Colorblind analysis

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Joint work with
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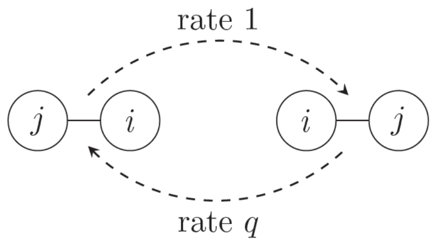
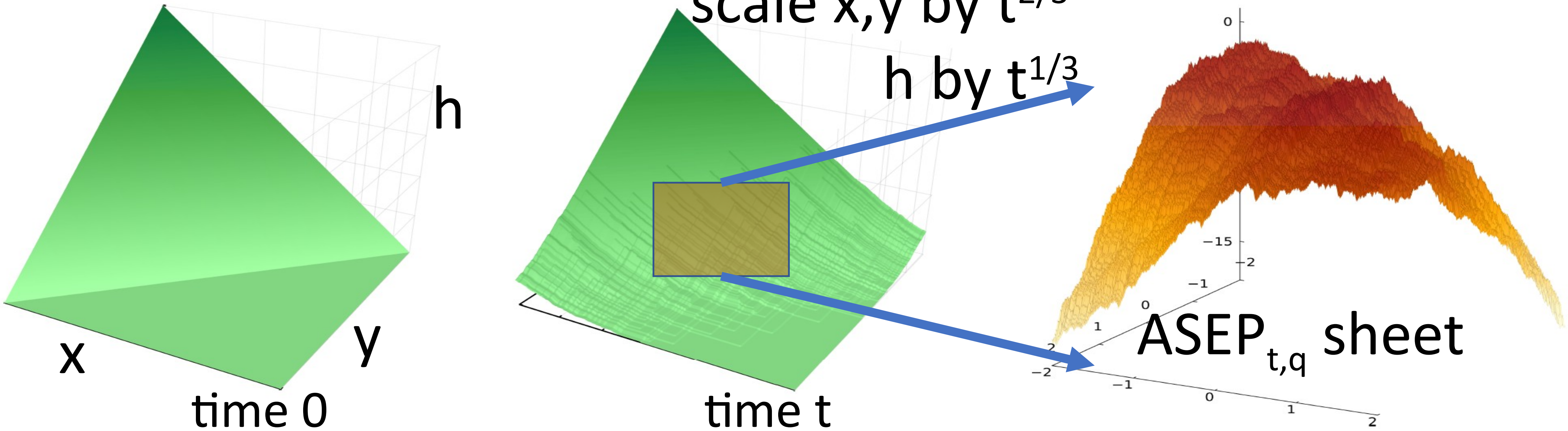
Mesoscopic scaling limit of colored ASEP

h(x;y,t) := # particles of color ≥ x to right of y at time t





<u>Thm</u>: The ASEP$_{t,q}$ sheet converges to the **Airy sheet** $\mathcal{S}(x;y)$.

Take home messages

- A full KPZ scaling limit may be overkill from a physics perspective, yet it implies the scaling limits for all sorts of observables of physical interest as corollaries.
- There is enhanced mathematical structure present in analysis of the full object compared to observables.
- We embed the full object into a larger object using the Yang-Baxter equation and extract the scaling limit of this.

Key ideas behind the theorem

- Work with colored stochastic six vertex (S6V) model
- Lecture 3 focuses on the uncolored model
  - Yang-Baxter embeds S6V height function in q-Boson model
  - Gibbs property, one-point GUE Tracy-Widom asymptotics and strong characterization yields Airy line ensemble limit
- Lecture 4 returns to the colored model
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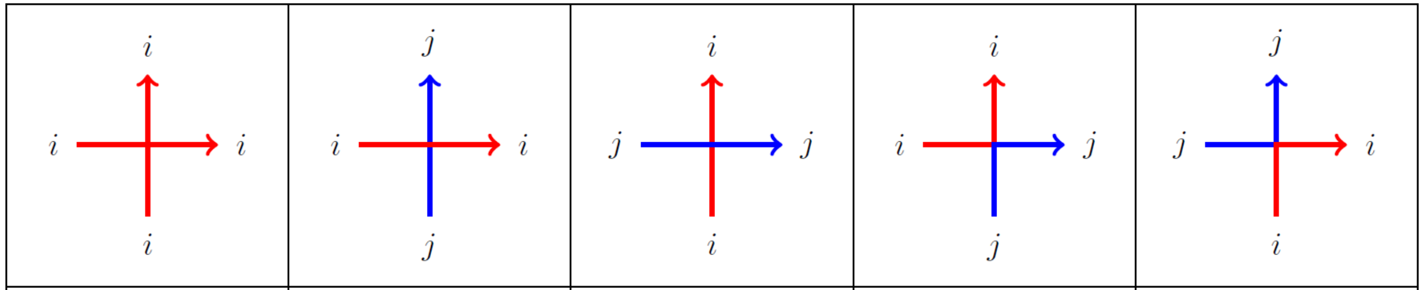
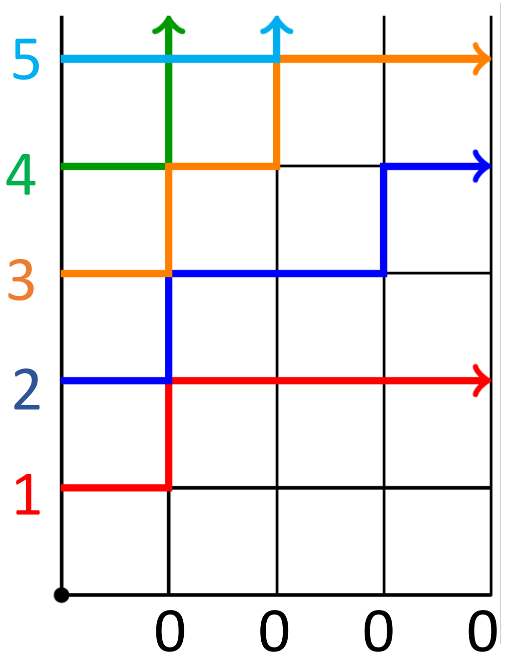
Colored stochastic six vertex (S6V) model

[Kulish-Reshetikhin-Sklyanin '81] [Bazhanov '85], [Jimbo '86], [Kuniba-Mangazeev-Maruyama-Okado '16], [Borodin-Wheeler '18]



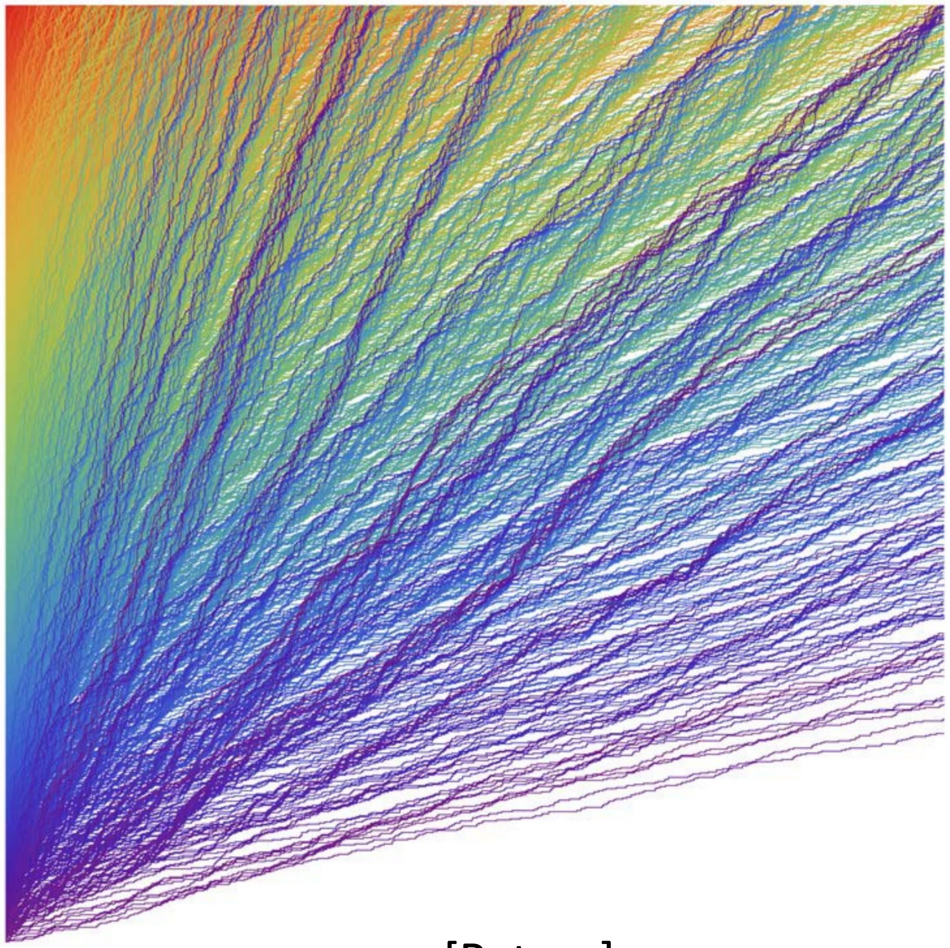
1	$\frac{q(1-z)}{1-qz}$	$\frac{1-z}{1-qz}$	$\frac{1-q}{1-qz}$	$\frac{z(1-q)}{1-qz}$



'Quantum' parameter q

'Spectral' parameter z

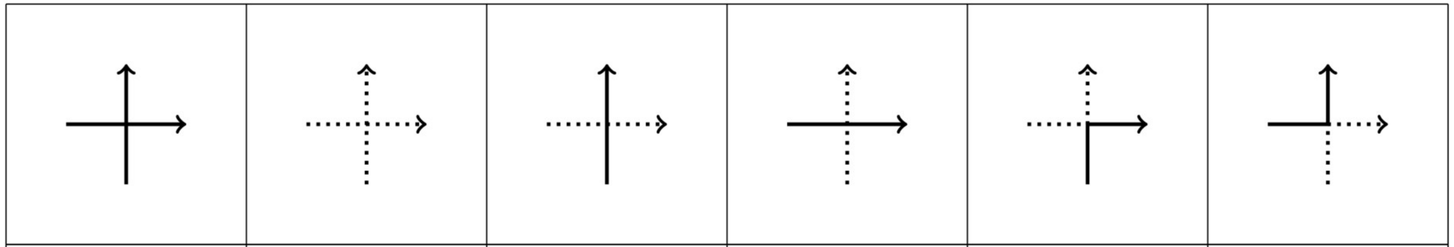
ASEP is recovered around the diagonal as z → 1



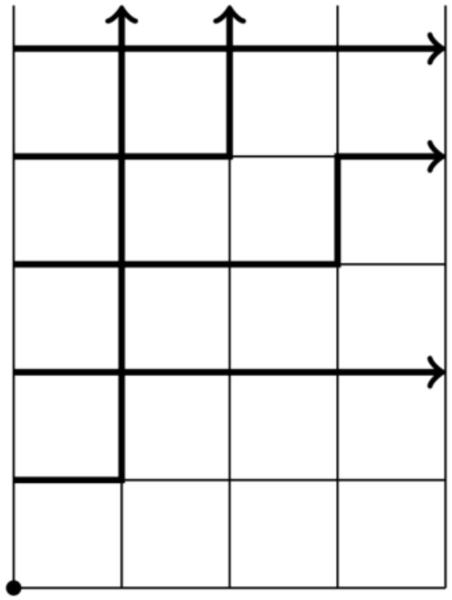
[Petrov]

Uncolored stochastic six vertex (S6V) model

[Pauling ‘35], [Lieb ‘67], [Gwa-Spohn ’93], [Bukman-Shore ’94], [Borodin-C-Gorin ‘14]



1	1	$\frac{q(1-z)}{1-qz}$	$\frac{1-z}{1-qz}$	$\frac{1-q}{1-qz}$	$\frac{z(1-q)}{1-qz}$



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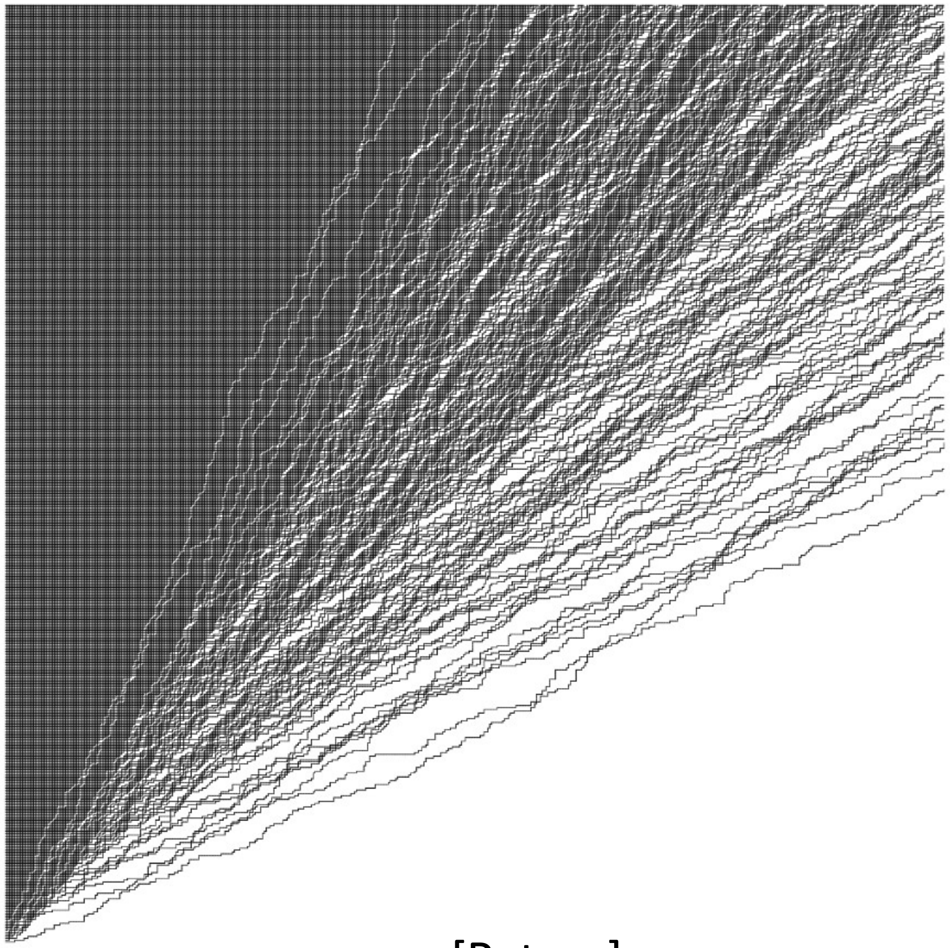

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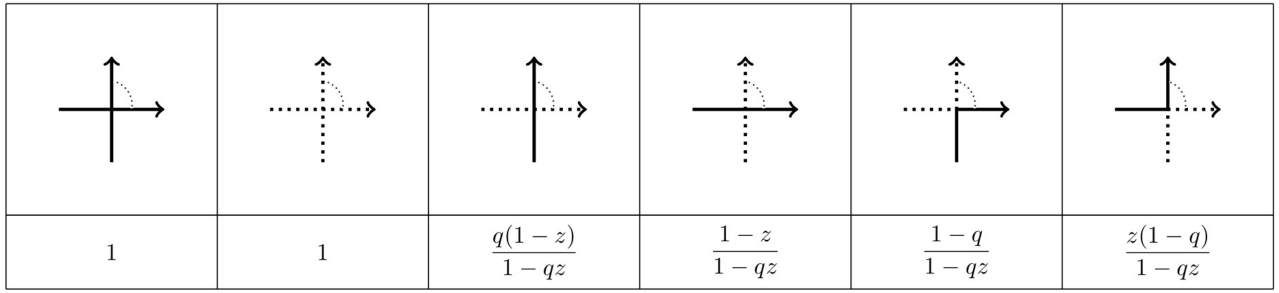
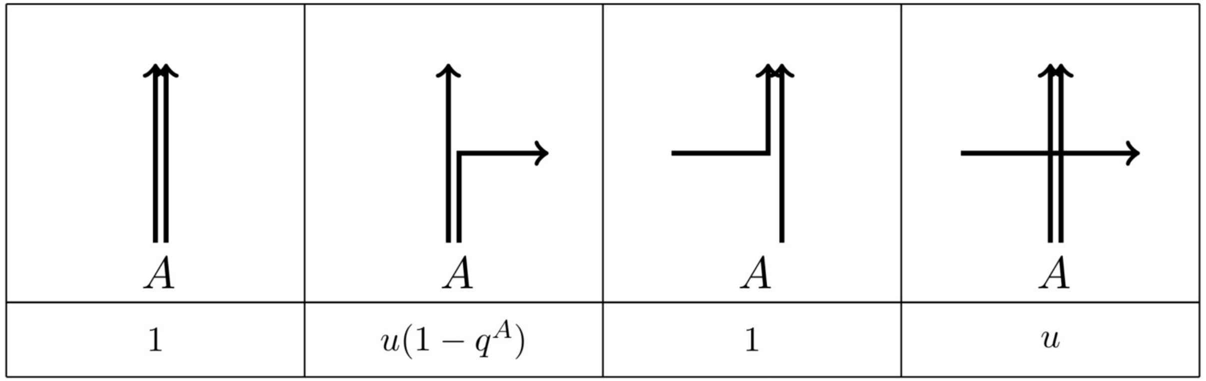
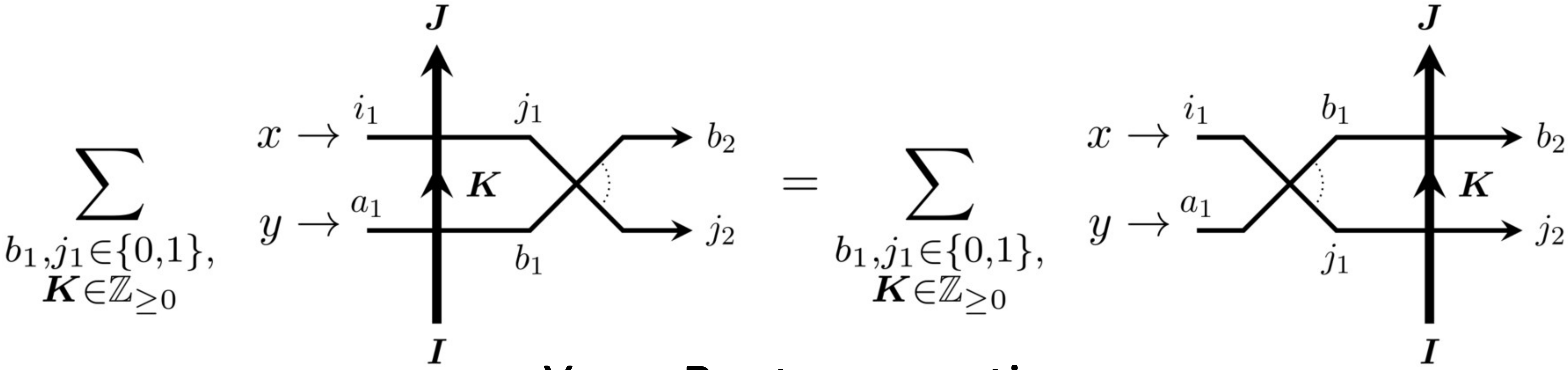
Yang-Baxter for uncolored S6V and q-Boson weights



S6V weights



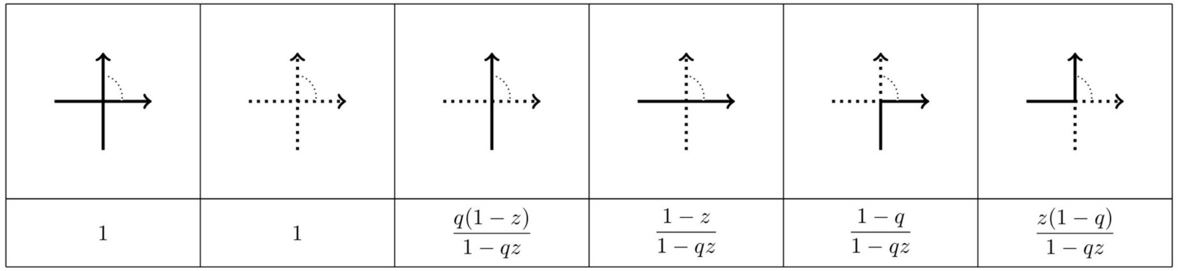
q-Boson weights

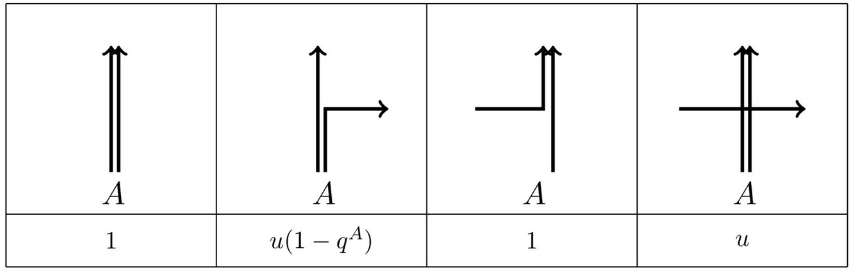


Yang-Baxter equation

Relating the uncolored q-Boson and S6V models

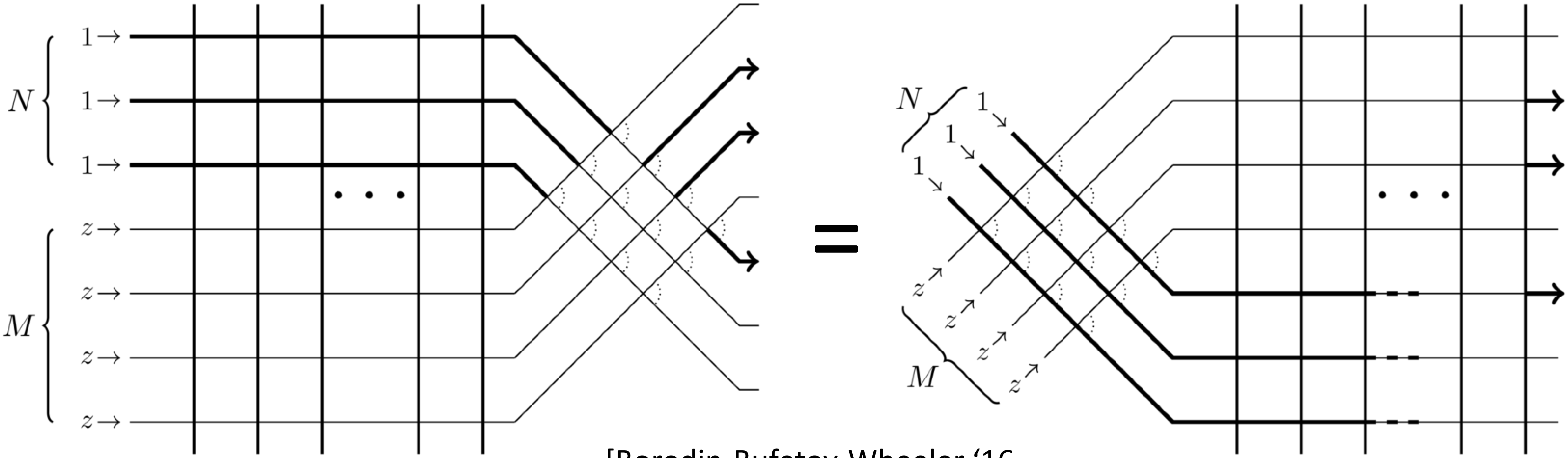
S6V weights





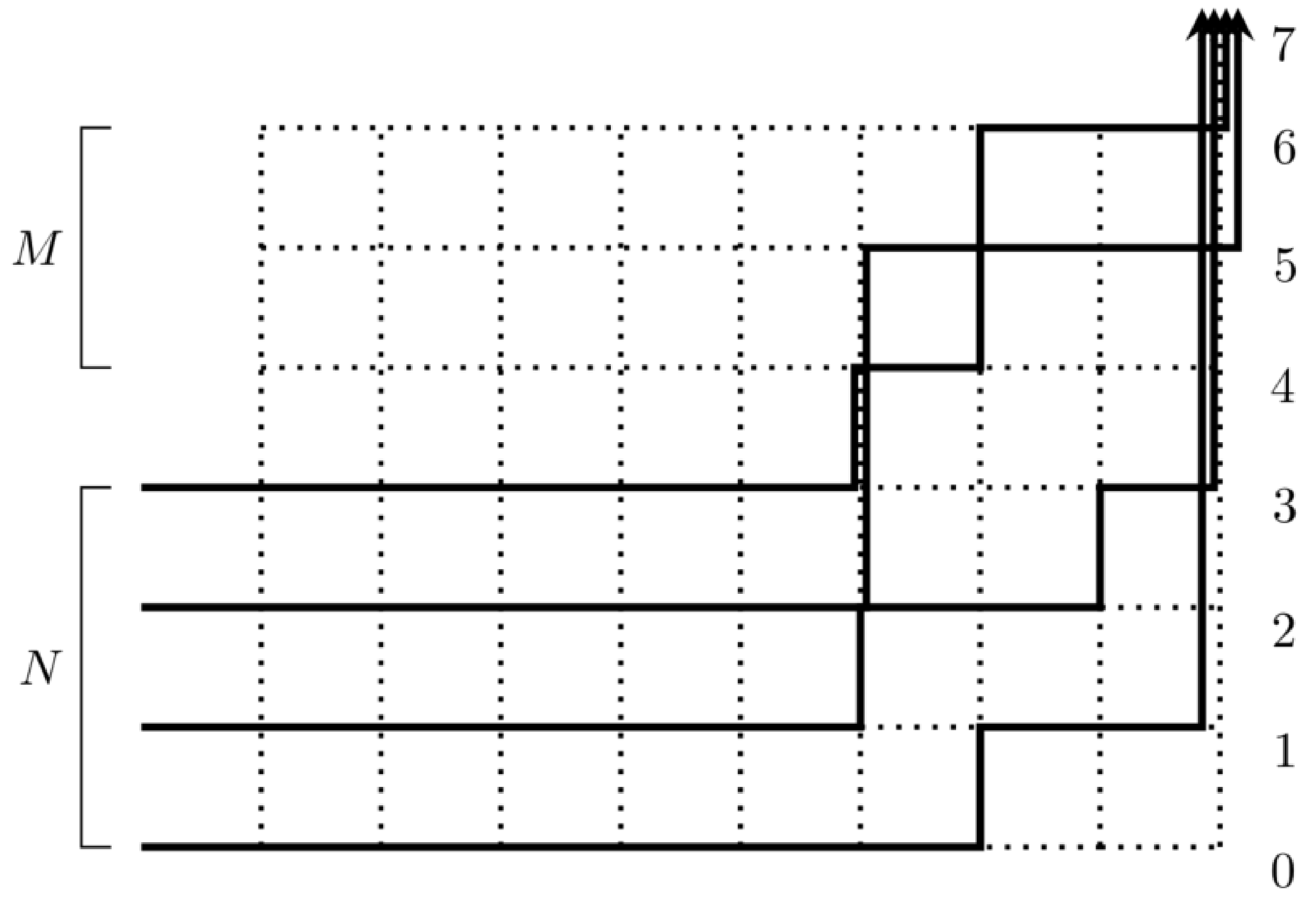
q-Boson weights

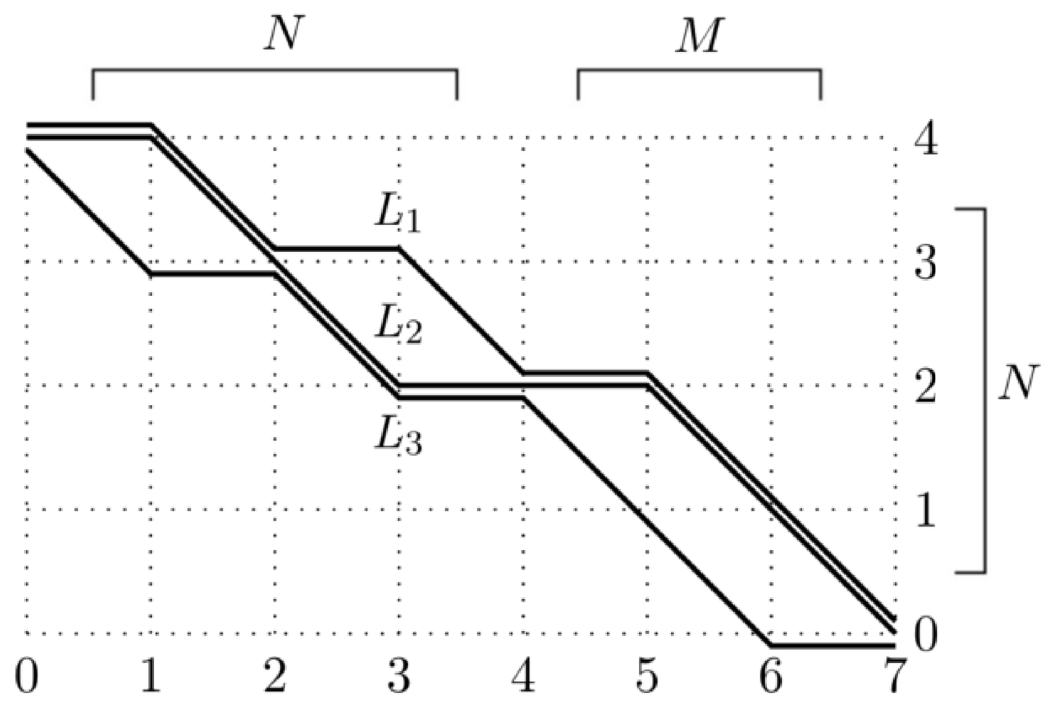
trivial weight S6V model trivial weight q-Boson model



[Borodin-Bufetov-Wheeler ‘16

Uncolored q-Boson model as a line ensemble





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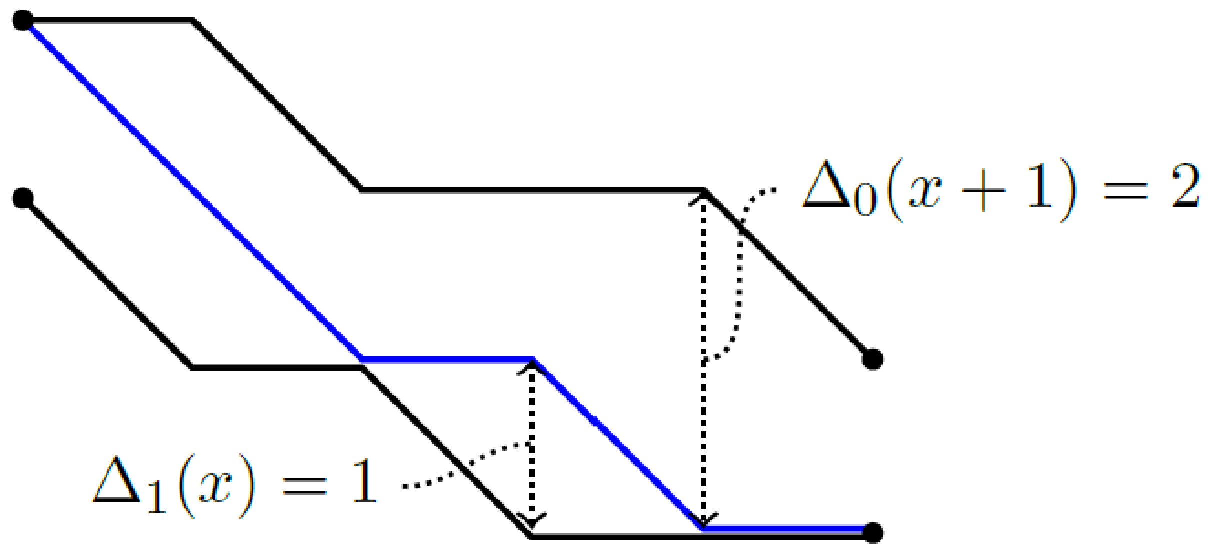
Uncolored q-Boson Gibbs property

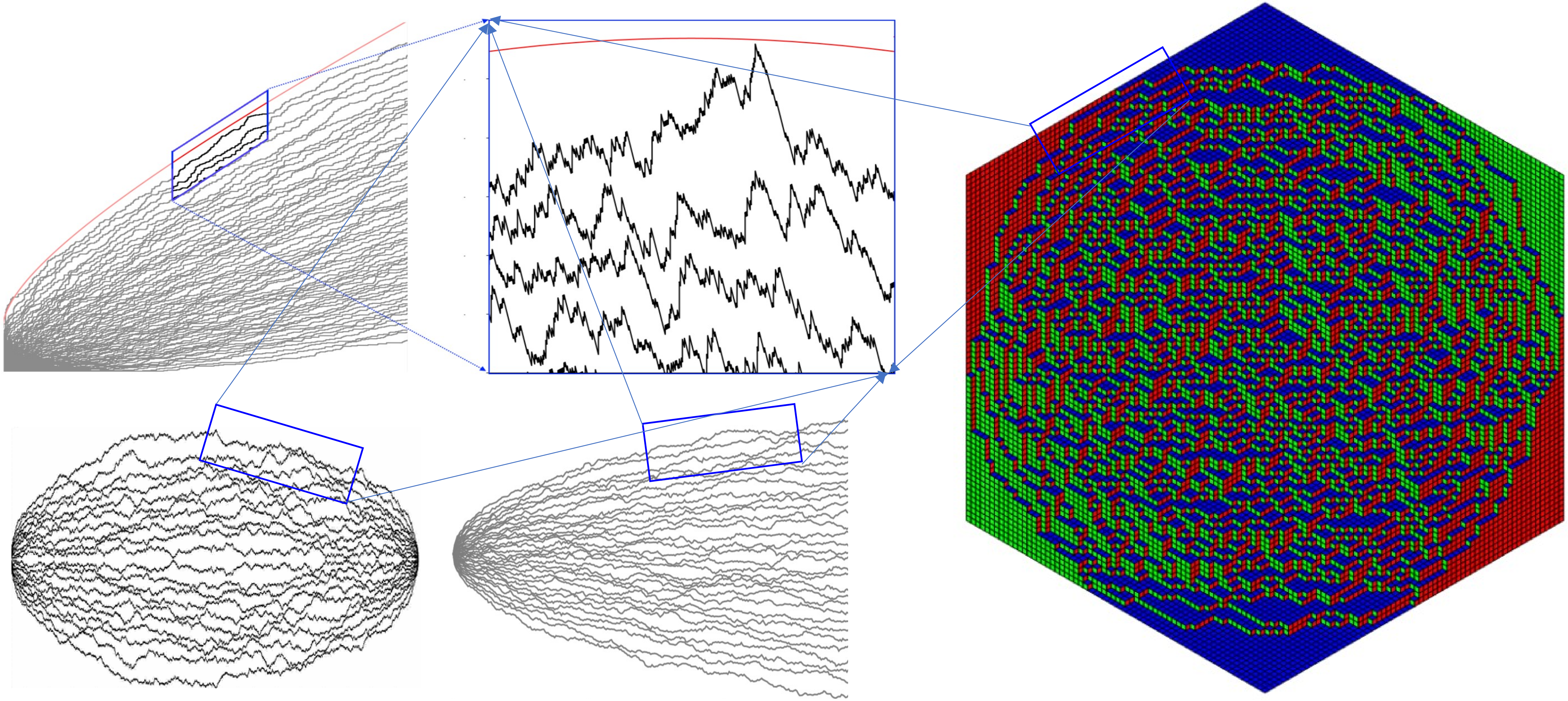
$\boldsymbol{L}^{(1)}$



$$\text{Marginal} \propto \begin{matrix}\text{Non-crossing} \\ \text{Bernoulli bridges}\end{matrix} \times \prod_{i=0}^{k} \prod_{x=a+1}^{b} \left(1 - q^{\Delta_i(x-1)} \mathbb{1}_{\Delta_i(x)=\Delta_i(x-1)-1}\right)$$

q=0 q>0

Non-intersecting Gibbsian line ensembles

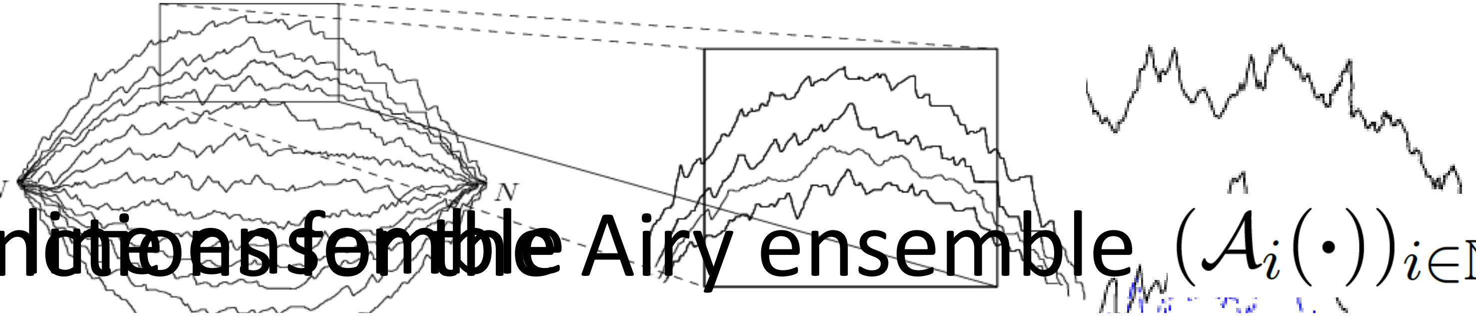


Free fermion / determinantal structure enables scaling limits

Parabolic Airy line ensemble at the edge

[Prahofer-Spohn ’02], [C-Hammond ‘11]

Scaling at the edge transversally by $N^{2/3}$ and perpendicularly by $N^{1/3}$ leads to a universal limit $\mathcal{P}_i(x) := \mathcal{A}_i(x) - x^2$.



The parabolic Airy line ensemble inherits the scaling limit of the Gibbs properties of the prelimits:

It enjoys the non-intersecting Brownian Gibbs property.

The correlation functions of the Airy ensemble $(\mathcal{A}_i(\cdot))_{i\in\mathbb{N}}$ are

$$\mathrm{d}\mathbb{P}\Big((t_j, y_j) \in \mathcal{A} \text{ for } j = 1, \ldots, m\Big) = \det\Big[K^{\mathrm{ext}}_{\mathrm{Ai}}((y_i, t_i); (y_j, t_j))\Big]_{1\le i,j\le m} \prod_{j=1}^{m} \mathrm{d}y_j$$

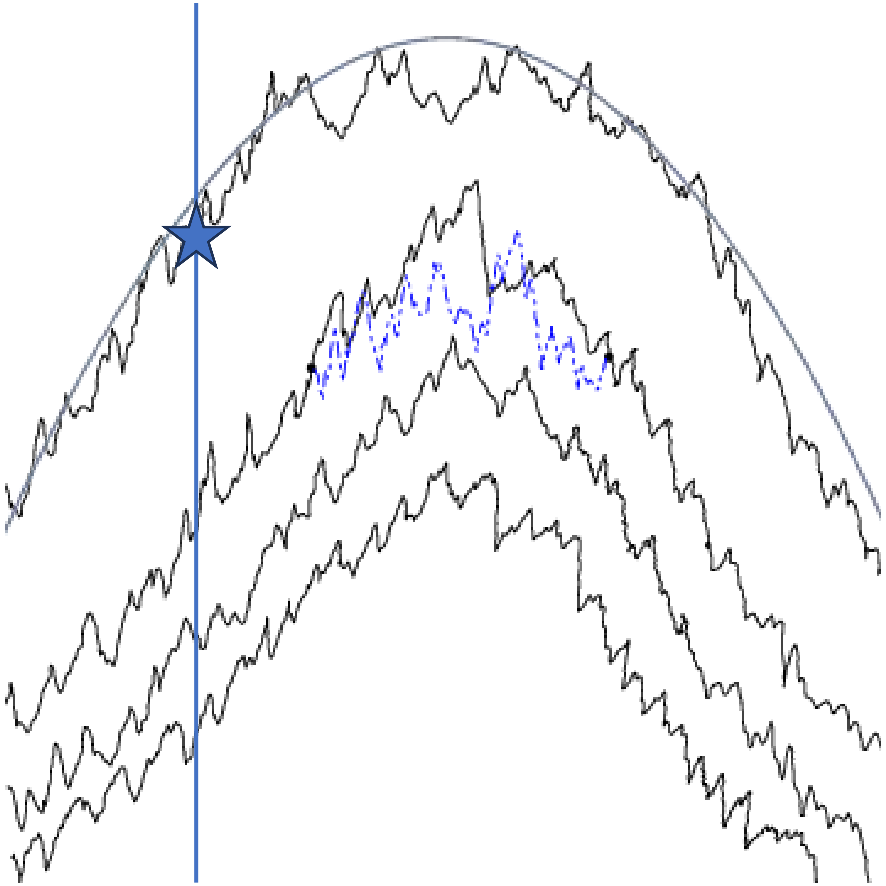
where

$$K^{\mathrm{ext}}_{\mathrm{Ai}}((x,t);(y,s)) := \begin{cases} \int_0^\infty e^{-\lambda(t-s)}\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)\,\mathrm{d}\lambda & t \ge s \\ -\int_{-\infty}^0 e^{-\lambda(t-s)}\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)\,\mathrm{d}\lambda & t < s \end{cases}$$

Airy line ensemble strong characterization

[Aggarwal-Huang '23]

<u>Thm</u>: Any $\mathbb{N}$-indexed line ensemble that enjoys the non-intersecting Brownian Gibbs property and whose top curve has a parabolic limit shape is the parabolic Airy line ensemble up to a random height shift.



One-point GUE Tracy-Widom fluctuations of the top curve anywhere implies the line ensemble is the Airy line ensemble.

Back to the q-Boson line ensemble

[Borodin-C-Gorin '16], [C-Dimitrov '18]
GUE Tracy-Widom fluctuations & HL Gibbs property

→

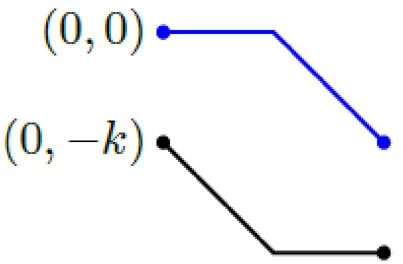
[Aggarwal-C-Hegde '24]
Tightness at edge & Brownian Gibbs property for limits

→

[Aggarwal-Huang '23]
Identification of limit as Airy line ensemble

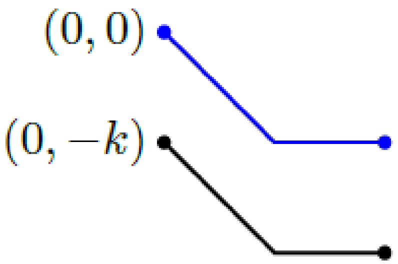
Lack of FKG inequality requires major reworking of theory of Gibbsian line ensembles to only use 'weak monotonicity'

[C-Dimitrov '18]



$W(B,g) = 1 - q^{k+1}$

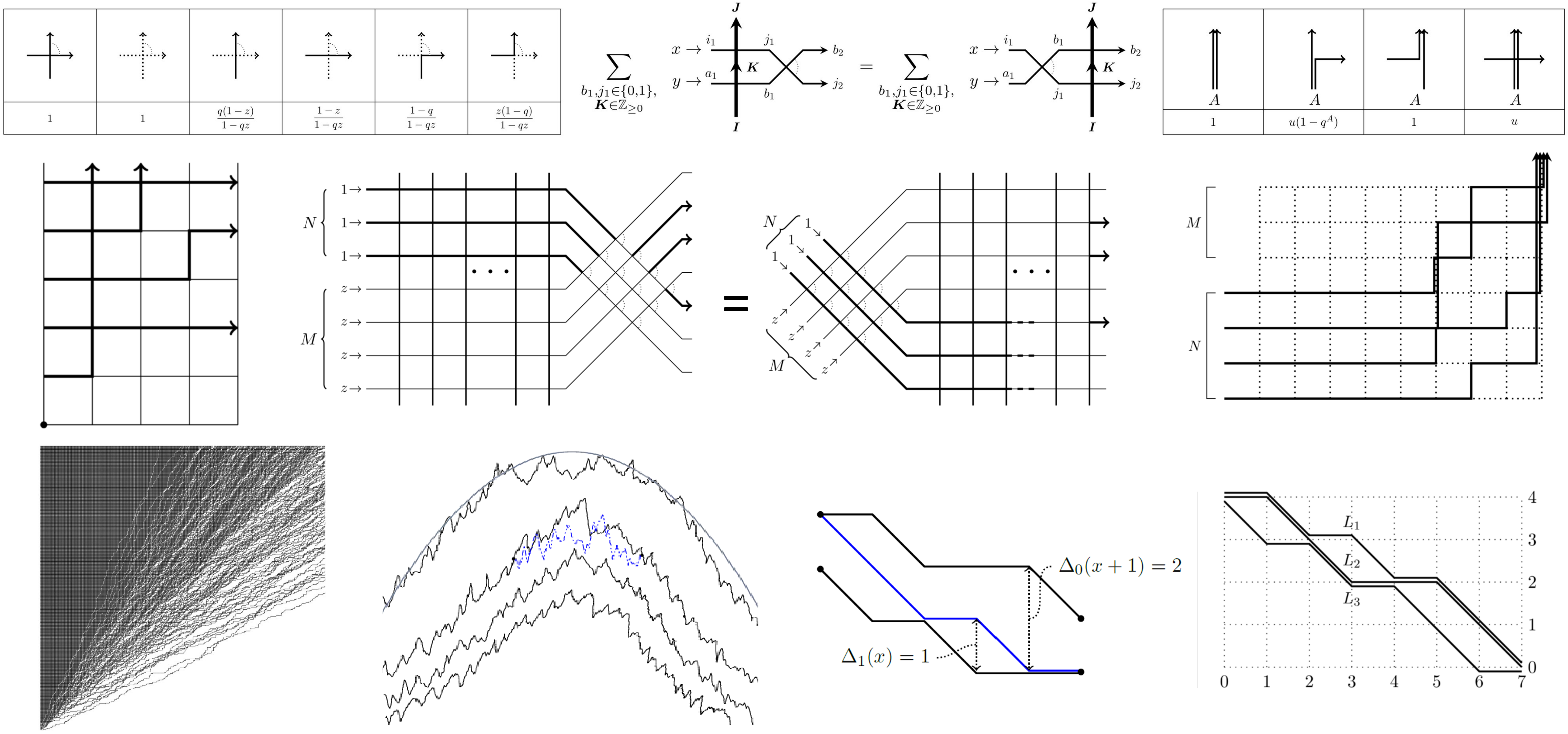
$\mathbb{P}(B) = \frac{1-q^{k+1}}{2-q^{k+1}}$



$W(B,g) = 1$

$\mathbb{P}(B) = \frac{1}{2-q^{k+1}}$

Airy line ensemble and the q-Boson model



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