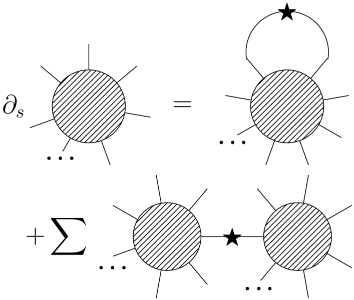
The non-perturbative side of the Kardar-Parisi-Zhang equation



Léonie Canet

Presentation outline

1. The KPZ fixed point in $d > 1$

2. The functional and non-perturbative renormalisation group

3. The chef's surprise: unpredicted scaling in $d = 1$

Acknowledgments

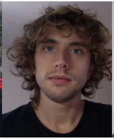
Collaborators:




B. Delamotte	N. Wschebor	H. Chaté	M. Brachet
LPTMC Paris	Montevideo	CEA Paris	ENS Paris

PhD students and post-docs:




T. Kloss M. Tarpin C. Fontaine F. Vercesi L. Gosteva

References:

LC, B. Delamotte, H. Chaté, N. Wschebor, PRL **104** (2010), PRE **84** (2011)

T. Kloss, **LC**, N. Wschebor, PRE **86** (2012), PRE **89** (2014), PRE **90** (2014)

M. Tarpin, **LC**, N. Wschebor, Phys. Fluids **30** (2018)

D. Squizzato, **LC**, PRE **100** (2019)

C. Fontaine, F. Vercesi, M. Brachet, **LC**, PRL **131** (2023)

Outline

1. The KPZ fixed point in $d > 1$
2. The functional and non-perturbative renormalisation group
3. The chef's surprise: unpredicted scaling in $d = 1$

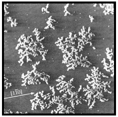
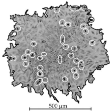
Stochastic interface growth and self-organised criticality

- KPZ equation describes kinetic roughening of growing interfaces
  - generic scale-invariance
  - universality

Halpin-Healy, Zhang, Phys. Rep. **254** (1995)

Krug, Adv. Phys. **46** (1997)



- correlation function takes Family-Vicsek scaling form

$$C(t,\mathbf{x}) = \left\langle (h(t,\mathbf{x}) - h(0,0))^2 \right\rangle \sim \begin{cases} |\mathbf{x}|^{2\chi} & t = 0 \\ t^{2\beta} & \mathbf{x} = 0 \end{cases}$$

$\longrightarrow$ collapse onto a universal scaling function

$$C(t,\mathbf{x}) \sim |\mathbf{x}|^{2\chi} F\left(t/|\mathbf{x}|^{z}\right), \quad z = \chi/\beta$$

- Galilean invariance

  $\longrightarrow$ in all dimension

  $$\chi + z = 2$$

- time-reversal symmetry

  $\longrightarrow$ in one dimension

  $$\chi = \frac{1}{2}, z = \frac{3}{2}$$

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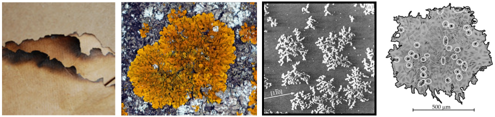
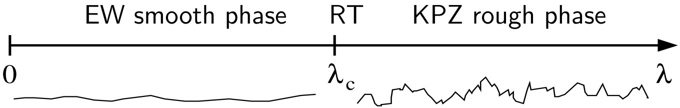


- dimension $1 \leq d \leq 2$: interface always rough

$\longrightarrow$ criticality without fine-tuning (attractive fixed-point)

- dimension $d > 2$: roughening phase transition

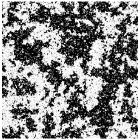


EW smooth phase	RT	KPZ rough phase
$\chi = (2-d)/2$	$\chi = 0$	no exact
$z = 2$	$z = 2$	results

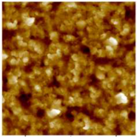
Critical phenomena and Renormalisation Group (RG)

▶ kinetic roughening is a non-equilibrium critical phenomena

- scale invariance, self-similarity
- universality
- anomalous critical exponents



second order phase transition

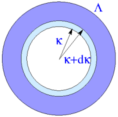


2D KPZ interface

Halpin-Healy Palasantzas EPL **105** (2014)

▶ criticality arises from fluctuations at all scales . . .

$\Longrightarrow$ Wilson's Renormalisation Group Wilson, Kogut, Phys. Rep. C **12** (1974)



- progressive integration of fluctuation modes
- sequence of scale-dependent effective models

scale invariance $\Longleftrightarrow$ fixed point of the Renormalisation Group

Field theory for the KPZ equation

- KPZ equation : a stochastic Langevin equation

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2}\left(\nabla h\right)^2 + \eta$$

$$\left\langle \eta(t,\mathbf{x})\eta(t',\mathbf{x}')\right\rangle = 2D\delta(t-t')\delta^d(|\mathbf{x}-\mathbf{x}'|)$$

- Martin-Siggia-Rose-Janssen-de Dominicis formalism

Martin, Siggia, Rose, PRA **8** (1973), Janssen, Z. Phys. B **23** (1976), de Dominicis, J. Phys. Paris **37** (1976)

$$\mathcal{Z}[j,\bar{j}] = \int \mathcal{D}h\,\mathcal{D}\bar{h}\; e^{-\mathcal{S}_{\rm KPZ}[h,\bar{h}] + \int_{t,\mathbf{x}}\{j\,h + \bar{j}\,\bar{h}\}}$$

$$\mathcal{S}_{\rm KPZ}[h,\bar{h}] = \int_{t,\mathbf{x}} \Big\{ \bar{h}\,\underbrace{\Big[\partial_t h - \nu\nabla^2 h - \frac{\lambda}{2}\left(\nabla h\right)^2\Big]}_{\text{deterministic part}} - \underbrace{D\,\bar{h}^2}_{\text{noise}} \Big\}$$

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$$\mathcal{S}_{\rm KPZ}[h,\bar{h}] = \int_{t,\mathbf{x}} \Big\{ \underbrace{\bar{h}\,\Big[\partial_t h - \nabla^2 h - \frac{\sqrt{g}}{2}\big(\nabla h\big)^2\Big]}_{\text{deterministic part}} - \underbrace{\bar{h}^2}_{\text{noise}} \Big\}$$

$\longrightarrow$ by rescaling time and fields : one coupling $g = \lambda^2 D/\nu^3$

The Kardar-Parisi-Zhang equation: A non-perturbative fixed-point

- perturbative Renormalisation Group

$\longrightarrow$ expansion at small coupling g

  - at 1-loop order Kardar, Parisi, Zhang, PRL **56** (1986)
  - at 2-loop order Frey and Täuber, PRE **50** (1994)

    Sun and Plischke, PRE **49** (1994)

    Teodorovich, JETP 82 **268** (1996)

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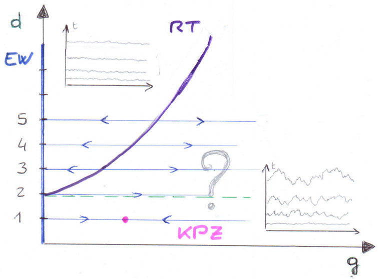
    Teodorovich, JETP 82 **268** (1996)

▷ at fixed d: g^* diverges when $d \to 2$

▷ near $d = 2 + \epsilon$: RT fixed point

$$z = 2 + \mathcal{O}(\epsilon^3)$$
$$\chi = 0 + \mathcal{O}(\epsilon^3)$$

... and nothing else !



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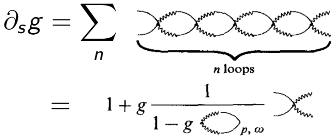
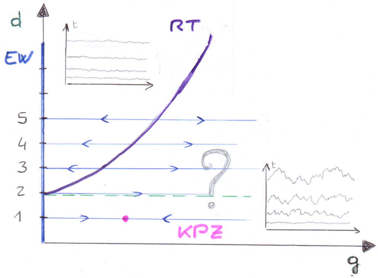
    Teodorovich, JETP 82 **268** (1996)
  - resummed to all-order

    Wiese, PRE **56** (1997), J. Stat. Phys. **93** (1998)





fails to find the KPZ strong-coupling fixed-point !

The Kardar-Parisi-Zhang equation in $d > 1$

- mostly numerical approaches
  - discrete models Tang *et al.* (1992)
    Marinari *et al.* (2012), Kelling and Ódor (2011)
    Halpin-Healy (2013), Pagani, Parisi (2015)
  - direct integrations Miranda and Reis (2008)
  - real space NRG Castellano *et al.* (1999)

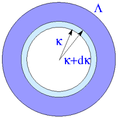
d	χ
2	0.384
3	0.304
4	0.256

- few analytical approaches
  - perturbative functional RG $d_c \simeq 2.5$
    Le Doussal and Wiese, PRE **72** (2005)
  - Mode-Coupling theory $d_c = 4$
    Frey, Tauber, Hwa, PRE **53** (1996), Colaiori and Moore, PRL **86** (2001)
  - Self-Consistent expansion $d_c = \infty$
    Schwartz and Perlsman, PRE **85** (1992), Schwartz and Katzav, J. Stat. Mech (2008)

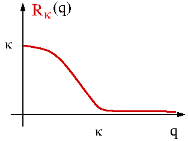
Outline

Functional and non-perturbative Renormalisation Group

- based on Wilson's RG ideas



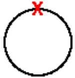
- progressive integration of fluctuation modes
- sequence of scale-dependent effective models



$\Longrightarrow$ Effective average action Γ_κ instead of effective action $\mathcal{S}_\kappa$

- exact RG equation for effective average action Wetterich, Phys. Lett. B **301** (1993)

$$\partial_\kappa \Gamma_\kappa = \frac{1}{2}\mathrm{Tr}\int_{\mathbf{q}} \partial_\kappa R_\kappa(\mathbf{q})\left[\Gamma_\kappa^{(2)} + R_\kappa\right]^{-1}(-\mathbf{q})$$



- complementary and accurate approximation schemes
  - derivative expansion
  - vertex expansion

Dupuis, *et al*, Phys. Rep. 910 (2021)

Ising 3D

	ν	η
conformal bootstrap	0.629971(4)	0.0362978(20)
FRG $\mathcal{O}(\partial^6)$	0.63007(10)	0.03648(18)
RG 6-loop	0.6304(13)	0.0335(25)

Balog, Chaté, Delamotte, Wschebor, PRL **103** (2019)

Functional and non-perturbative Renormalisation Group

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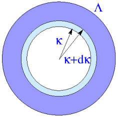

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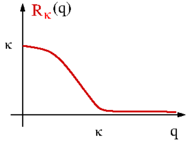

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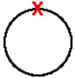

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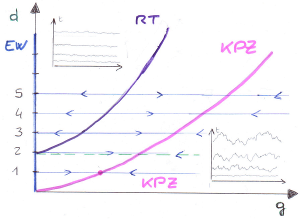
Functional Renormalisation Group: The KPZ fixed-point

- simplest ansatz captures strong-coupling fixed-point in all d

$$\Gamma_\kappa[\psi,\tilde{\psi}]=\int_{t,\mathbf{x}}\left\{\tilde{\psi}\Big(\partial_t\psi-\frac{\lambda}{2}(\nabla\psi)^2-\nu_\kappa\,\nabla^2\psi\Big)-\,D_\kappa\tilde{\psi}^2\right\}$$

one coupling $g_\kappa = \lambda^2 D_\kappa/\nu_\kappa$

two anomalous dimensions $\eta^D_\kappa = -\partial_s \ln D_\kappa$, $\eta^\nu_\kappa = -\partial_s \ln \nu_\kappa$



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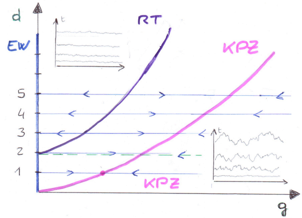
two anomalous dimensions $\eta_\kappa^D=-\partial_s\ln D_\kappa$, $\eta_\kappa^\nu=-\partial_s\ln\nu_\kappa$

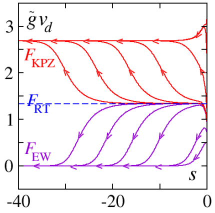


- refined ansatz for quantitative description

$$\Gamma_\kappa[\psi,\tilde\psi]=\int_{t,\mathbf{x}}\Big\{\tilde\psi f_\kappa^\lambda(D_t,\nabla)\Big[\partial_t\psi-\frac{\lambda}{2}(\nabla\psi)^2\Big]-\tilde\psi f_\kappa^\nu(D_t,\nabla)\,\nabla^2\psi-f_\kappa^D(D_t,\nabla)\tilde\psi^2\Big\}$$

one coupling $g_\kappa=\lambda^2 D_\kappa/\nu_\kappa$

full functions f_κ^ν, f_κ^D, f_κ^λ of $D_t=\partial_t-\lambda\nabla\psi\cdot\nabla$ and ∇



example of flow in $d=3$

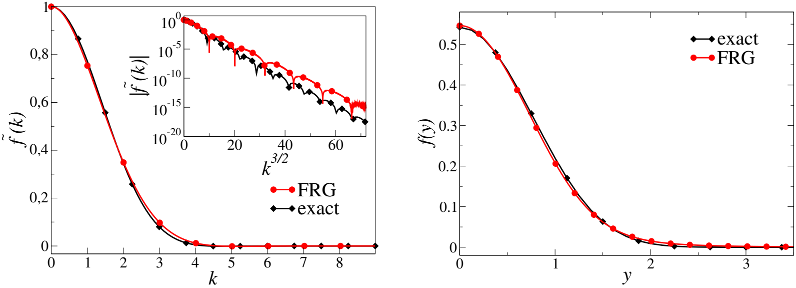
LC, Chaté, Delamotte, Wschebor, PRL **104**, (2010), PRE **84**, (2011)

Functional Renormalisation Group for KPZ: Universal scaling functions

- generic scaling in all d

$$C(t, \mathbf{p}) = \left\langle h(t, \mathbf{p}) h(0, -\mathbf{p}) \right\rangle_c = \frac{1}{p^{d-2\chi}} \tilde{f}(tp^z)$$

- very accurate results in $d = 1$



LC, Chaté, Delamotte, Wschebor PRE **84**, (2011), Prähofer and Spohn, J. Stat. Phys. **115** (2004)

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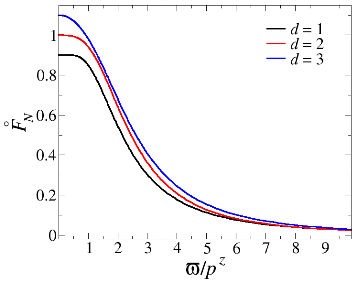
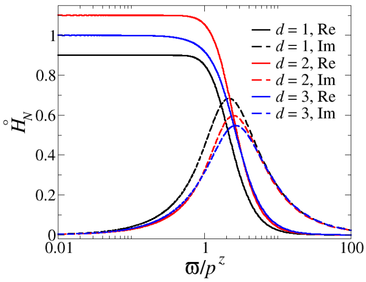
$$C(t,\mathbf{p}) = \left\langle h(t,\mathbf{p})h(0,-\mathbf{p})\right\rangle_c = \frac{1}{p^{d-2\chi}}\,\tilde{f}(tp^z)$$

- scaling functions for correlations and response in $d = 2$ and 3





Kloss, LC, Wschebor, PRE **86** (2012)

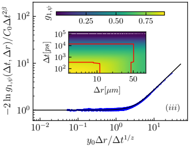
Comparison with numerics
Other results for non-integrable cases

- universal amplitude ratio in $d = 2$ from large-scale simulations
  - FRG: $R = 0.940$ Kloss, LC, Wschebor, PRE **86** (2012)
  - numerics: $R = 0.944 \pm 0.031$ Halpin-Healy, PRE **88** (2013)

- universal scaling function

for 2D exciton-polariton condensates

K. Deligiannis, *et al*, PRR **4** (2022)



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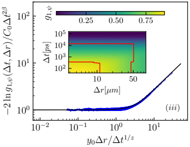
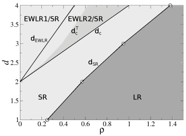


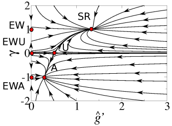
- KPZ equation with correlated noise

- long-range spatial noise Kloss, LC, Delamotte, Wschebor, PRE **89** (2014)
- short-range spatial noise Mathey, Agoritsas, Kloss, Lecomte, LC, PRE **95** (2017)
- long-range temporal noise Squizzato, LC, PRE **100** (2019)



- KPZ equation with anisotropy

Kloss, LC, Wschebor, PRE **90** (2014)



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1D Kardar-Parisi-Zhang equation: exact results

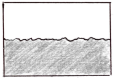
- universal height distribution for the KPZ equation

- curved geometry – droplet (TW-GUE)

  Sasamoto, Spohn, PRL **104** (2010)

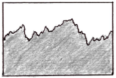
  Amir, Corwin and Quastel, Commun. Pure Appl. Math. **64** (2011)

  Calabrese, Le Doussal, Rosso, EPL **90** (2010)



- flat geometry (TW-GOE)

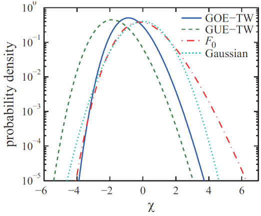
  Calabrese, Le Doussal, PRL **106** (2011),



- Brownian geometry (Baik-Rains)

  Imamura, Sasamoto, PRL (2012)

  Borodin, Corwin, Ferrari, Vetö, Math. Phys. Ann. Geom. **18** (2015)



- two-point correlation function: Airy processes

Prahöfer, Spohn, J. Stat. Phys. (2004), Sasamoto, J. Phys. A (2005), Imamura, Sasamoto, PRL (2012)

- large deviations for atypical large fluctuations

Le Doussal, Majumdar, Schehr, EPL **113** (2016)

- yet it still reserves its surprises !

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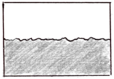
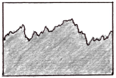


  - flat geometry (TW-GOE)

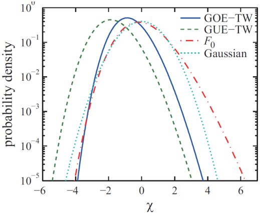
    Calabrese, Le Doussal, PRL **106** (2011),



  - Brownian geometry (Baik-Rains)

    Imamura, Sasamoto, PRL (2012)

    Borodin, Corwin, Ferrari, Vetö, Math. Phys. Ann. Geom. **18** (2015)



- two-point correlation function: Airy processes

Prahöfer, Spohn, J. Stat. Phys. (2004), Sasamoto, J. Phys. A (2005), Imamura, Sasamoto, PRL (2012)

- large deviations for atypical large fluctuations

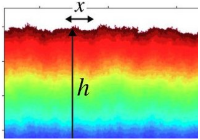
Le Doussal, Majumdar, Schehr, EPL **113** (2016)

- **yet it still reserves its surprises !**

The Kardar-Parisi-Zhang and the Burgers equations

- KPZ equation for stochastically growing interfaces

$$\partial_t h - \frac{\lambda}{2}\left(\nabla h\right)^2 = \nu \nabla^2 h + \sqrt{D}\eta$$



η: stochastic Gaussian noise with correlations

$$\langle \eta(t,\mathbf{x})\eta(t',\mathbf{x}')\rangle = 2\delta(t-t')\delta^d(\mathbf{x}-\mathbf{x}')$$

Takeuchi *et al* Sci. Rep. **1** (2011)

$\Longrightarrow$ exact mapping to:

- Burgers equation for randomly stirred fluids Burgers (1948)

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = \nu\nabla^2\mathbf{v} + \mathbf{f}$$

$\mathbf{f}$: stochastic Gaussian forcing at large scale

for $\mathbf{v} = -\lambda\nabla h$ with $\nabla\times\mathbf{v} = 0$ and $\mathbf{f} = -\lambda\sqrt{D}\nabla\eta$

The KPZ - Burgers equation in the inviscid limit

The Galerkin-truncated Burgers equation: Crossover from inviscid-thermalised to Kardar-Parisi-Zhang scaling

C. Cartes[1], E. Tirapegui[2], R. Pandit[3] and M. Brachet[4]

Phil. Trans. A **380** (2022)

Family-Vicsek Scaling of Roughness Growth in a Strongly Interacting Bose Gas

Kazuya Fujimoto,[1,2] Ryusuke Hamazaki,[3,4] and Yuki Kawaguchi[2]

Phys. Rev. Lett. **124** (2020)

(c)

Model	α	β	z
KPZ	$1/2$	$1/3$	$3/2$
EW	$1/2$	$1/4$	2
BHM ($\nu \gg 1$)	0.517 ± 0.030	0.255 ± 0.012	2.07 ± 0.20
BHM ($\nu \simeq 1/2$)	0.500 ± 0.003	0.489 ± 0.004	1.00 ± 0.01

this Letter

Anomalous ballistic scaling in the tensionless or inviscid Kardar-Parisi-Zhang equation

Enrique Rodríguez-Fernández,[1,*] Silvia N. Santalla,[2,†] Mario Castro,[3,‡] and Rodolfo Cuerno[1,§]

Phys. Rev. E **106** (2022)

observation of an unpredicted scaling $z = 1$ in the limit $\nu \to 0$

- note: $z = 1$ scaling also predicted in $d \to \infty$, $\text{Re} \to \infty$ in Burgers

Bouchaud, Mézard, Parisi, PRE **52** (1995)

The KPZ - Burgers equation in the inviscid limit

- simulation of 1D Burgers equation

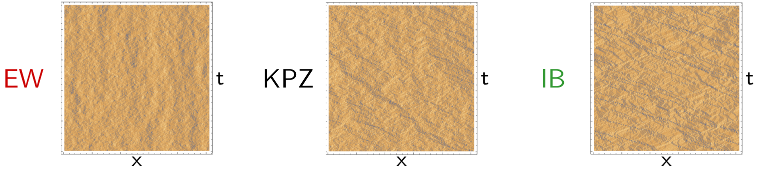
Cartes, Tirapegui, Pandit, Brachet, Phil. Trans. A **380** (2022)

$$\partial_t v + \lambda v \partial_x v = \nu \partial_x^2 v + \sqrt{D} \partial_x f$$

$\longrightarrow$ spectral (Galerkin) truncation preserves all the symmetries

- Galilean invariance
- time-reversal symmetry

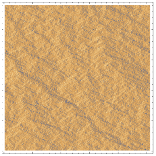
- observation of three scaling regimes



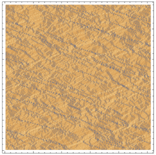
The KPZ - Burgers equation in the inviscid limit

EW

KPZ

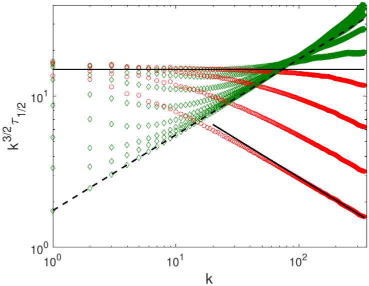


IB



- Decorrelation time from the two-point function $C(t,k)$

$$\tau_{1/2}(k) \text{ such that } C(\tau_{1/2}(k),k) = \frac{1}{2}C(0,k)$$



Inviscid Burgers ($\nu = 0$): $z = 1$

Kardar-Parisi-Zhang: $z = 3/2$

Edwards-Wilkinson ($\lambda = 0$): $z = 2$

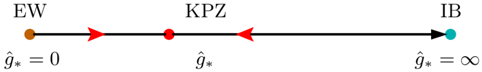
Renormalisation Group analysis

- KPZ-Burgers equation: one coupling constant $g = \lambda^2 D/\nu^3$ (or Re)

inviscid limit $\nu \to 0 \iff$ infinite coupling limit $g \to \infty$

non-perturbative side also in $d = 1$!

- possible scenario :



$\Longrightarrow$ scaling $z = 1$ controlled by UV fixed point IB

Renormalisation Group analysis

- KPZ-Burgers equation: one coupling constant $g = \lambda^2 D/\nu^3$ (or Re)

  inviscid limit $\nu \to 0 \Longleftrightarrow$ infinite coupling limit $g \to \infty$

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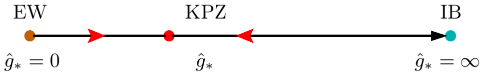
- possible scenario :



$\Longrightarrow$ scaling $z = 1$ controlled by UV fixed point IB

Functional Renormalisation Group approach

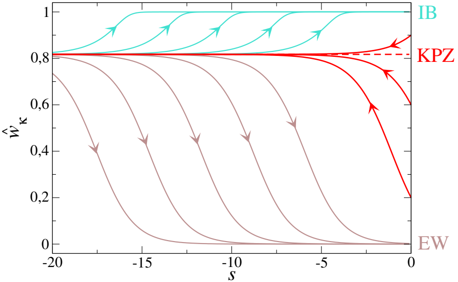
- simplest approximation:
  - effective parameters ν_κ, D_κ, λ $\Longrightarrow$ $\hat{g}_\kappa$
  - define $\eta_\kappa = -\partial_s \ln \nu_\kappa$ with $s = \ln(\kappa/\Lambda)$ (RG 'time')
  - define $\hat{w}_\kappa = \hat{g}_\kappa/(1 + \hat{g}_\kappa) \in [0, 1]$

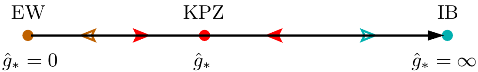
Functional Renormalisation Group: Existence of IB fixed-point

- FRG flow equation for $\hat{w}_\kappa$

$$\partial_s \hat{w}_\kappa = \hat{w}_\kappa \left(1 - \hat{w}_\kappa\right) \left(2\eta_\kappa - 1\right) \qquad \text{with} \qquad \eta_\kappa = 0 \ \text{ for } \ \hat{w}_\kappa = 0$$

- 3 fixed point solutions
  - KPZ: $0 < \hat{w}_* < 1$
    $\eta_* = 1/2 \ , \ z_{\mathrm{KPZ}} = 3/2$
    IR stable, UV unstable
  - EW: $\hat{w}_* = 0$
    $\eta_* = 0 \ , \ z_{\mathrm{EW}} = 2$
    UV stable, IR unstable
  - IB: $\hat{w}_* = 1$
    η_* to be determined, $z_{\mathrm{IB}} = ?$
    UV stable, IR unstable





Fontaine, Vercesi, Brachet, LC, PRL **131** (2023)

Functional Renormalisation Group: Numerical solution in the IR

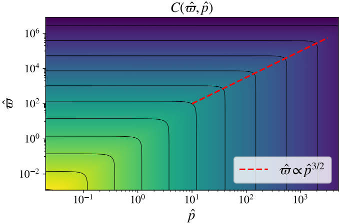
- refined ansatz (NLO) for quantitative description

Kloss, LC, Wschebor PRE **86** (2012)

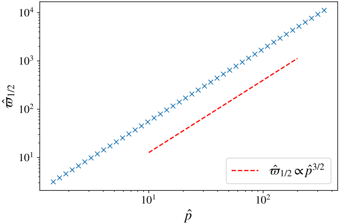
$$\Gamma[\psi,\tilde\psi]=\int_{t,\mathbf{x}}\left\{\tilde\psi\Big(\partial_t\psi-\frac{g_\kappa}{2}(\nabla\psi)^2-f_\kappa(\partial_t,\nabla)\,\nabla^2\psi\Big)-f_\kappa(\partial_t,\nabla)\tilde\psi^2\right\}$$

$\longrightarrow$ compute correlation $\quad C_\kappa(\varpi,p)=\dfrac{2f_\kappa(\varpi,p)}{\varpi^2+p^4 f_\kappa^2(\varpi,p)}$

correlation function

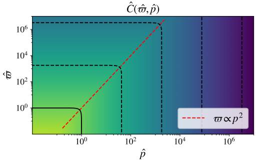


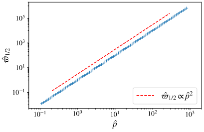
half-frequency: $C(\varpi_{1/2},p)=\frac{1}{2}C(0,p)$



KPZ fixed-point in the IR for any initial condition g_Λ

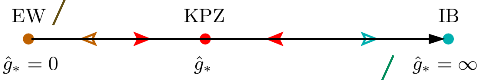
Functional Renormalisation Group: Probing the UV fixed points





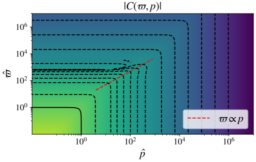
EW scaling $z = 2$

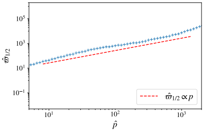
initial condition $\hat{g}_\Lambda \ll \hat{g}_*$



initial condition $\hat{g}_\Lambda \gg \hat{g}_*$

new IB scaling $z \simeq 1$!

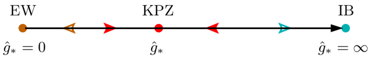


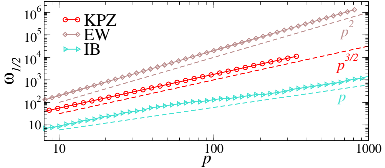


Fontaine, Vercesi, Brachet, LC, PRL **131** (2023)

Functional Renormalisation Group: Summary of numerical results

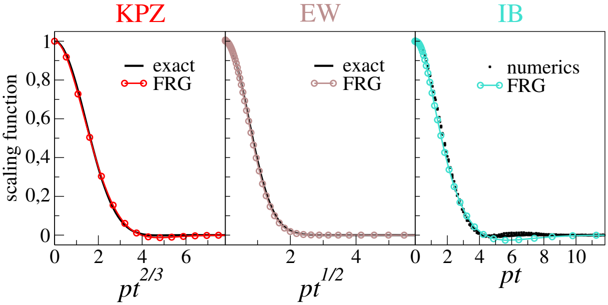
- 3 fixed-points with different z
  - KPZ: stable, controls the IR
  - EW, IB: unstable, control the UV





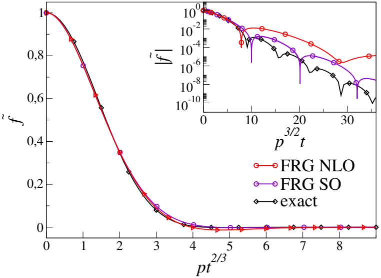
Fontaine, Vercesi, Brachet, LC, PRL **131** (2023)

and different scaling functions

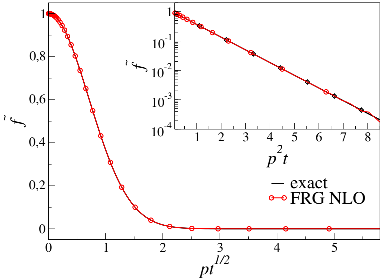


Functional Renormalisation Group: Zooming in the tails of the scaling functions

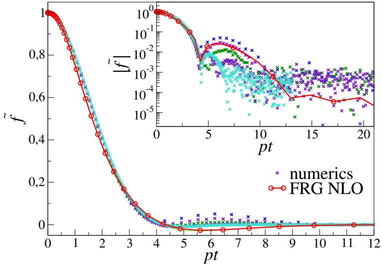
KPZ



EW



IB



Can we rigorously demonstrate that $z = 1$?

Space-time correlations from Functional Renormalisation Group

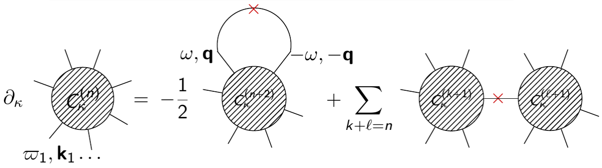
- space-time *n*-point connected correlation functions

$$C^{(n)}_{\alpha_1\ldots\alpha_n}(\{t_i,\mathbf{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1,\mathbf{x}_1)\cdots v_{\alpha_n}(t_n,\mathbf{x}_n)\Big\rangle_c$$

- exact (but infinite hierarchy of) FRG flow equations for $C^{(n)}$
  - derived from flow equation for generating functional $\mathcal{W}_\kappa = \ln \mathcal{Z}_\kappa$

$$\partial_\kappa \mathcal{W}_\kappa = -\frac{1}{2}\mathrm{Tr}\int_{t_x,t_y,\mathbf{x},\mathbf{y}} \partial_\kappa[R_\kappa]_{\alpha\beta}(\mathbf{x}-\mathbf{y})\left\{\frac{\delta^2\mathcal{W}_\kappa}{\delta j_\alpha(t_x,\mathbf{x})\delta j_\beta(t_y,\mathbf{y})} + \frac{\delta\mathcal{W}_\kappa}{\delta j_\alpha(t_x,\mathbf{x})}\frac{\delta\mathcal{W}_\kappa}{\delta j_\beta(t_y,\mathbf{y})}\right\}$$

Polchinski, Nucl. Phys. B **231** (1984), Wetterich, Phys. Lett. B **301** (1993)



Analytical solution with FRG: A detour by Navier-Stokes
Extended symmetries and Ward identities

- Field theory for stochastic Navier-Stokes equation

$$\mathcal{S}_{\rm NS} = \int_{t,\mathbf{x}} \bar{v}_\alpha \underbrace{\left[\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha + \frac{1}{\rho}\partial_\alpha \pi - \nu \nabla^2 v_\alpha\right]}_{\text{equation of motion}} + \bar{\pi} \underbrace{\left[\partial_\alpha v_\alpha\right]}_{\text{incompressibility}} - \int_{t,\mathbf{x},\mathbf{x}'} \bar{v}_\alpha \underbrace{\left[N_L(|\mathbf{x}-\mathbf{x}'|)\right]}_{\text{forcing}} \bar{v}_\alpha$$

- existence of extended symmetries
  - time-dependent Galilean invariance: $\mathcal{G} = \begin{cases} \mathbf{x} \to \mathbf{x} + \vec{\epsilon}(t) \\ \mathbf{v} \to \mathbf{v} - \dot{\vec{\epsilon}}(t) \end{cases}$
    - well-known
  - time-dependent shift symmetry: $\mathcal{R} = \begin{cases} \delta \bar{v}_\alpha(t,\vec{x}) & = \bar{\epsilon}_\alpha(t) \\ \delta \bar{p}(t,\vec{x}) & = v_\beta(t,\vec{x})\bar{\epsilon}_\beta(t) \end{cases}$
    - not identified yet!

$\Longrightarrow$ compensation between variations of $\bar{v}_\alpha v_\beta \partial_\beta v_\alpha$ and $\bar{\pi}\partial_\alpha v_\alpha$

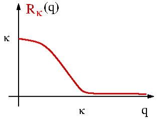
LC, Delamotte, Wschebor, Phys. Rev. E **91** (2015)

infinite set of local in time exact Ward identities
for all vertices $\Gamma_\kappa^{(m,n)}$ with a $\mathbf{q}_i = 0$

Analytical solution with FRG: A detour by Navier-Stokes
Exact closure in the large wave-number limit

▶ flow for $C^{(n)}_{\alpha_1\ldots\alpha_n}(\{t_i,\mathbf{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1,\mathbf{x}_1)\cdots v_{\alpha_n}(t_n,\mathbf{x}_n)\Big\rangle_c$

(1) large wave-number expansion: all $|\mathbf{k}_i|$ and $\left|\sum_i \mathbf{k}_i\right| \gg \kappa$



▷ $\partial_\kappa R_\kappa(\mathbf{q})$: $|\mathbf{q}| \lesssim \kappa \Longrightarrow |\vec{q}| \ll |\vec{k}_i|$

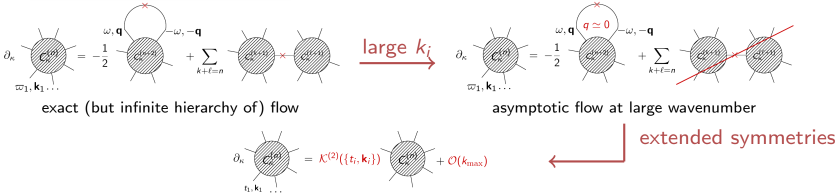
$\Longrightarrow$ set $\vec{q} = 0$ in all vertices

asymptotically exact for $|\mathbf{k}_i| \gg \kappa \sim L^{-1}$

Blaizot, Wschebor, Mendez-Galain, Phys. Lett B **832** (2006), Tarpin, LC, Wschebor, Phys. Fluids **30** (2018)

Analytical solution with FRG: A detour by Navier-Stokes
Exact closure in the large wave-number limit

- flow for $C^{(n)}_{\alpha_1\ldots\alpha_n}(\{t_i,\mathbf{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1,\mathbf{x}_1)\cdots v_{\alpha_n}(t_n,\mathbf{x}_n)\Big\rangle_c$



closed flow at large wavenumber

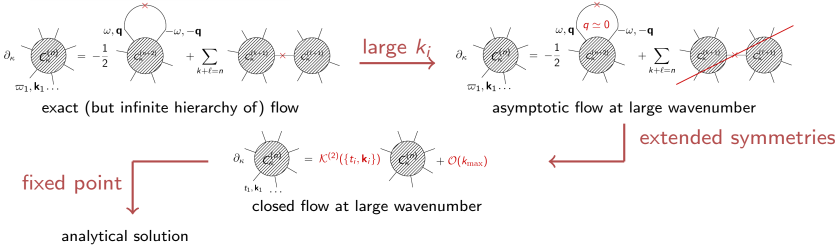
(2) Ward identities related to extended symmetries

- time-dependent Galilean invariance: $\mathcal{G} = \begin{cases} \mathbf{x} \to \mathbf{x} + \vec{\epsilon}(t) \\ \mathbf{v} \to \mathbf{v} - \dot{\vec{\epsilon}}(t) \end{cases}$
- time-dependent shift symmetry: $\mathcal{R} = \begin{cases} \delta\bar{v}_\alpha(t,\vec{x}) & = \bar{\epsilon}_\alpha(t) \\ \delta\bar{p}(t,\vec{x}) & = v_\beta(t,\vec{x})\bar{\epsilon}_\beta(t) \end{cases}$
  of response fields

LC, Delamotte, Wschebor, Phys. Rev. E **91** (2015)

Analytical solution with FRG: A detour by Navier-Stokes
Exact asymptotic form of correlations

- flow for $C^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\mathbf{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1,\mathbf{x}_1)\cdots v_{\alpha_n}(t_n,\mathbf{x}_n)\Big\rangle_c$



$C^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\mathbf{x}_i\}) =$ **K41** $\times$ **dominant term**

(3) solution at the fixed point

$$C^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\mathbf{k}_i\}) \propto \begin{cases} \exp\left(-\alpha_0 \frac{L^2}{\tau^2} \left|\sum_\ell \mathbf{k}_\ell t_\ell\right|^2 + \mathcal{O}(|\mathbf{k}_{\max}|L)\right) & t \ll \tau \\ \exp\left(-\alpha_\infty \frac{L^2}{\tau} \, |t| \, \sum_{k\ell} \mathbf{k}_k \cdot \mathbf{k}_\ell + \mathcal{O}(|\mathbf{k}_{\max}|L)\right) & t \gg \tau \end{cases}$$

M. Tarpin, LC, N. Wschebor, Phys. Fluids **30** (2018), LC, J. Fluid Mech. *Perspectives* **950** (2022)

Analytical solution with FRG: A detour by Navier-Stokes
Exact asymptotic form of correlations

- solution at the fixed-point: prediction of two time regimes

$$C^{(2)}(\{t,\mathbf{k}\}) \propto \begin{cases} \exp\Big(-\alpha_0\,\big|\mathbf{k}t\big|^2 + \mathcal{O}(|\mathbf{k}|L)\Big) & t \ll \tau_0 \\ \exp\Big(-\alpha_\infty\,|t||\mathbf{k}|^2 + \mathcal{O}(|\mathbf{k}|L)\Big) & t \gg \tau_0 \end{cases}$$

  - for $C^{(2)}$, at small times $\tau_a \propto k^{-1} \neq k^{-2/3} \Longrightarrow$ random sweeping
  - rigorous and generalised for any n-point correlations
  - prediction of a new regime at large time

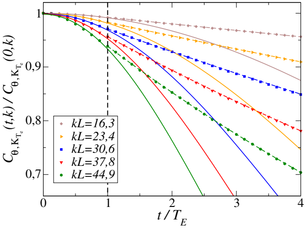
- extensive comparisons with simulations
  - in Navier-Stokes turbulence
  - in passive scalar turbulence

Gorbunova, Balarac, LC, Eyink, Rossetto, Phys. Fluids **33** (2021)

Gorbunova, Pagani, Balarac, LC, Rossetto, PRF **6** (2021)

C. Pagani, LC, Phys. Fluids **33** (2021)

LC, J. Fluid Mech. **950** (2022)



Analytical solution with FRG: Navier-Stokes vs Burgers
To be incompressible or not to be

- Action for Navier-Stokes equation (incompressible)

$$\mathcal{S}_{\rm NS} = \int_{t,\mathbf{x}} \left\{ \bar{v}_\alpha \left[\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha + \frac{\partial_\alpha \pi}{\rho} - \nu \nabla^2 v_\alpha \right] + \bar{\pi} \left[\partial_\alpha v_\alpha \right] \right\} - \int_{t,\mathbf{x},\mathbf{x}'} \bar{v}_\alpha \left[N(|\mathbf{x}-\mathbf{x}'|) \right] \bar{v}_\alpha$$

- Action for 1D Burgers-KPZ equation (pressureless)

$$\mathcal{S}_{\rm Burgers} = \int_{t,x} \left\{ \bar{v} \left[\partial_t v + v \partial_x v - \nu\, \partial_x^2 v \right] - D \left(\partial_x \bar{v} \right)^2 \right\}$$

- existence of extended symmetries
  - gauged Galilean invariance for both: $\mathcal{G} : \begin{cases} \mathbf{x} \to \mathbf{x} + \vec{\epsilon}(t) \\ \mathbf{v} \to \mathbf{v} - \dot{\vec{\epsilon}}(t) \end{cases}$
  - gauged shift symmetry for NS: $\mathcal{R} : \begin{cases} \delta \bar{v}_\alpha(t,\vec{x}) & = \bar{\epsilon}_\alpha(t) \\ \delta \bar{p}(t,\vec{x}) & = v_\beta(t,\vec{x}) \bar{\epsilon}_\beta(t) \end{cases}$

$\Longrightarrow$ incompressibility: variations of $\bar{v}_\alpha v_\beta \partial_\beta v_\alpha$ and $\bar{\pi} \partial_\alpha v_\alpha$ compensate

  - gauged shift symmetry for Burgers: $\mathcal{R} : \delta \bar{v}(t,\vec{x}) = \bar{\epsilon}_\alpha(t)$

Analytical solution with FRG: Navier-Stokes vs Burgers
To be incompressible or not to be

- Action for Navier-Stokes equation (incompressible)

$$\mathcal{S}_{\rm NS} = \int_{t,\mathbf{x}} \left\{ \bar{v}_\alpha \left[\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha + \frac{\partial_\alpha \pi}{\rho} - \nu \nabla^2 v_\alpha \right] + \bar{\pi} \left[\partial_\alpha v_\alpha \right] \right\} - \int_{t,\mathbf{x},\mathbf{x}'} \bar{v}_\alpha \left[N(|\mathbf{x}-\mathbf{x}'|) \right] \bar{v}_\alpha$$

- Action for 1D Burgers-KPZ equation (pressureless)

$$\mathcal{S}_{\rm Burgers} = \int_{t,x} \left\{ \bar{v} \left[\partial_t v + v \partial_x v - \nu\, \partial_x^2 v \right] - D \left(\partial_x \bar{v} \right)^2 \right\}$$

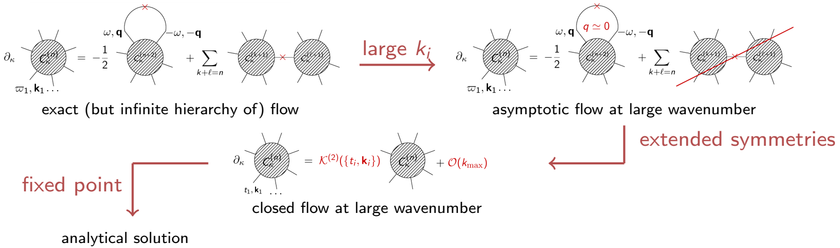
- existence of extended symmetries
  - gaugeded Galilean invariance for both: $\mathcal{G} : \begin{cases} \mathbf{x} \to \mathbf{x} + \vec{\epsilon}(t) \\ \mathbf{v} \to \mathbf{v} - \dot{\vec{\epsilon}}(t) \end{cases}$
  - gauged shift symmetry for NS: $\mathcal{R} : \begin{cases} \delta \bar{v}_\alpha(t,\vec{x}) & = \bar{\epsilon}_\alpha(t) \\ \delta \bar{p}(t,\vec{x}) & = v_\beta(t,\vec{x}) \bar{\epsilon}_\beta(t) \end{cases}$

$\Longrightarrow$ incompressibility: variations of $\bar{v}_\alpha v_\beta \partial_\beta v_\alpha$ and $\bar{\pi} \partial_\alpha v_\alpha$ compensate

  - gauged shift symmetry for Burgers: $\mathcal{R} : \delta \bar{v}(t,\vec{x}) = \bar{\epsilon}_\alpha(t)$

Analytical solution with FRG: Exact asymptotic solution for inviscid Burgers

- Exact closure of the flow of $C(t,p)$ in the limit of large wavenumbers



- solution at the fixed-point at large p (UV):

$$C(t,p) \propto \begin{cases} \exp\Big(-\alpha_0\,(pt)^2 + \mathcal{O}(pL)\Big) & t \ll \tau_0 \\ \exp\Big(-\alpha_\infty\, p^2|t| + \mathcal{O}(pL)\Big) & t \gg \tau_0 \end{cases}$$

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Analytical solution with FRG: Exact asymptotic solution for inviscid Burgers

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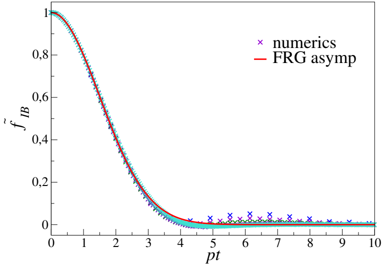
- proof of $z = 1$ scaling at small t
- analytical form of the scaling function
- crossover at large t in numerics ?
- exact mathematical solution for the pdf?



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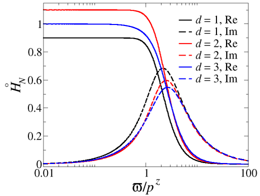
Summary of results from FRG

Strong-coupling fixed point of the KPZ equation in $d > 1$

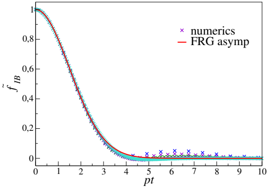
- critical exponents
- universal scaling functions
- presence of correlated noise, anisotropy



Unpredicted scaling $z = 1$ for inviscid KPZ-Burgers in $d = 1$

- numerical evidence probing the UV
- exact asymptotic solution:

  $z = 1$ and scaling function



Perspectives in KPZ with FRG

- inviscid KPZ-Burgers fixed point in $d > 1$

$\Longrightarrow$ see poster by Liuba Gosteva

- IB in deterministic Kuramoto-Sivashinsky and complex Ginzburg-Landau equations

$\Longrightarrow$ see talk by Francesco Vercesi

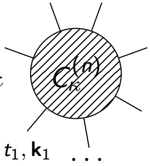
- KPZ in open quantum systems

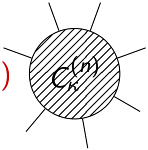
$\Longrightarrow$ see poster by Martina Zündel

Thank you for attention !

Analytical solution with FRG: Exact closure in the large wavenumber limit

closed flow equation for all $C^{(n)}(\{t_i, \mathbf{k}_i\})$ in the limit $|\mathbf{k}_i| \gg L^{-1}$

∂_κ 

$= \mathcal{K}^{(2)}(\{t_i, \mathbf{k}_i\})$ 

$+ \mathcal{O}(k_{\max})$

$$\mathcal{K}^{(2)}(\{t_i, \mathbf{k}_i\}) = \frac{1}{3} \int_\omega J^{(2)}(\omega) \sum_{k,\ell} \frac{\vec{k}_k \cdot \vec{k}_\ell}{\omega^2} \left(e^{i\omega(t_k - t_\ell)} - e^{i\omega t_k} - e^{-i\omega t_\ell} + 1 \right)$$

with the non-linear part hidden in

$$J^{(2)}(\omega) = -\int_{\mathbf{q}} \left\{ 2\kappa\partial_\kappa N_\kappa(\mathbf{q}) \, |G_\kappa(\omega, \mathbf{q})|^2 - 2\kappa\partial_\kappa R_\kappa(\mathbf{q}) \, C_\kappa(\omega, \mathbf{q}) \Re G_\kappa(\omega, \mathbf{q}) \right\}$$

Tarpin, LC, Wschebor, Phys. Fluids **30**, 055102 (2018)