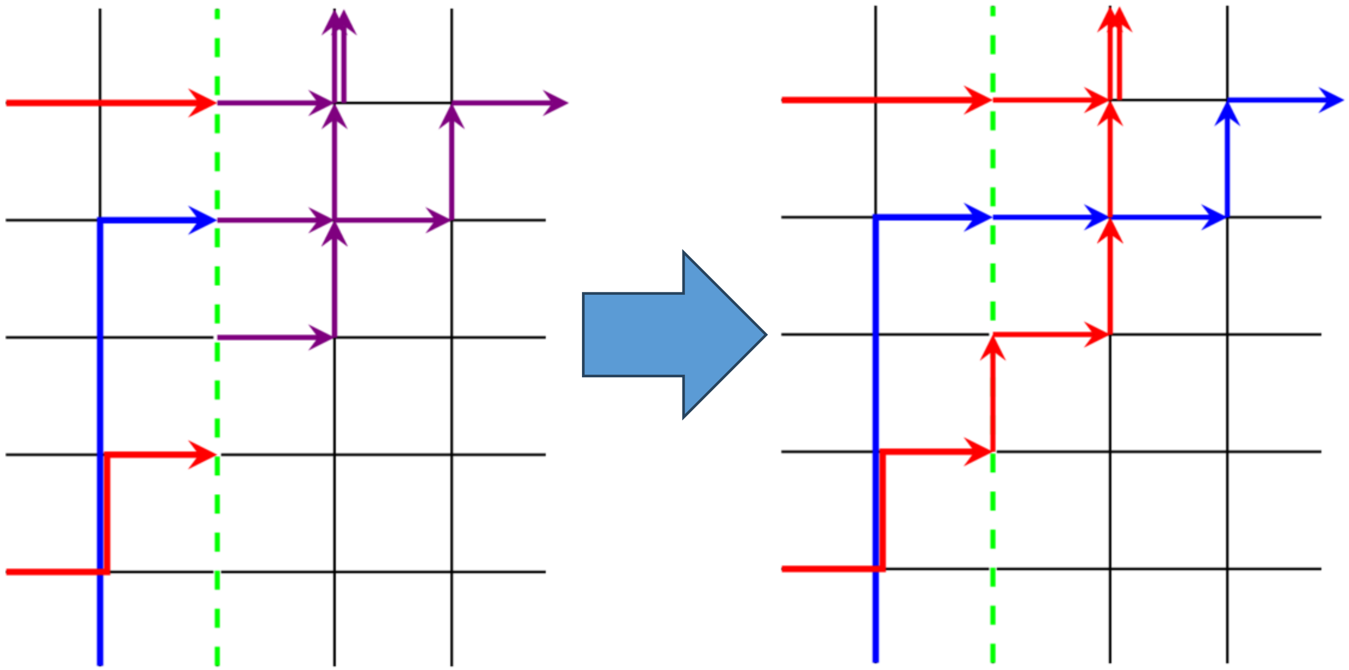
Origin of intercolor q-Boson Gibbs property when q=0



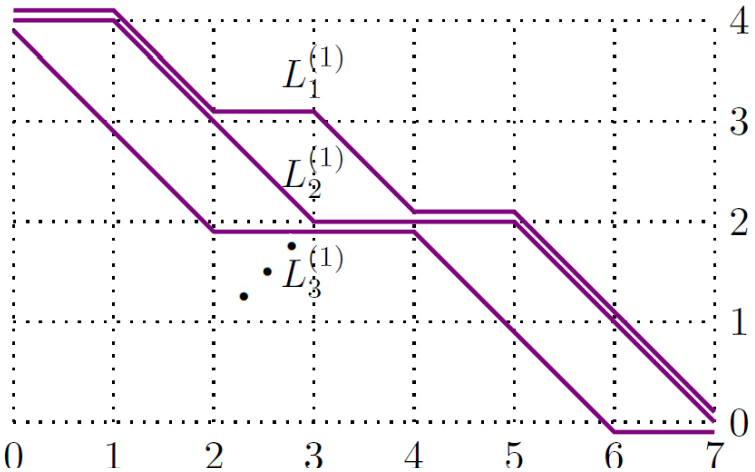
1	$u(1-q^{A_i})q^{A_{[i+1,N]}}$	1	$u(1-q^{A_j})q^{A_{[j+1,N]}}$	0	$uq^{A_{[i+1,N]}}$

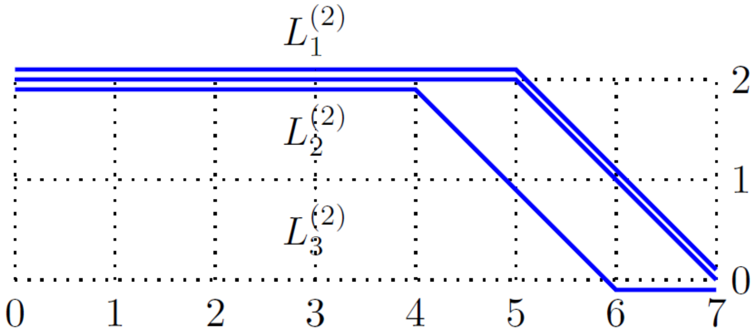
When q=0, the highest arrow incoming to each vertex must exit right.

When q>0 there is a probabilistic penalty.



Iterating intercolor Gibbs property yields (q=0)

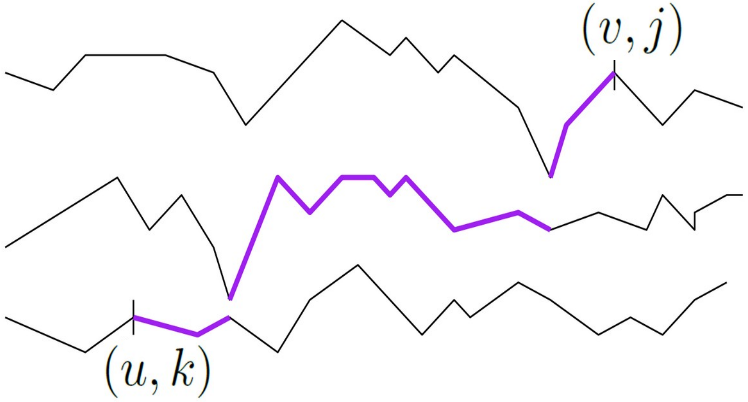
Given $\boldsymbol{L}^{(1)}$ 

, $\boldsymbol{L}^{(2)}$ 

satisfies $$L_j^{(2)}(y) = \sup_{z \le y} \left(L_{k+1}^{(2)}(z) + \boldsymbol{L}^{(1)}[(z,k) \to (y,j)] \right)$$

where

$$\boldsymbol{f}[(u,k) \to (v,j)] = \sup_{t_k < \ldots < t_{j-1}} \sum_{i=j}^{k} \left(f_i(t_{i-1}) - f_i(t_i) \right)$$



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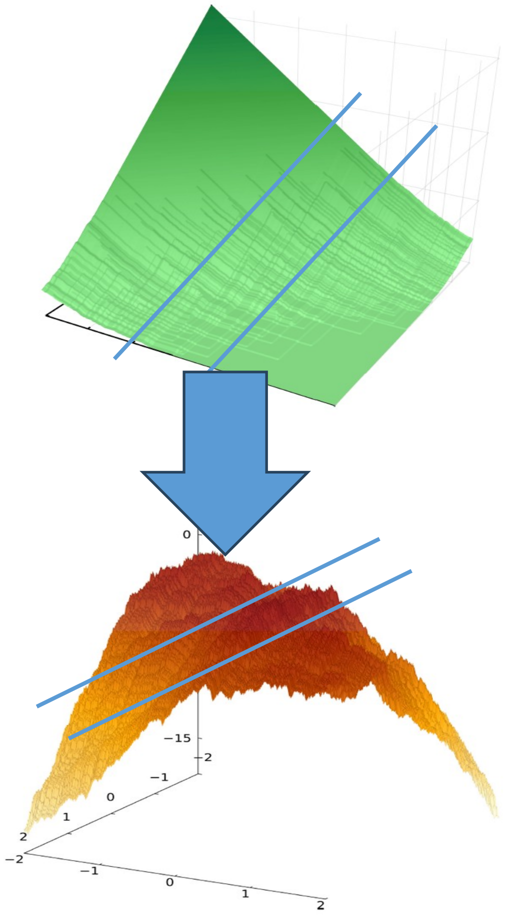
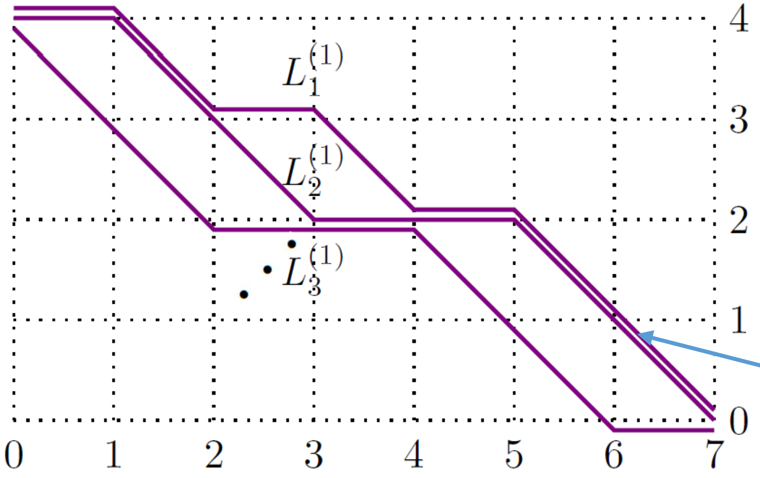
Putting it all together

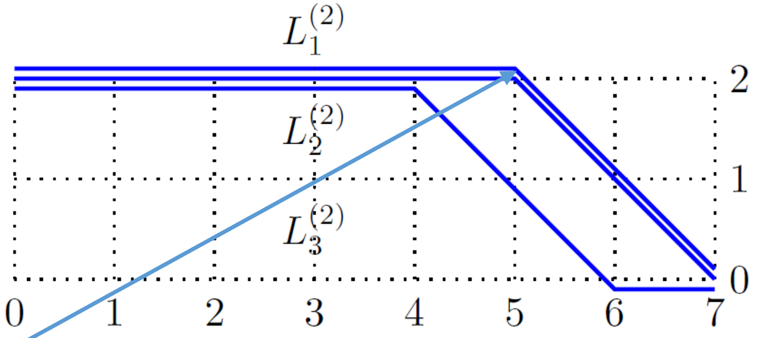
Goal





variational

formula

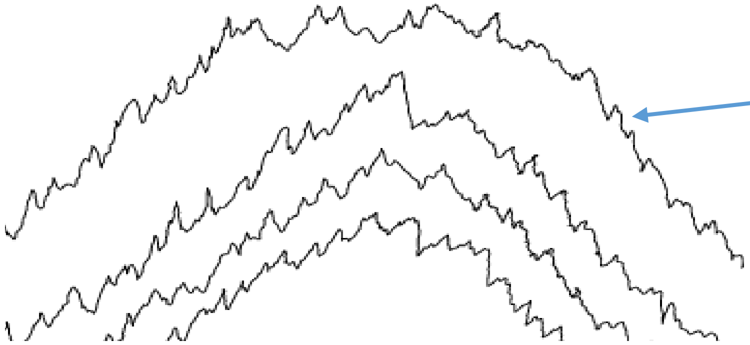


S6V sheet

convergence (Lecture 3)

convergence (jointly)

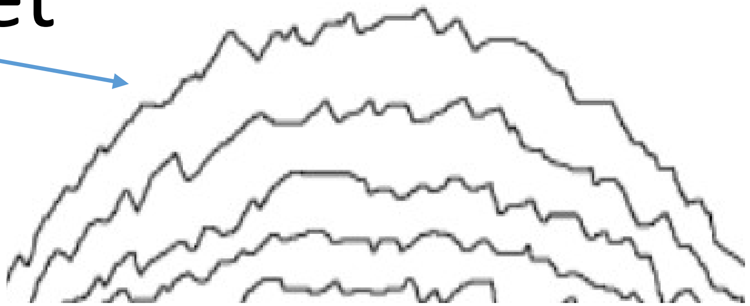
Airy sheet



Airy line ensemble

variational

formula



Take home messages

- A full KPZ scaling limit may be overkill from a physics perspective, yet it implies the scaling limits for all sorts of observables of physical interest as corollaries.
- There is enhanced mathematical structure present in analysis of the full object compared to observables.
- We embed the full object into a larger object using the Yang-Baxter equation and extract the scaling limit of this.