Les Houches Doctorate School
KPZ: New trends in theories and experiments
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Les Houches, France

Universität
zu Köln

Driven Open Quantum Matter: From Micro- to Macrophysics

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ML4Q

CRC183
ENTANGLED STATES OF MATTER

CRC1238
Control and Dynamics
of Quantum Materials

Outline

Keldysh theory general: A. Kamenev, *Field theory or non-equilibrium systems*, Cambridge University Press

Review: L. Sieberer, M. Buchhold, SD, *Keldysh Field Theory for Driven Open Quantum Systems*, Reports on Progress in Physics (2016);

L. Sieberer, M. Buchhold, J. Marino, SD, *Universality in Driven Open Quantum Matter*, arxiv (2023)

1. From the Lindblad equation to the Lindblad-Keldysh functional integral

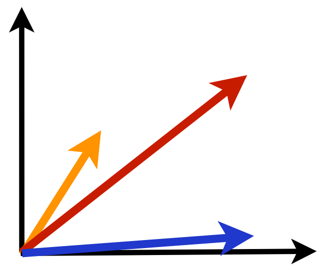
- Lindblad equation for driven open quantum matter
- construction of Lindblad-Keldysh functional integral

$$\partial_t \rho = -i[H, \rho] + \mathcal{L}[\rho]$$

$$\mathrm{e}^{\mathrm{i}\Gamma[\Phi]} = \int \mathcal{D}\delta\Phi \mathrm{e}^{\mathrm{i}S_M[\Phi+\delta\Phi]}$$

2. KPZ equation in exciton-polariton condensates

- background: semiclassical limit, classifying eq. vs. non-eq. states
- from XP to KPZ: absence of algebraic order out of equilibrium
- compact KPZ and non-equilibrium phase transition

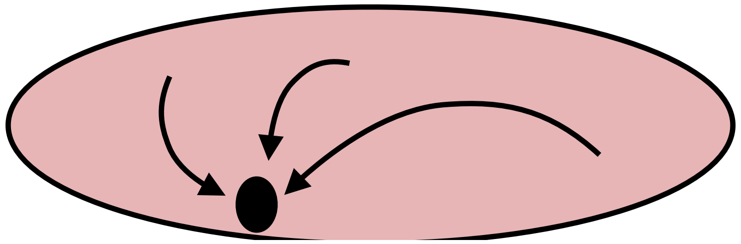


3. Macroscopic non-equilibrium phenomena from weak non-equilibrium drive

- non-equilibrium O(N) models: phase structure, limit cycles
- novel non-equilibrium criticality at onset of a limit cycle
- route towards KPZ via breaking of time translation symmetry

4. Principles of universality in driven open quantum matter

- the principles: eq. vs. non-eq.; pure vs. mixed states; weak vs. strong symmetries
- application: 1D KPZ in open vs. closed systems



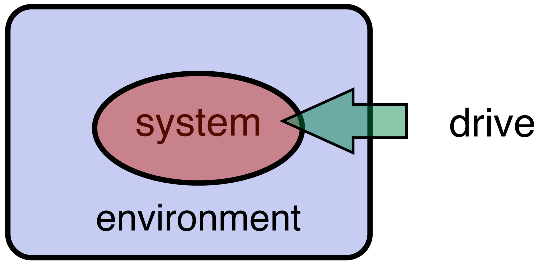
5. Quantum aspects: Topology in driven open quantum matter

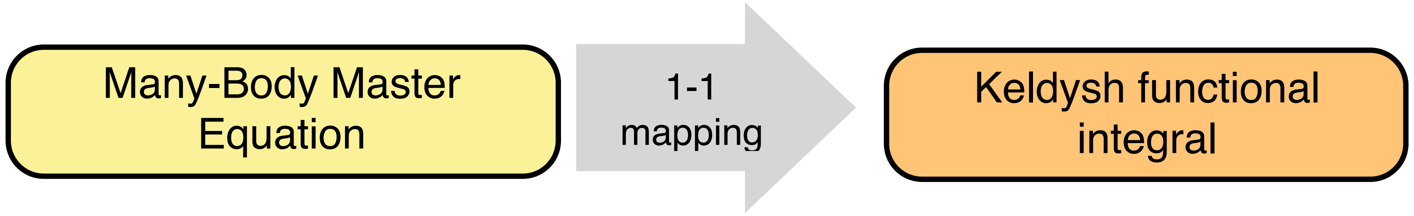
- topological dark states in Lindblad evolution
- universality of topological response: pure states, mixed states

$$\frac{\theta}{4\pi} \int A_+ \, dA_+ - A_- \, dA_-$$

1. From the Lindblad equation to the Lindblad-Keldysh functional integral

- Lindblad equation for driven open quantum matter
- construction of Lindblad-Keldysh functional integral
- structural properties




Many-Body Master Equation

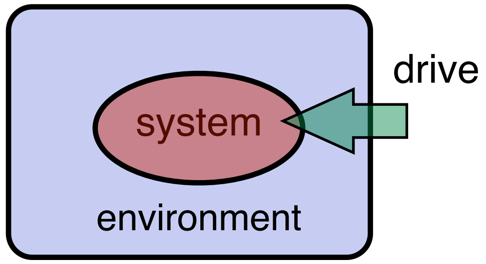
1-1 mapping

Keldysh functional integral

$$\partial_t \hat{\rho} = -i(\hat{H} - \sum_i \gamma_i \hat{L}_i^\dagger \hat{L}_i)\hat{\rho} + \text{h.c.} + 2\sum_i \gamma_i \hat{L}_i \hat{\rho} \hat{L}_i^\dagger$$

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+, \Phi_-]}$$

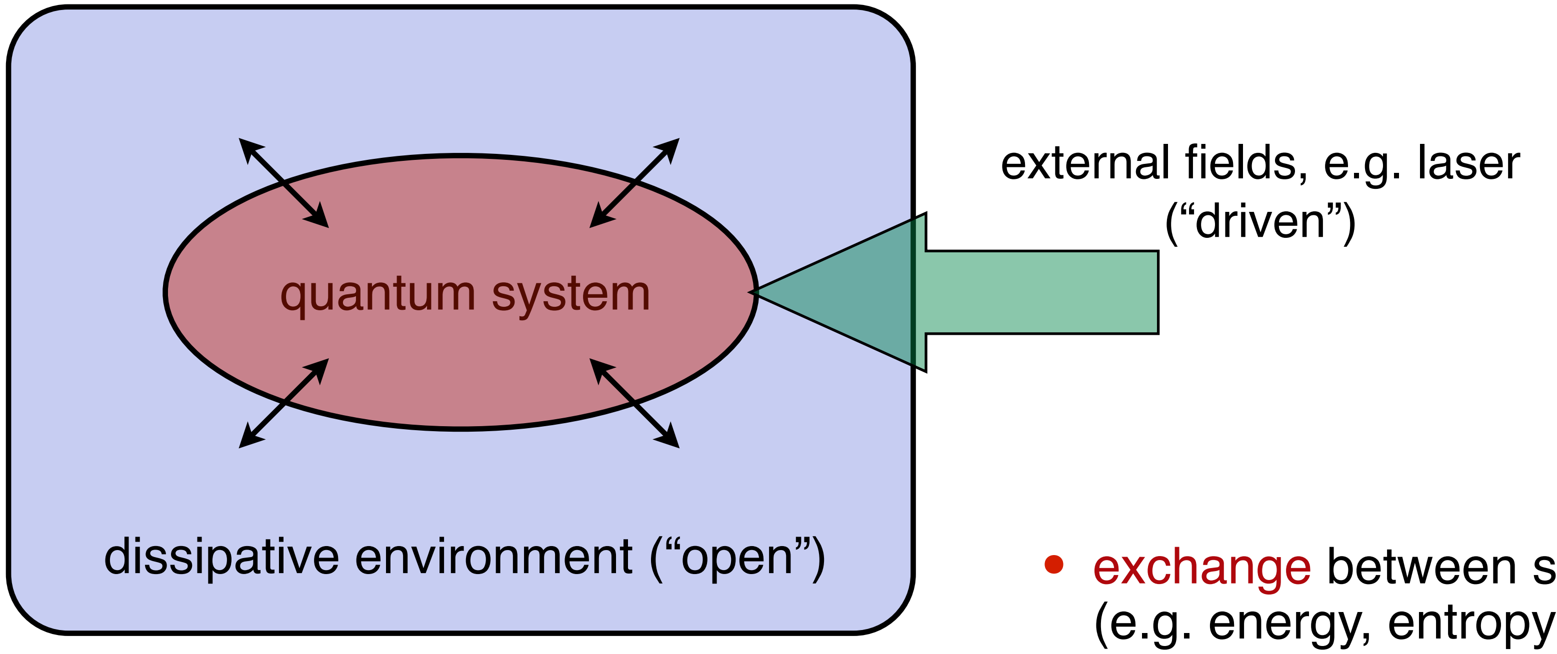
Lindblad quantum master equation: From few to many degrees of freedom



$$\partial_t \rho = -i[H,\rho] + \kappa \sum_i L_i \rho L_i^\dagger - \tfrac{1}{2}\{L_i^\dagger L_i, \rho\}$$

What is a driven open quantum system?

- quantum optics:



- exchange between system and bath
  (e.g. energy, entropy, particle number)

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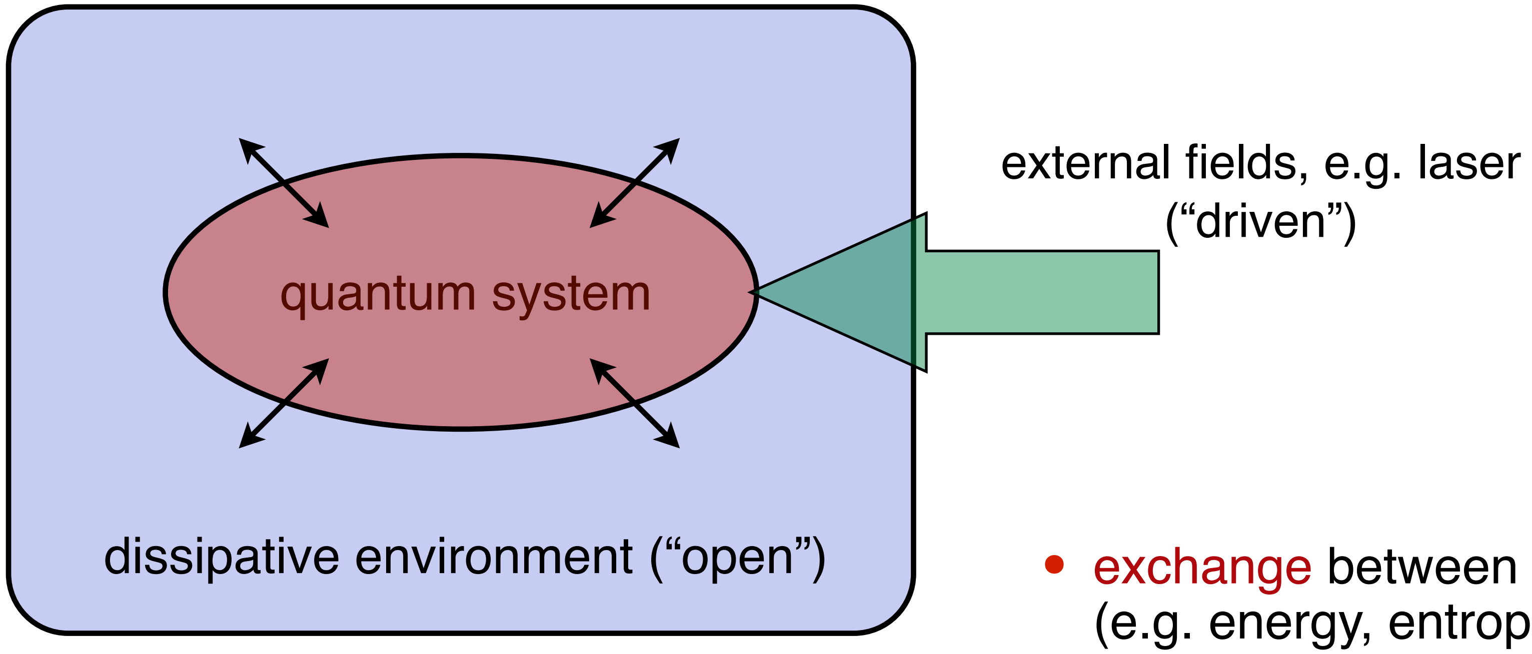


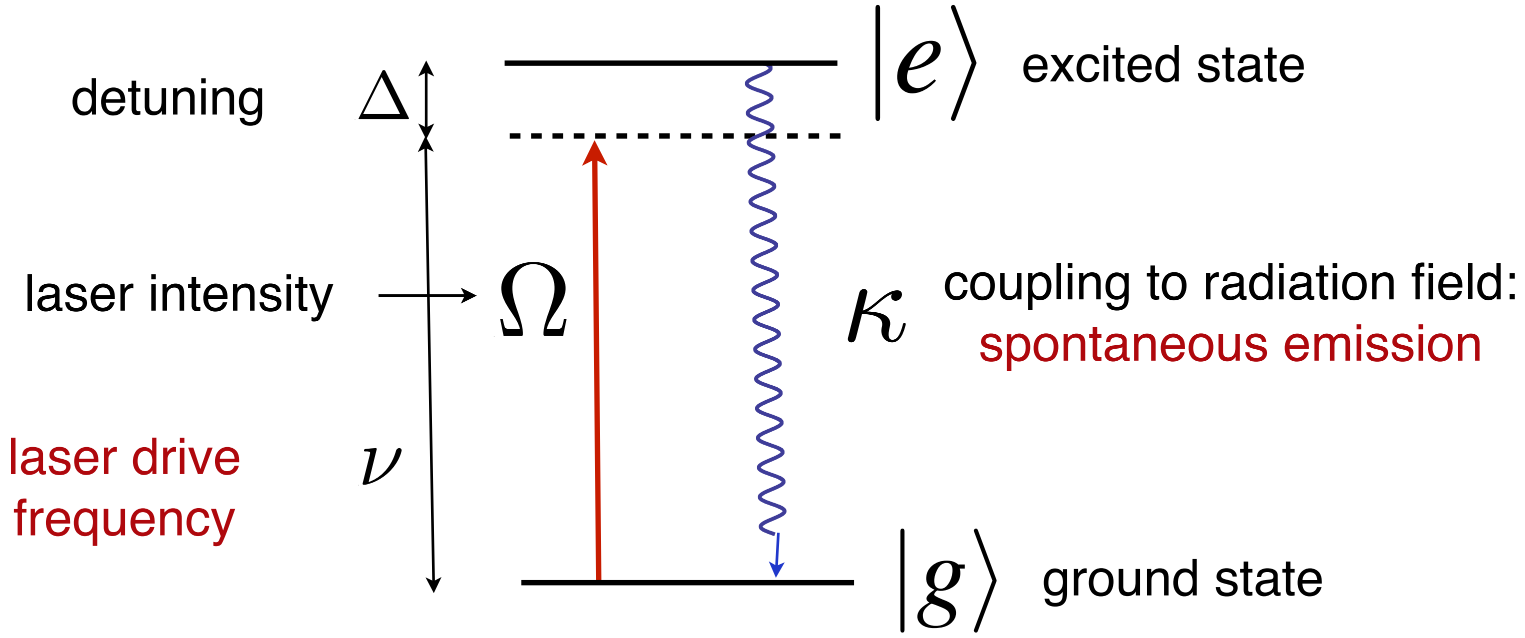
quantum system

dissipative environment (“open”)

external fields, e.g. laser (“driven”)

- exchange between system and bath (e.g. energy, entropy, particle number)

- example: laser driven atom coupled to the radiation field (two-level system)



detuning Δ

$|e\rangle$ excited state

laser intensity → Ω

κ coupling to radiation field: spontaneous emission

laser drive frequency ν

$|g\rangle$ ground state

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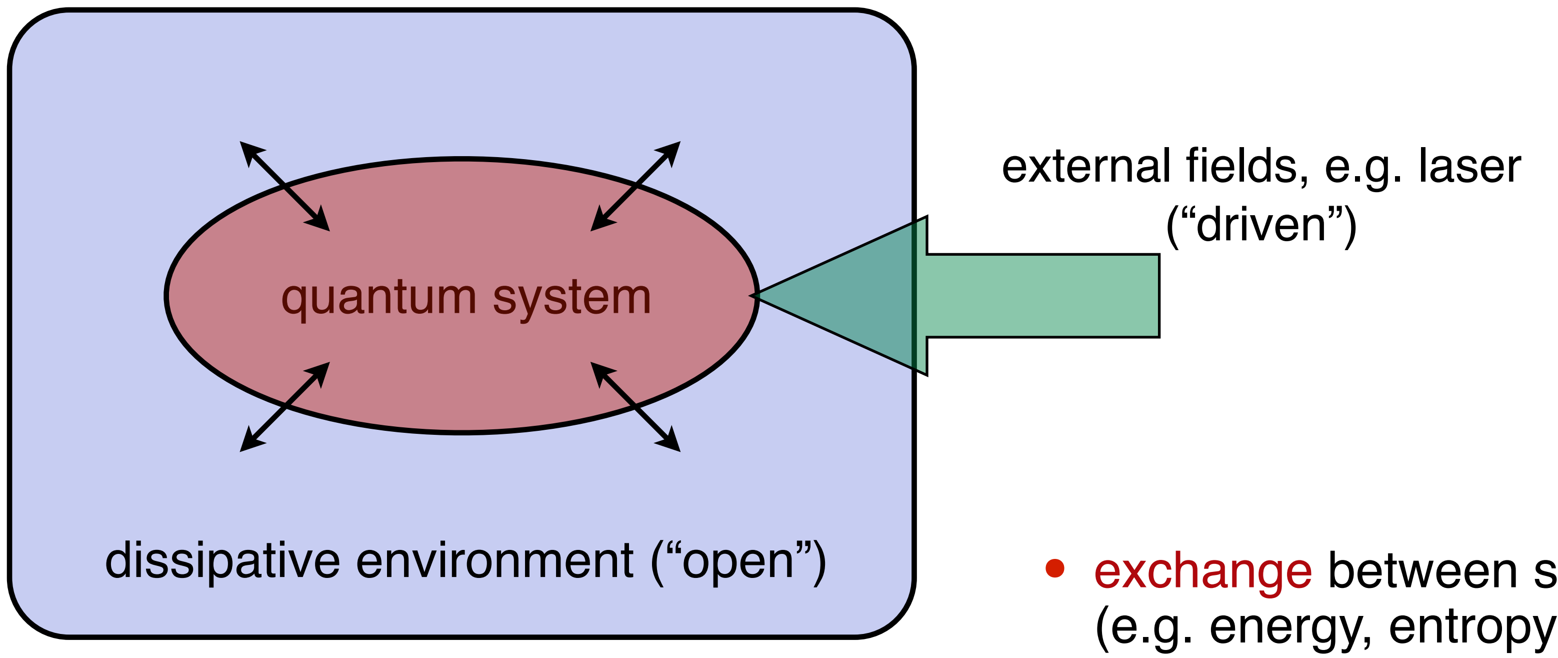

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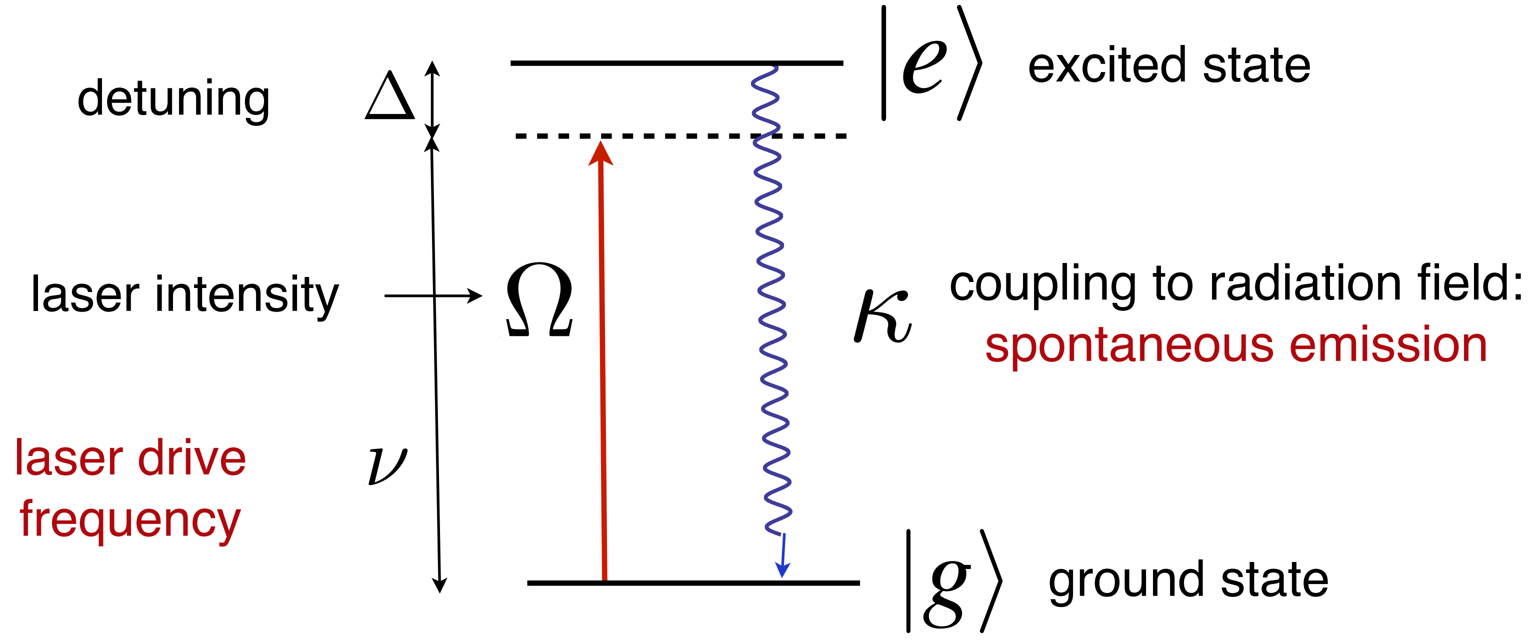

detuning Δ

$|e\rangle$ excited state

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κ coupling to radiation field:
spontaneous emission

laser drive
frequency ν

$|g\rangle$ ground state

- simple fact: drive essential to access
  upper level

- implications:

  - no minimisation of energy
  - no guarantee for detailed balance
  - no obedience of the second law of
    thermodynamics (state purification)

Driven open quantum systems: microscopic description

- quantum master equation

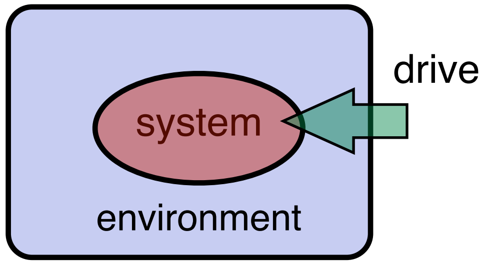
Lindblad operators

$$\partial_t \hat{\rho} = \underbrace{-i\left[\hat{H},\hat{\rho}\right]}_{\text{coherent evolution}} + \sum_i \underbrace{\gamma_i[2\hat{L}_i\hat{\rho}\hat{L}_i^\dagger - \hat{L}_i^\dagger\hat{L}_i\hat{\rho} - \hat{\rho}\hat{L}_i^\dagger\hat{L}_i]}_{\text{driven-dissipative evolution}}$$

lattice site, spin …

$$\equiv \hat{\mathcal{L}}[\hat{\rho}]$$

Lindbladian; also: Liouvillian



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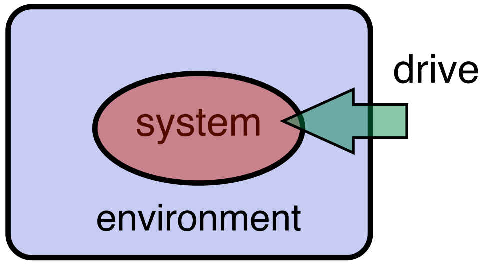

- derivation from system-bath setting: second order time dependent perturbation theory

- starting point: system-bath setting

$$\hat{H}_{\mathrm{t}} = \hat{H} + \hat{H}_{\mathrm{b}} + \hat{H}_{\text{s-b}}$$

$$\hat{H}_{\mathrm{b}} = \sum_\mu \epsilon_\mu \hat{b}_\mu^\dagger \hat{b}_\mu$$

$$\hat{H}_{\text{s-b}} = \sum_\mu g_\mu e^{-i\nu t}\, \hat{L}\, \hat{b}_\mu^\dagger + \mathrm{h.\ c.}$$

(for single L)

- bath: infinitely many mode modes than system

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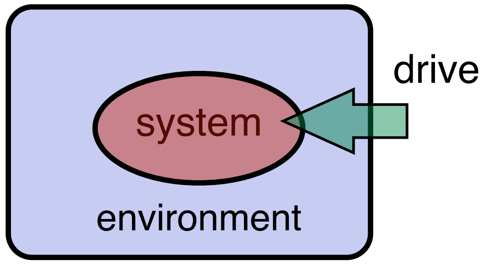

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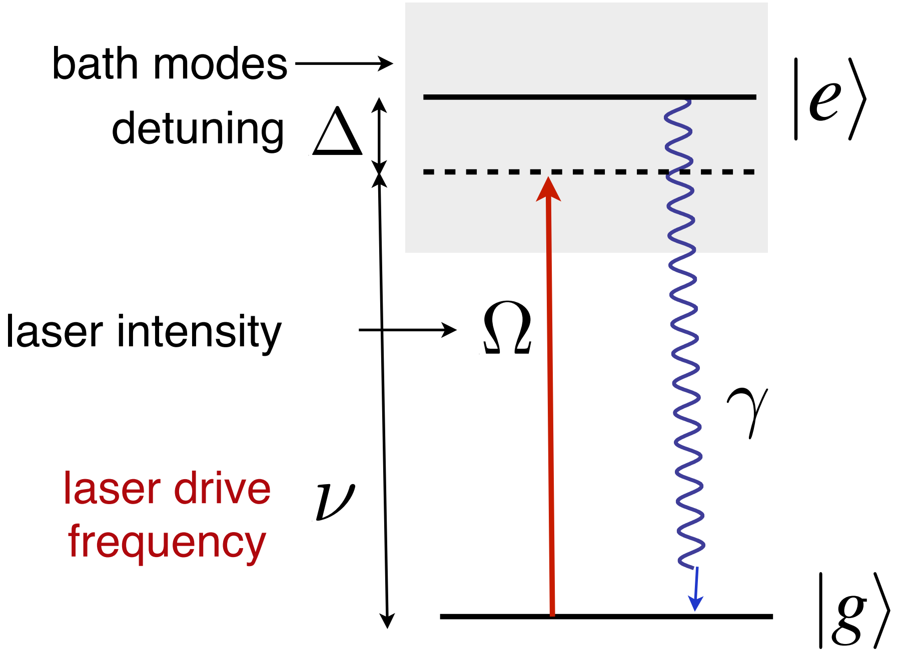
- example: two-level system



$$H = (|e\rangle, |g\rangle) \begin{pmatrix} \Delta & \Omega \\ \Omega & 0 \end{pmatrix} \begin{pmatrix} \langle e| \\ \langle g| \end{pmatrix}$$

$$\hat{L} = |g\rangle\langle e| = \sigma^-$$

$$g_\mu \sim \Omega$$

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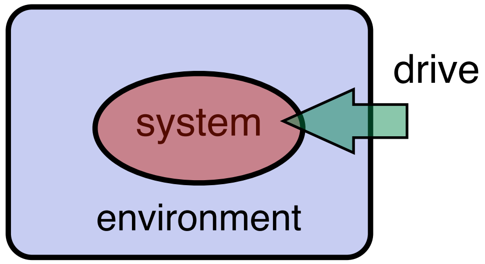

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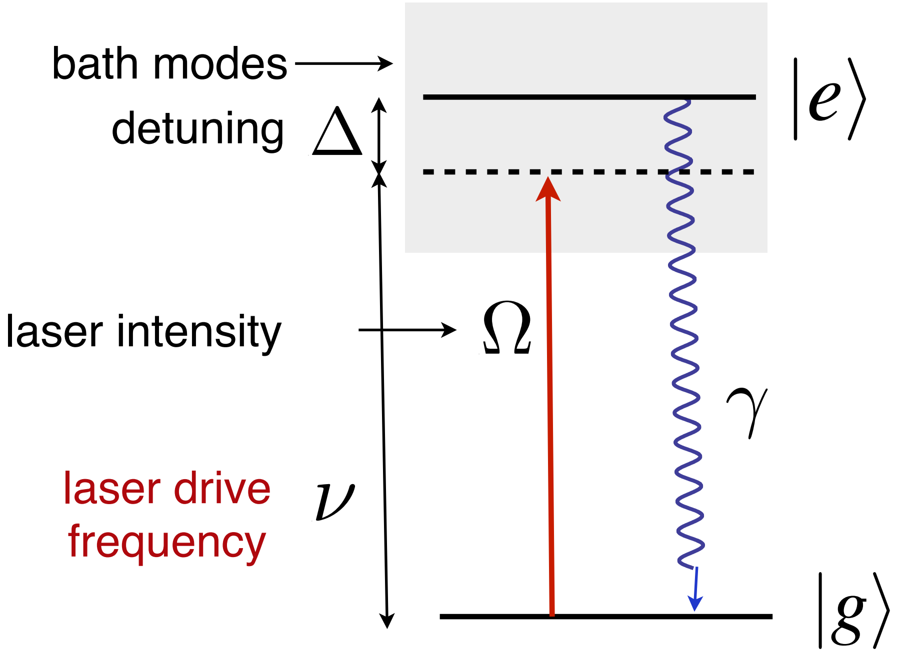

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(for single L)

- bath: infinitely many mode modes than system

- 3 approximations:
  - Born: weak system-bath coupling -> bath state unaffected by system (2nd order pert. th.)
  - Markov: system evolution slow wrt bath -> time-local evolution
  - rotating wave: drive ν selects relevant energy regimes

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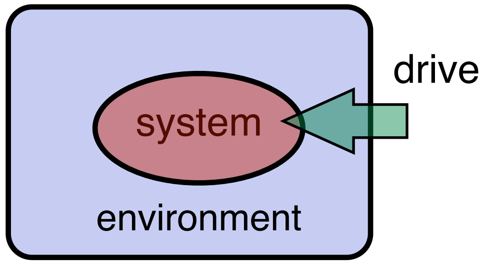

$$\equiv \hat{\mathcal{L}}[\hat{\rho}]$$ Lindbladian; also: Liouvillian

- derivation from ‘symmetry’ (i.e. implementing key physical requirements)
  - Lindbladian defines a dynamical map $\hat{\rho}(t + \Delta t) = \hat{\rho}(t) + \Delta t \cdot \hat{\mathcal{L}}[\hat{\rho}]$
  - with properties
    - Hermiticity: $\hat{\rho}(t)^\dagger = \hat{\rho}(t) \implies \hat{\rho}^\dagger(t + \Delta t) = \hat{\rho}(t + \Delta t)$ since $\hat{\mathcal{L}}[\hat{\rho}]^\dagger = \hat{\mathcal{L}}[\hat{\rho}]$
    - complete positivity: $\hat{\rho}(t) \geq 0 \implies \hat{\rho}(t + \Delta t) \geq 0$
    - trace preservation / probability conservation $\partial_t \mathrm{tr} \hat{\rho}(t) = 0$ since $\mathrm{tr} \hat{\mathcal{L}}[\hat{\rho}] = 0$

Göran Lindblad

➡ up to a unitary transformation (above: diagonal form in index i), $\hat{\mathcal{L}}[\hat{\rho}]$is the most general time-local generator with these properties

G. Lindblad, Commun. Math. Phys. (1976)

Nielsen & Chuang, Chap. 8

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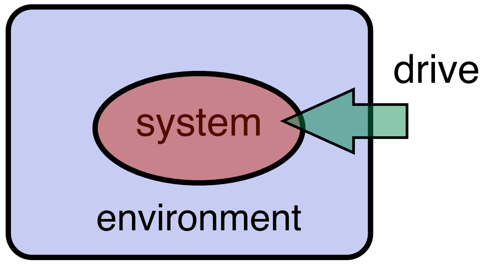

- interpretation:

$$\partial_t \hat\rho = -i(\hat H - i\sum_i \gamma_i \hat L_i^\dagger \hat L_i)\hat\rho + \mathrm{h.c.} + 2\sum_i \gamma_i \hat L_i \hat\rho \hat L_i^\dagger$$

energy

decay (dissipation)

ensures probability conservation
(fluctuation)

“$E - i\Gamma$”

$$\partial_t \mathrm{tr}\hat\rho(t) = 0$$

Driven open quantum systems: microscopic description

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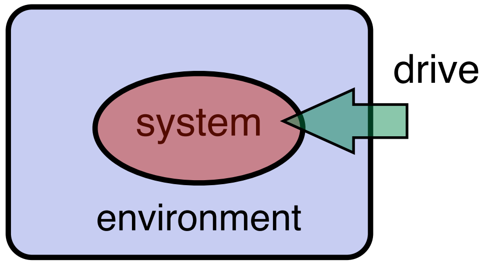

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- So far: few degrees of freedom in the “system”
- Question: What if we replace few by many degrees of freedom?

➡ The interface of quantum optics and many-body physics

➡ Quantum Optics:
coherent and driven-dissipative dynamics on equal footing

➡ Many-Body Physics:
continuum of spatial degrees of freedom

➡ Statistical Mechanics:
physics at the largest distances

microphysics → macrophysics

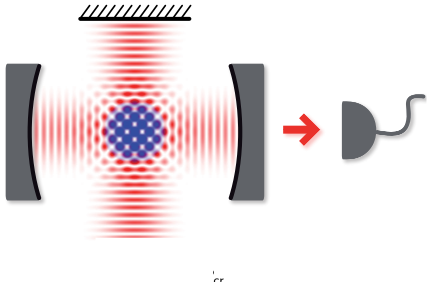
The interface of quantum optics and many-body physics

➡ Quantum Optics ➡ Many-Body Physics ➡ Statistical Mechanics

microphysics → macrophysics

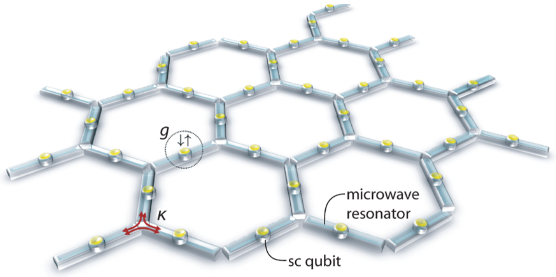
- The experimental platforms: light-matter systems realize driven open quantum matter

Atoms



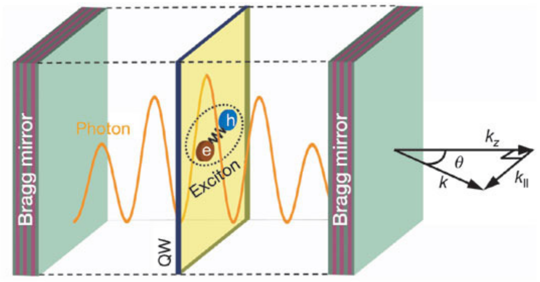
Bose-Einstein condensate in a cavity
Baumann et al., Nature (2010)

Light



Microcavity arrays
Houck, Türeci, Koch, Nat. Phys. (2012)

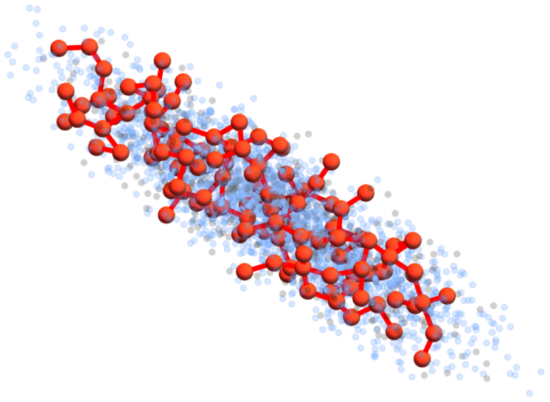
Solids



Exciton-polariton condensates
Kasprzak et al., Nature (2006)

and more:

- polar molecules
- nano-mechanics
- photon BECs
- driven quantum dots



driven-dissipative Rydberg gases

S. Helmrich, A. Arias, G. Lochhead, M. Buchhold, SD, S. Whitlock, Nature (2020); T. Wintermantel, … SD, S. Whitlock, Nat. Comm. (2021)

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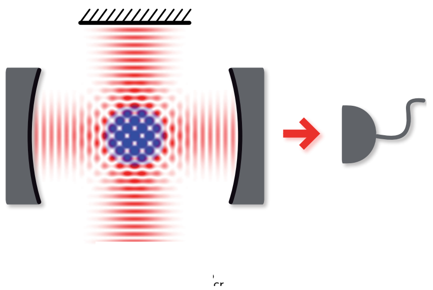

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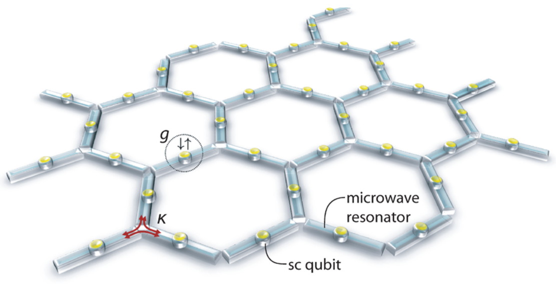

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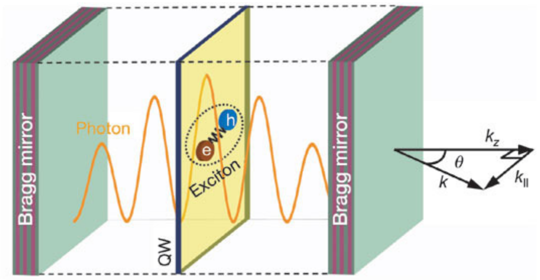

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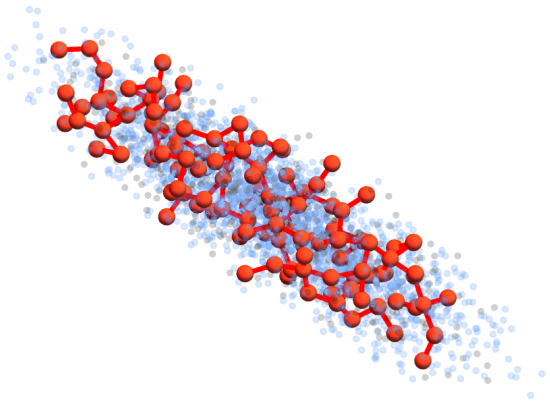

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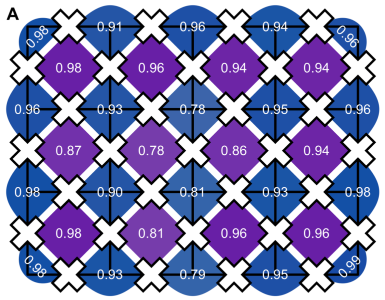
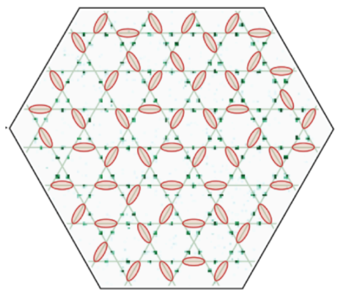
Quantum devices / NISQ Platforms



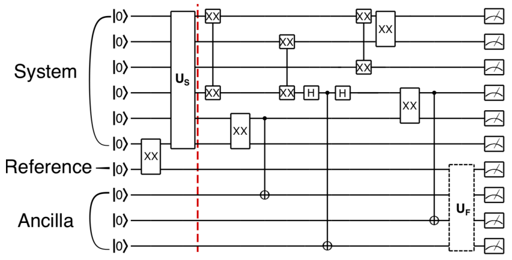
superconducting circuits

K. Satzinger et al. Science (2021)



Rydberg tweezers

G. Semeghini et al. Science (2021)



trapped ions

C. Noel et al. Nat. Phys. (2022)

The interface of quantum optics and many-body physics

➡ Quantum Optics ➡ Many-Body Physics ➡ Statistical Mechanics

microphysics → macrophysics

- The experimental platforms: light-matter systems realize driven open quantum matter

Atoms

Light

Solids

and more:

challenges to theory:

➡ novel out-of-equilibrium phenomena?

➡ efficient theoretical tools?

➡ experimental platforms?

- polar molecules
- nano-mechanics
- photon BECs
- driven quantum dots

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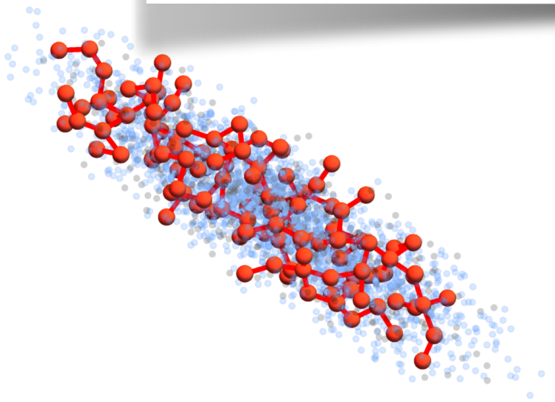

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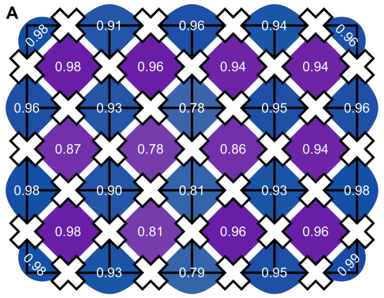

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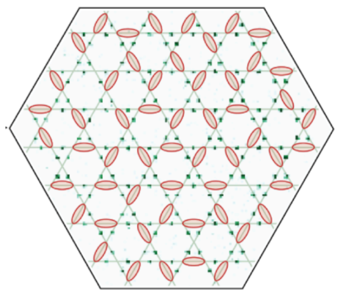
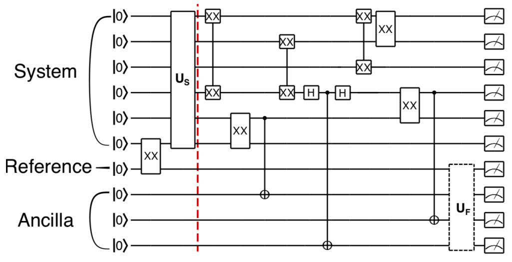


Rydberg tweezers

G. Semeghini et al. Science (2021)



trapped ions

C. Noel et al. Nat. Phys. (2022)

A workhorse model: Lindbladian formulation

- generic microscopic many-body model: ‘Lindblad ϕ^4 theory’

$$\partial_t \rho = -i[H, \rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho]$$

many-body system

$$H = \int_{\mathbf{x}} \hat{\phi}^\dagger_{\mathbf{x}} \, (-\frac{\triangle}{2M} - \mu) \, \hat{\phi}_{\mathbf{x}} + \frac{\lambda}{2} (\hat{\phi}^\dagger_{\mathbf{x}} \hat{\phi}_{\mathbf{x}})^2$$

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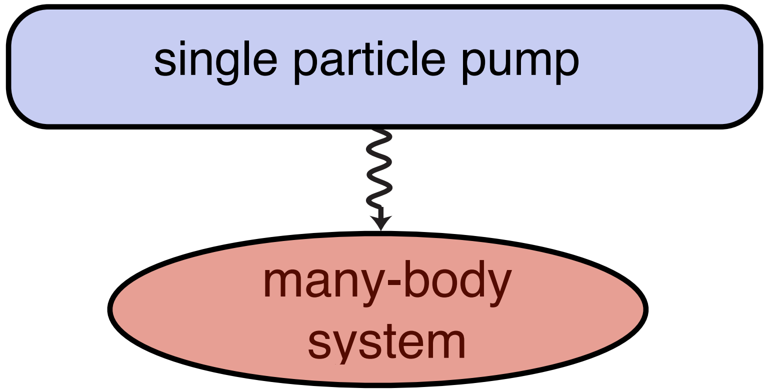

single particle pump

many-body system

$$\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho]$$

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$$\mathcal{D}[\rho] = \gamma_p \int_{\mathbf{x}} \underbrace{[\hat{\phi}^\dagger_{\mathbf{x}} \, \rho \, \hat{\phi}_{\mathbf{x}} - \frac{1}{2}\{\hat{\phi}_{\mathbf{x}} \hat{\phi}^\dagger_{\mathbf{x}}, \rho\}]}_{\text{single particle pump}} \quad +$$

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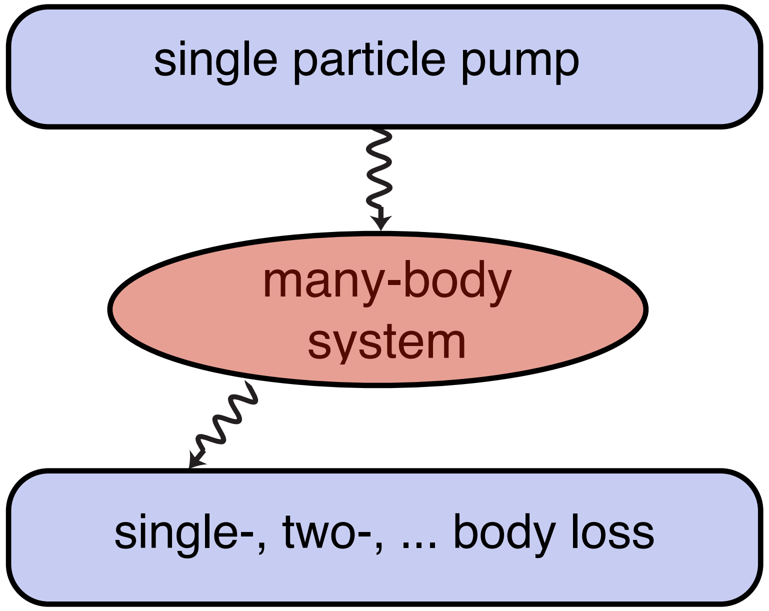

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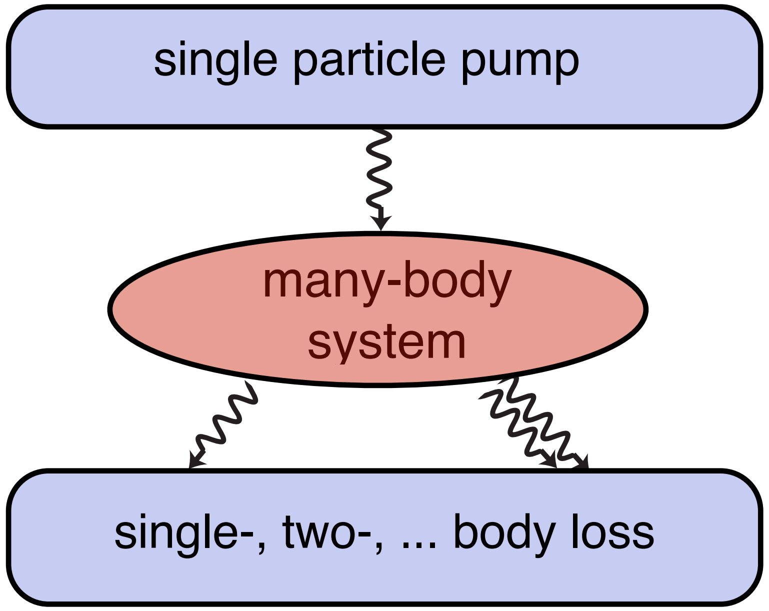

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single particle pump

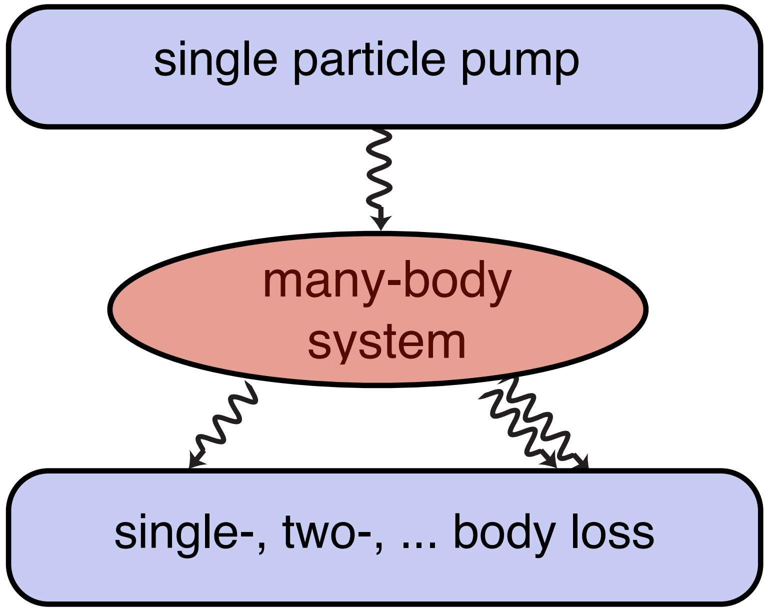
single particle loss

$$\kappa \int_{\mathbf{x}} [\hat{\phi}^2_{\mathbf{x}}\, \rho\, \hat{\phi}^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^{\dagger\,2}_{\mathbf{x}}\hat{\phi}^2_{\mathbf{x}}, \rho\}]$$

two particle loss

A workhorse model: Lindbladian formulation

- generic microscopic many-body model: ‘Lindblad ϕ^4 theory’



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single particle pump

single particle loss

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two particle loss

- basic physics: mean field theory
  - study evolution of $\phi(\mathbf{x},t) \equiv \langle\hat\phi(\mathbf{x})\rangle(t) = \mathrm{tr}[\hat\phi(\mathbf{x})\hat\rho(t)]$ $\quad \hat\rho = \prod_{\mathbf{x}} \hat\rho(\mathbf{x}), \quad \hat\rho(\mathbf{x}) = |\phi(\mathbf{x})\rangle\langle\phi(\mathbf{x})|$

coherent state

A workhorse model: Lindbladian formulation

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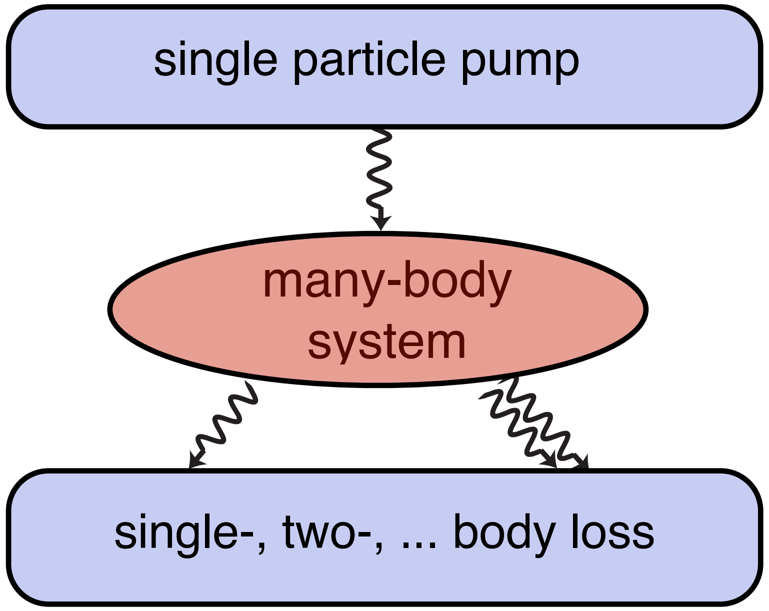

single particle pump

many-body system

single-, two-, ... body loss

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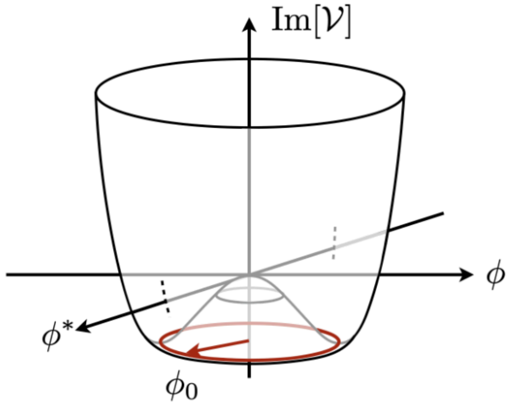
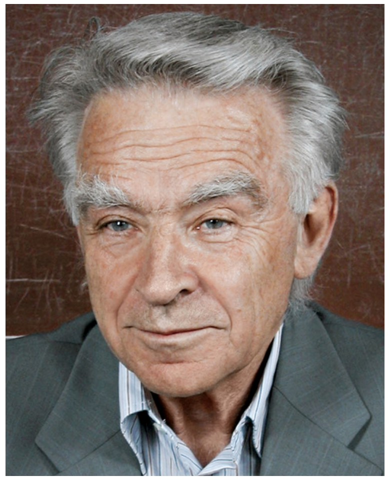
  - homogenous limit $\phi(\mathbf{x},t) = \phi(t)$

$$i\partial_t \phi(t) = [-\mu - i(\gamma_l - \gamma_p) + (\lambda - i\kappa)|\phi(t)|^2]\phi(t)$$

➡ overdamped motion in potential landscape

➡ condensation / spontaneous U(1) symmetry breaking for $\gamma_l - \gamma_p < 0$





Leonid W. Keldysh

Keldysh functional integral for stationary states of driven open quantum systems

Many-Body Master Equation

1-1 mapping

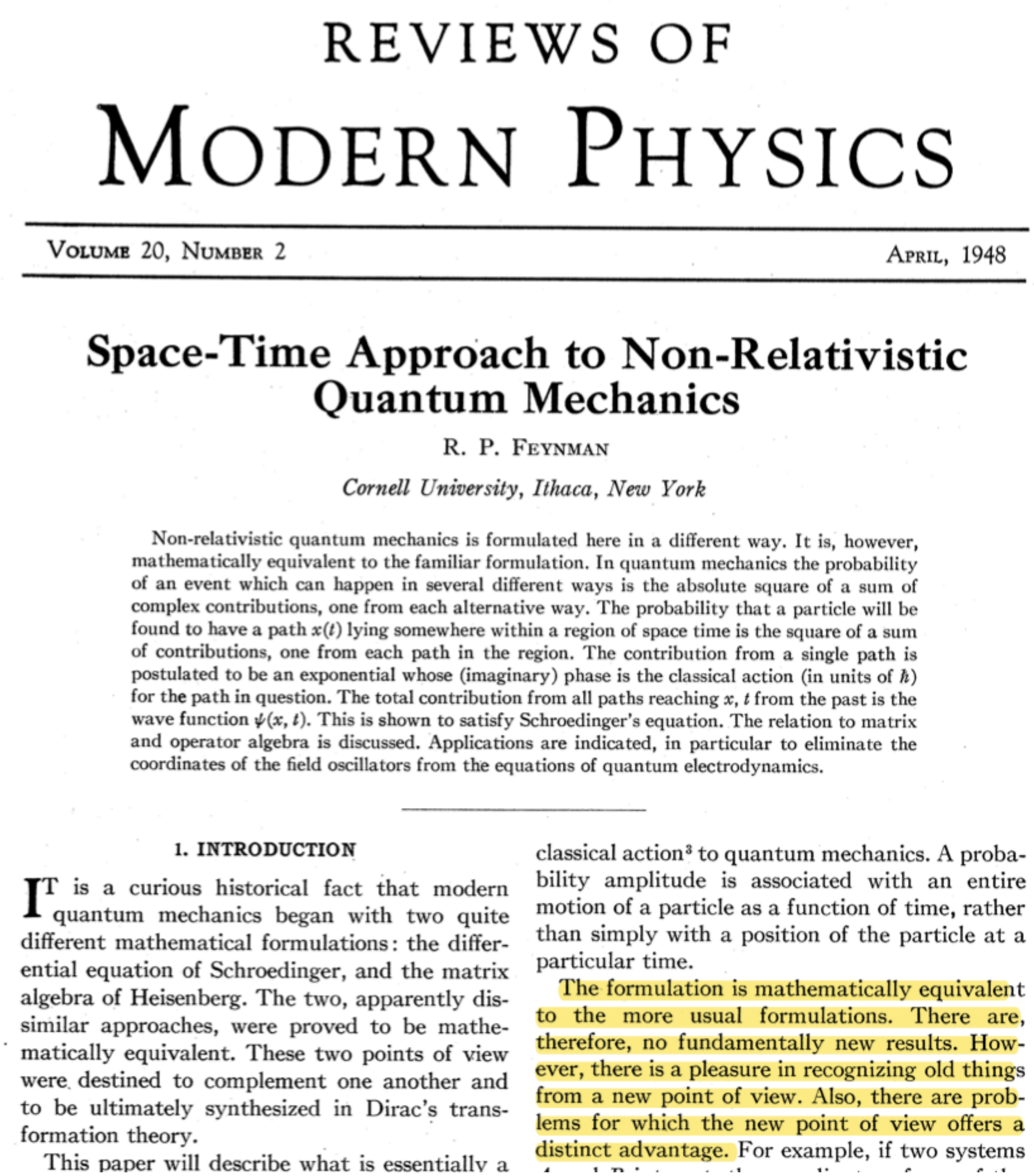
Keldysh functional integral

$$\partial_t \hat{\rho} = -i(\hat{H} - \sum_i \gamma_i \hat{L}_i^\dagger \hat{L}_i)\hat{\rho} + \mathrm{h.c.} + 2\sum_i \gamma_i \hat{L}_i \hat{\rho} \hat{L}_i^\dagger$$

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) e^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$$

Keldysh functional integrals: Why?

- Feynman's formulation of quantum mechanics



REVIEWS OF
MODERN PHYSICS

VOLUME 20, NUMBER 2 APRIL, 1948

Space-Time Approach to Non-Relativistic Quantum Mechanics

R. P. FEYNMAN

Cornell University, Ithaca, New York

Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be found to have a path $x(t)$ lying somewhere within a region of space time is the square of a sum of contributions, one from each path in the region. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of $\hbar$) for the path in question. The total contribution from all paths reaching x, t from the past is the wave function $\psi(x, t)$. This is shown to satisfy Schroedinger's equation. The relation to matrix and operator algebra is discussed. Applications are indicated, in particular to eliminate the coordinates of the field oscillators from the equations of quantum electrodynamics.

1. INTRODUCTION

IT is a curious historical fact that modern quantum mechanics began with two quite different mathematical formulations: the differential equation of Schroedinger, and the matrix algebra of Heisenberg. The two, apparently dissimilar approaches, were proved to be mathematically equivalent. These two points of view were destined to complement one another and to be ultimately synthesized in Dirac's transformation theory.

This paper will describe what is essentially a

classical action[3] to quantum mechanics. A probability amplitude is associated with an entire motion of a particle as a function of time, rather than simply with a position of the particle at a particular time.

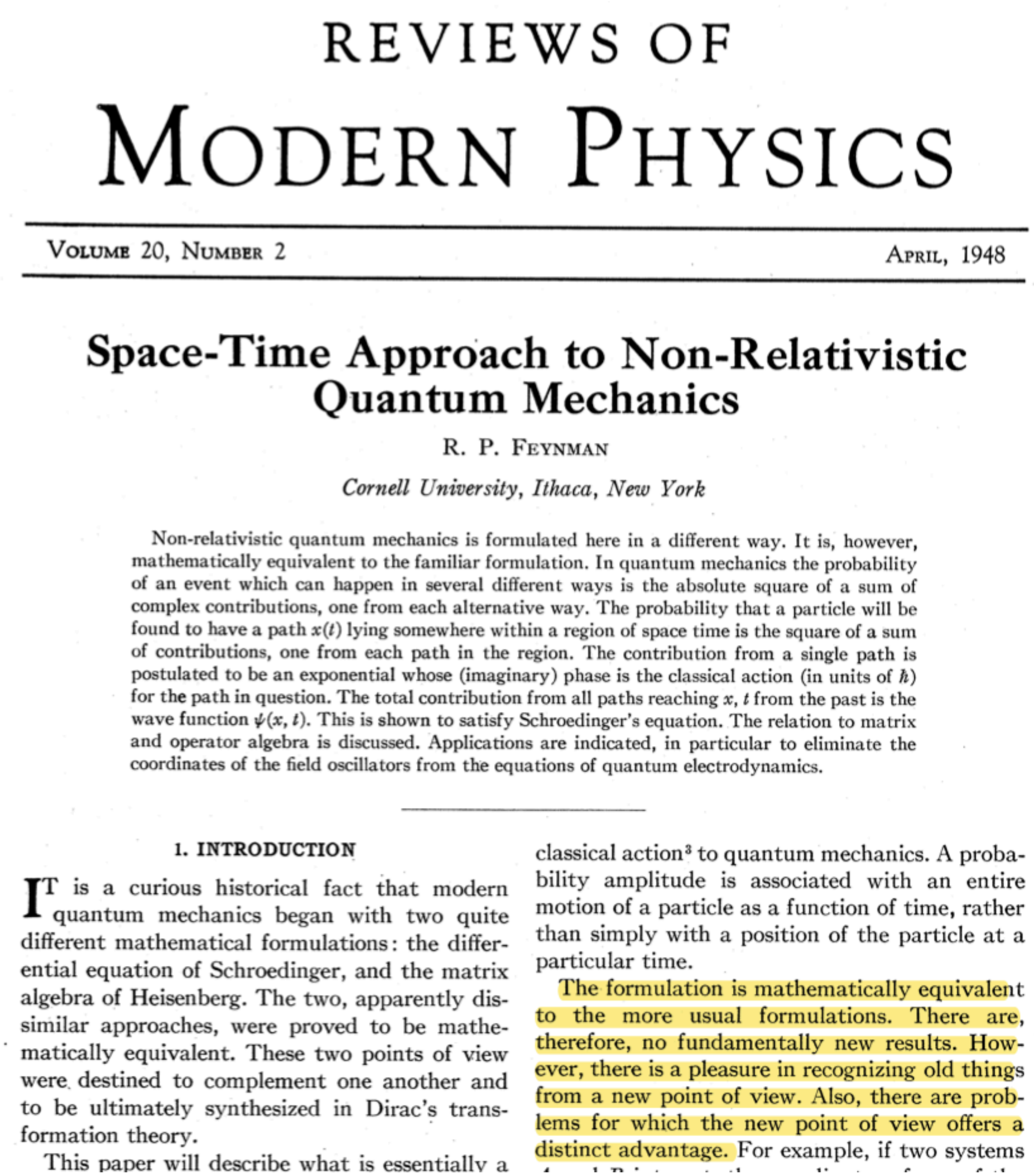
The formulation is mathematically equivalent to the more usual formulations. There are, therefore, no fundamentally new results. However, there is a pleasure in recognizing old things from a new point of view. Also, there are problems for which the new point of view offers a distinct advantage. For example, if two systems

- Useful language for systems with many degrees of freedom

- general: powerful techniques
  - diagrammatic perturbation theory;
  - collective variables;
  - renormalization group

Keldysh functional integrals: Why?

- Feynman's formulation of quantum mechanics



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- non-equilibrium Keldysh
  - closer to the real-time formulations of quantum mechanics
  - gives unified view on and principles (e.g. symmetries) for equilibrium and non-equilibrium systems
  - indispensable for many systems:
    - disorder
    - dissipation

    infinite harmonic baths!
  - open the powerful toolbox of quantum field theory for many-body non-equilibrium situations

Keldysh functional integral

more details: L. Sieberer, M. Buchhold, SD,
Keldysh Field Theory for Driven Open Quantum Systems,
Reports on Progress in Physics (2016)

- The basic idea in three steps:

$$\hbar = 1$$

$$U(t, t_0) = e^{-iH(t-t_0)}$$

1. Schrödinger equation: evolving a state vector

$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \quad \Rightarrow |\psi\rangle(t) = U(t, t_0)|\psi\rangle(t_0)$$

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$$\partial_t\rho(t) = -i[H,\rho(t)] \qquad \Rightarrow \rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0)$$

   - identical for pure (separable) states $\rho = |\psi\rangle\langle\psi|$

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3. The same is true for the Lindblad equation:

$$\partial_t\rho = -i[H,\rho] + \kappa\sum_i L_i\rho L_i^\dagger - \tfrac{1}{2}\{L_i^\dagger L_i,\rho\} \ \equiv \mathcal{L}[\rho]$$

$$\Rightarrow \rho(t) = e^{\mathcal{L}(t-t_0)}\rho(t_0)$$

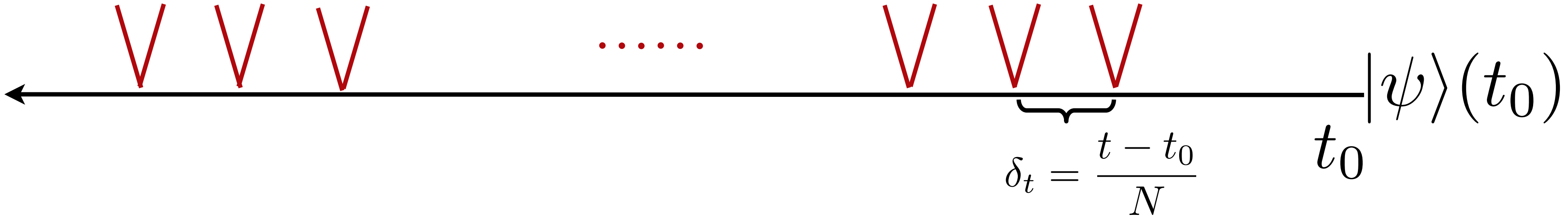
linear superoperator (acts from both sides on density matrix)

Keldysh functional integral (bosons)

fermions: L. Sieberer, M. Buchhold, J. Marino, SD, *Universality in Driven Open Quantum Matter*, arxiv (Dec 2023)

1. Functional integral idea:

➡ "Trotterization" of time interval and insertion of coherent states: $e^{iH(t-t_0)} = \lim_{N\to\infty} (1 + i\delta_t H)^N$



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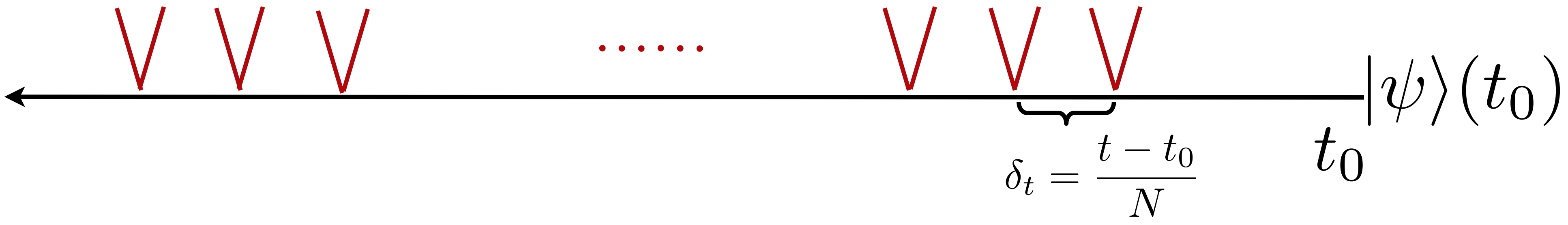

- one time step

coherent states (bosons):

$$a|\phi\rangle = \phi|\phi\rangle$$

$$\langle\phi'|\phi\rangle = e^{\phi'^*\phi}$$

$$\mathbf{1} = \int \frac{d\phi^* d\phi}{\pi} e^{-\phi^*\phi}|\phi\rangle\langle\phi|$$

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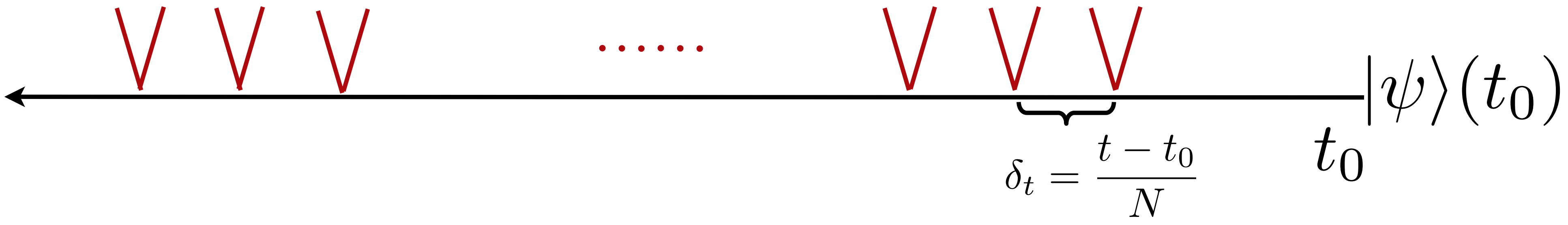

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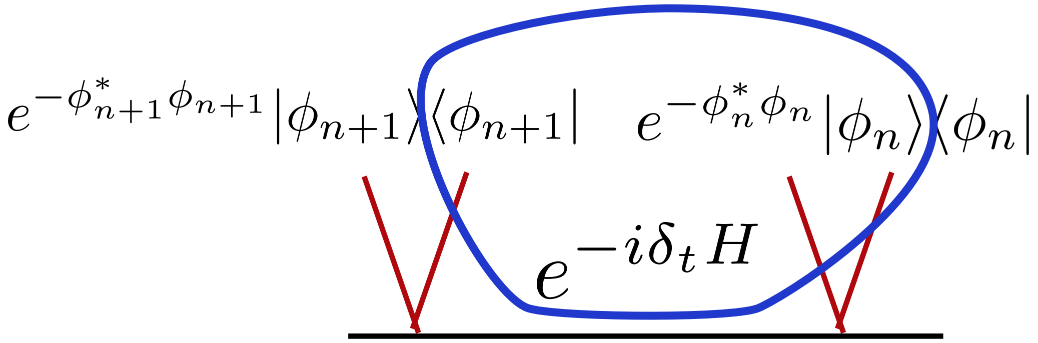

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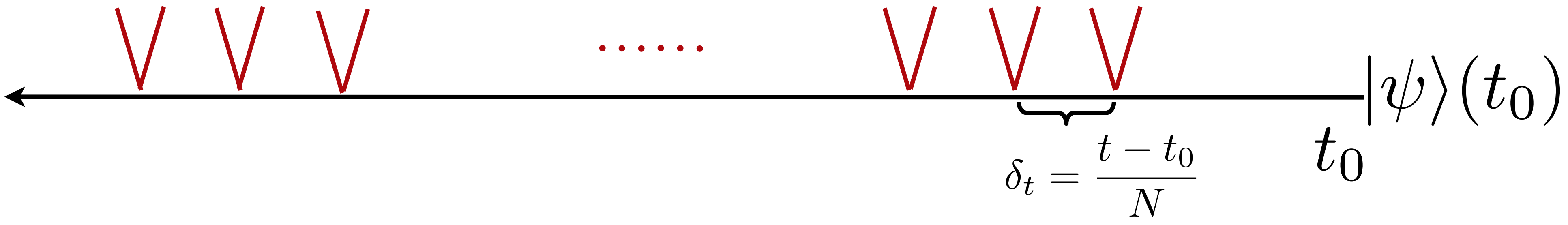
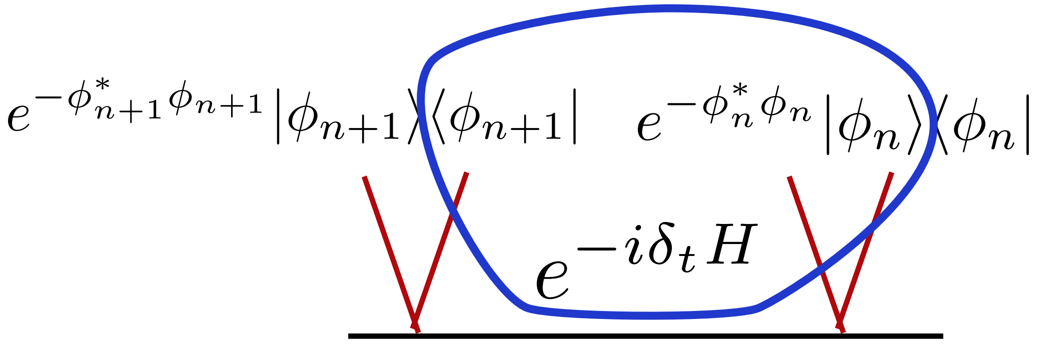


- one time step

$$e^{-\phi_n^*\phi_n}\langle\phi_{n+1}|e^{-i\delta_t H[a^\dagger,a]}|\phi_n\rangle$$



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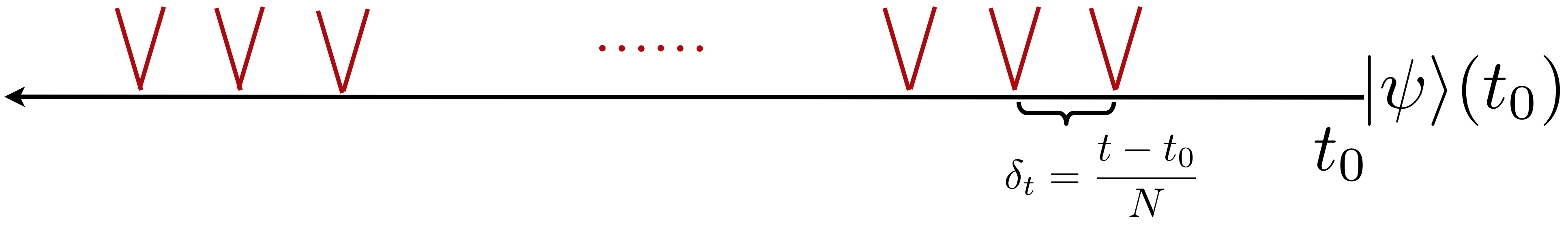

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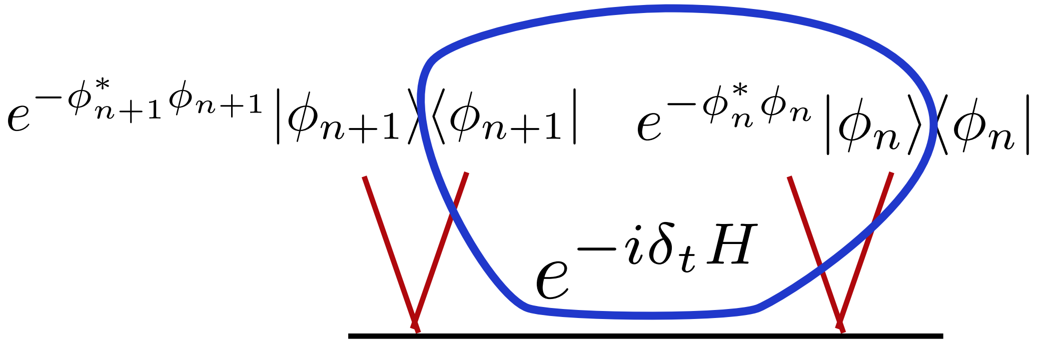

$$e^{-\phi^*_n\phi_n}\langle\phi_{n+1}|e^{-i\delta_t H[a^\dagger,a]}|\phi_n\rangle$$

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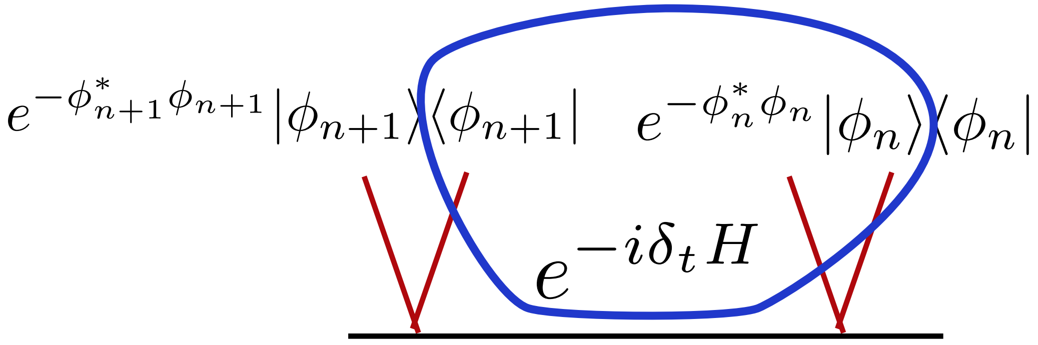

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H normally ordered

$$= e^{-\phi^*_n\phi_n}e^{+\phi^*_{n+1}\phi_n}(1-i\delta_t H[\phi^*_{n+1},\phi_n])$$

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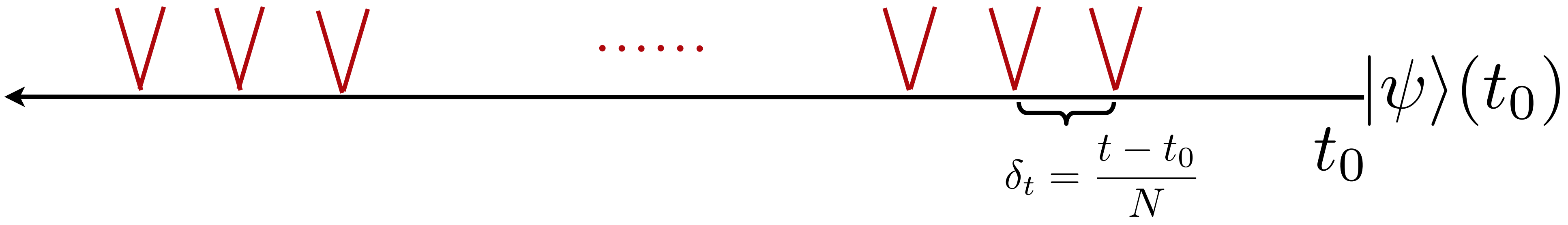

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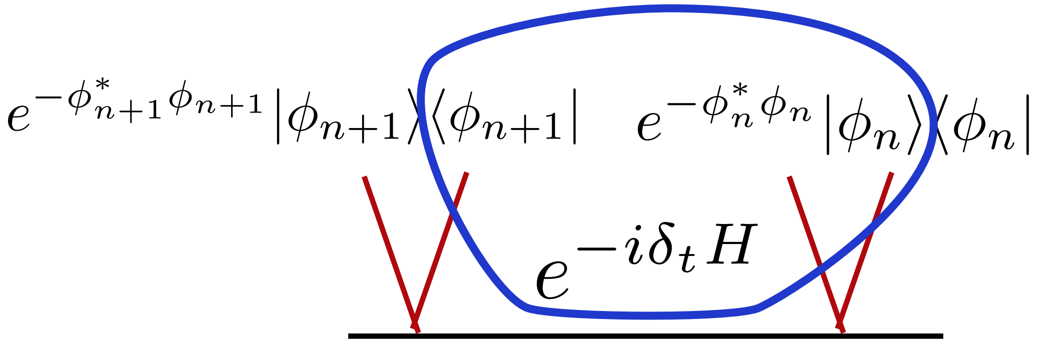

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H normally ordered

$$= e^{-\phi_n^*\phi_n}e^{+\phi_{n+1}^*\phi_n}(1-i\delta_t H[\phi_{n+1}^*,\phi_n])$$

$$\approx e^{i\delta_t[-i\frac{(\phi_{n+1}^*-\phi_n^*)}{\delta_t}\phi_n-H[\phi_{n+1}^*,\phi_n]]}$$

coherent states (bosons):

$$a|\phi\rangle = \phi|\phi\rangle$$

$$\langle\phi'|\phi\rangle = e^{\phi'^*\phi}$$

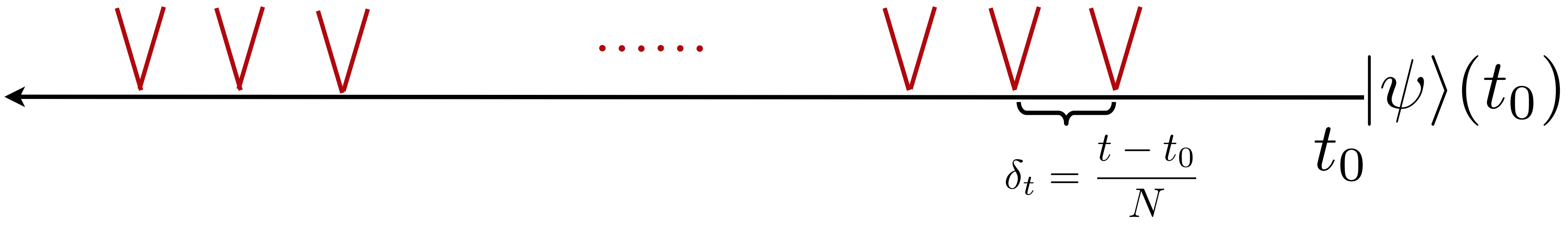
$$\mathbf{1} = \int\frac{d\phi^* d\phi}{\pi}e^{-\phi^*\phi}|\phi\rangle\langle\phi|$$

Keldysh functional integral (bosons)

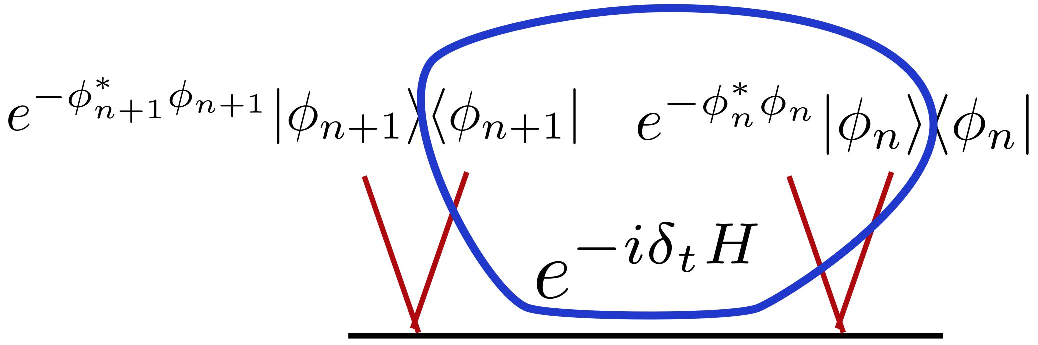
fermions: L. Sieberer, M. Buchhold, J. Marino, SD, *Universality in Driven Open Quantum Matter*, arxiv (Dec 2023)

1. Functional integral idea:

➡ “Trotterization” of time interval and insertion of coherent states: $e^{iH(t-t_0)} = \lim_{N\to\infty}(1+i\delta_t H)^N$



- one time step



$$e^{-\phi_n^*\phi_n}\langle\phi_{n+1}|e^{-i\delta_t H[a^\dagger,a]}|\phi_n\rangle$$

$$\approx e^{-\phi_n^*\phi_n}\langle\phi_{n+1}|1-i\delta_t H[a^\dagger,a]|\phi_n\rangle$$

H normally ordered

$$= e^{-\phi_n^*\phi_n}e^{+\phi_{n+1}^*\phi_n}(1-i\delta_t H[\phi_{n+1}^*,\phi_n])$$

$$\approx e^{i\delta_t[-i\frac{(\phi^*_{n+1}-\phi^*_n)}{\delta_t}\phi_n - H[\phi^*_{n+1},\phi_n]]}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$dt \quad -i\partial_t\phi^*(t)\cdot\phi(t) \qquad H[\phi^*(t),\phi(t)]$$

continuum limit

coherent states (bosons):

$$a|\phi\rangle = \phi|\phi\rangle$$

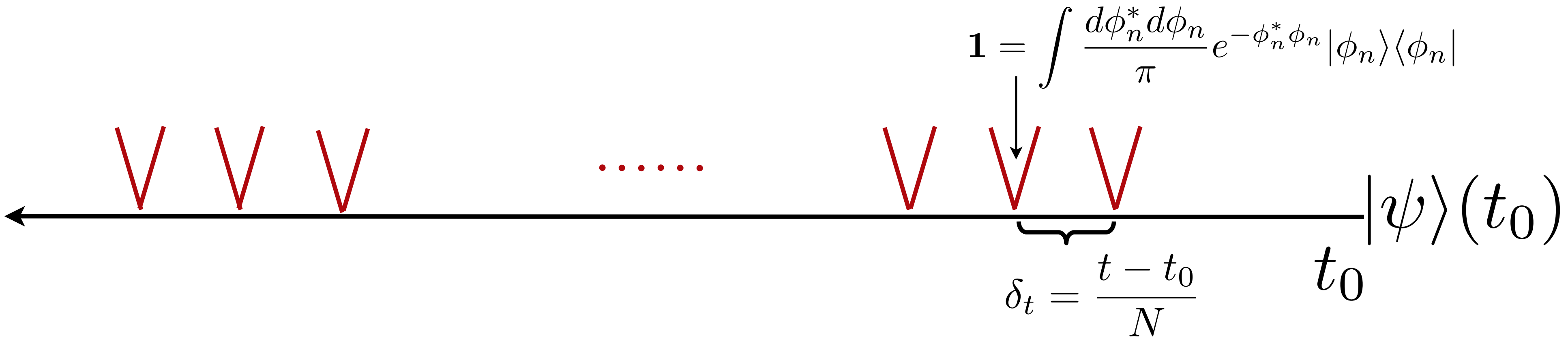
$$\langle\phi'|\phi\rangle = e^{\phi'^*\phi}$$

$$\mathbf{1} = \int\frac{d\phi^* d\phi}{\pi}e^{-\phi^*\phi}|\phi\rangle\langle\phi|$$

1. Functional integral idea (Feynman)

- "Trotterization" of the time interval and insertion of full sets of coherent states

$$e^{iH(t-t_0)} = \lim_{N\to\infty} (1 + i\delta_t H)^N$$



$$\mathbf{1} = \int \frac{d\phi_n^* d\phi_n}{\pi} e^{-\phi_n^*\phi_n} |\phi_n\rangle\langle\phi_n|$$

$$\delta_t = \frac{t - t_0}{N}$$

- continuum time limit $N \to \infty$

$$\underbrace{\int \prod_t \frac{d\phi^*(t)d\phi(t)}{\pi}}_{=: \int \mathcal{D}(\phi^*, \phi)} e^{i\int_{t_0}^{t_f} dt[-i\partial_t\phi^*(t)\cdot\phi(t) - H[\phi^*(t),\phi(t)]]}$$

functional integral measure

➡ operator H -> complex, time dependent functional H

2. Schrödinger vs. Heisenberg-von Neumann

- Schrödinger equation: evolving a state vector

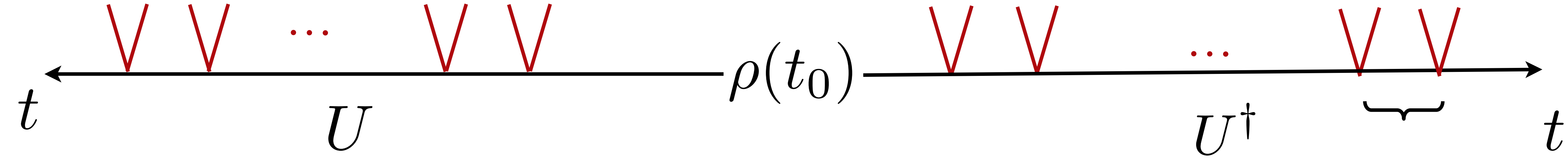
$$U(t, t_0) = e^{-iH(t-t_0)}$$

$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \quad \Rightarrow |\psi\rangle(t) = U(t, t_0)|\psi\rangle(t_0)$$

- Heisenberg-von Neumann equation: evolving a state (density) matrix

$$\partial_t\rho(t) = -i[H, \rho(t)] \quad \Rightarrow \rho(t) = U(t, t_0)\rho(t_0)U^\dagger(t, t_0)$$

- Second case: “Trotterization” on both sides:



$$e^{iH(t-t_0)} = \lim_{N\to\infty} (1 + i\delta_t H)^N \qquad \delta_t = \frac{t - t_0}{N}$$

➡ two sets of degrees of freedom for matrix evolution

3. Schrödinger vs. Lindblad

- Schrödinger equation: evolving a state vector

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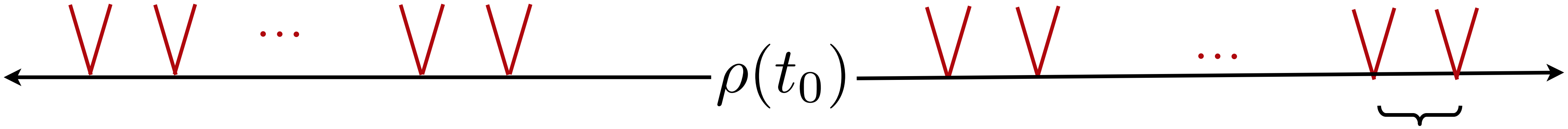
$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \qquad \Rightarrow |\psi\rangle(t) = U(t,t_0)|\psi\rangle(t_0)$$

- Lindblad equation: evolving a state (density) matrix

$$\partial_t\rho = -i[H,\rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho] \qquad \Rightarrow \rho(t) = e^{\mathcal{L}(t-t_0)}\rho(t_0)$$

- Identical program for Liouville generator of dynamics (left and right action on density matrix)



$$\rho(t) = e^{(t-t_0)\mathcal{L}}\ \rho_0 = \lim_{N\to\infty}\left(1+\delta_t\mathcal{L}\right)^N \rho_0 \qquad\qquad \delta_t = \frac{t-t_0}{N}$$

➡ two sets of degrees of freedom for matrix evolution

Keldysh partition function

- Schrödinger equation: evolving a state vector

$$U(t, t_0) = e^{-iH(t-t_0)}$$

$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \quad \Rightarrow |\psi\rangle(t) = U(t, t_0)|\psi\rangle(t_0)$$

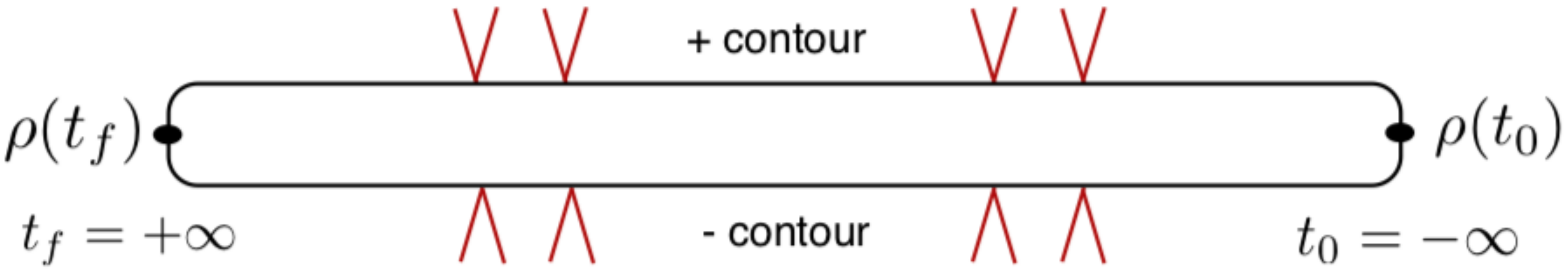
- Lindblad equation: evolving a state (density) matrix

$$\partial_t\rho = -i[H, \rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho] \qquad \Rightarrow \rho(t) = e^{\mathcal{L}(t-t_0)}\rho(t_0)$$

- final step: Keldysh “partition function”

$$Z = \mathrm{tr}\rho(t) = \mathrm{tr}\rho(t_0) = 1$$

$t_0 \to -\infty, t_f \to +\infty$



information on all stages; stationarity reached (boundary conditions irrelevant)

Detailed calculation: damped harmonic oscillator

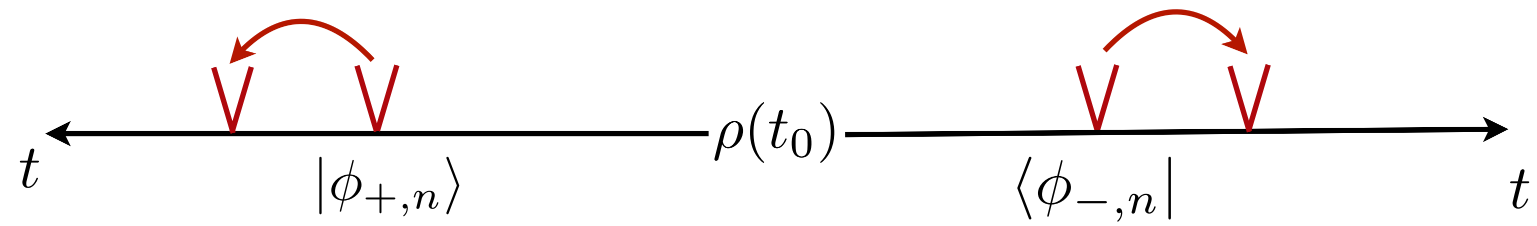
- key steps present for single degree of freedom, e.g.

$$\hat{\mathcal{L}}[\rho] = -i[\omega_0 \hat{a}^\dagger \hat{a}, \rho] + \kappa(2\hat{a}\rho\hat{a}^\dagger - \{\hat{a}^\dagger \hat{a}, \rho\})$$

$$\rho(t) = e^{(t-t_0)\mathcal{L}}\,\rho_0 = \lim_{N\to\infty}(1+\delta_t\mathcal{L})^N\,\rho_0$$

$$\delta_t = \frac{t-t_0}{N}$$

  - one time step



  - many time steps in temporal continuum limit

coherent states (bosons):

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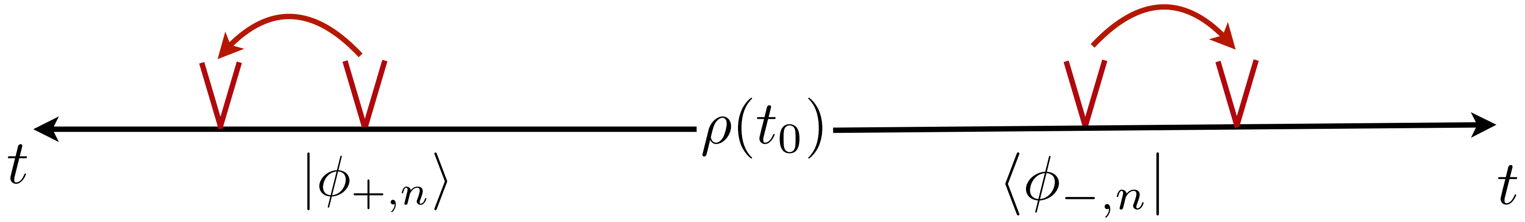

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- from a single free to many interacting degrees of freedom

$$\sigma = \pm$$

  - single d.o.f. $\phi_\sigma(t), \quad \int dt$ (0+1) dimensional field theory
  - many d.o.f.s, lattice $\phi_{\sigma,i}(t), \quad \int dt \sum_i$
  - many d.o.f.s, continuum $\phi_\sigma(t,\vec{x}), \quad \int dt d^d x$ (d+1) dimensional field theory

- works analogously for interactions (subtlety: operator ordering for non-linear L)
- left out: fermions (additional sign in odd-parity Lindblad operators)

L. Sieberer, M. Buchhold, J. Marino, SD, *Universality in Driven Open Quantum Matter*, arxiv (2023)

Keldysh functional integral for interacting many-body system

- Lindblad equation: $$\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho]$$

$$= -i(H\rho - \rho H) + \kappa \sum_i (L_i \rho L_i^\dagger - \tfrac{1}{2} L_i^\dagger L_i \rho - \tfrac{1}{2}\rho L_i^\dagger L_i)$$

- equivalent Keldysh functional integral:

$$Z = \int \mathcal{D}(\Phi_+,\Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$$

$$\Phi_\pm = \begin{pmatrix} \phi_\pm \\ \phi_\pm^* \end{pmatrix}$$

$$S_M[\Phi_+,\Phi_-] = \int dt(\phi_+^* i\partial_t \phi_+ - \phi_-^* i\partial_t \phi_- - i\mathcal{L}[\Phi_+,\Phi_-])$$

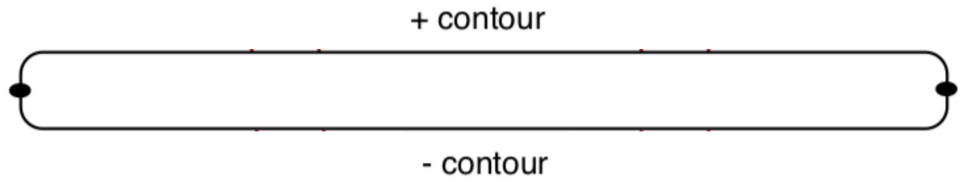
$$\boxed{\mathcal{L}[\Phi_+,\Phi_-] = -i\left(H_+ - H_-\right) - \kappa \sum_i \left(L_{i,+} L_{i,-}^\dagger - \tfrac{1}{2} L_{i,+}^\dagger L_{i,+} - \tfrac{1}{2} L_{i,-}^\dagger L_{i,-}\right)}$$

$$H_\pm = H(\Phi_\pm) \text{ etc.}$$

➡ recognize Lindblad structure

➡ simple translation table (for contour normal ordered Lindbladian)

- operator right of density matrix -> - contour
- operator left of density matrix -> + contour

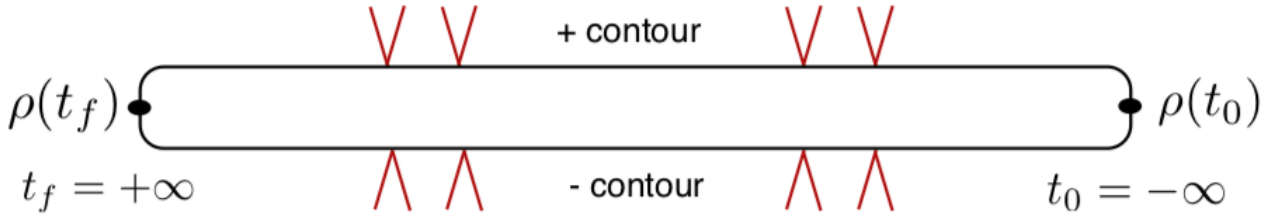


➡ can be operated on lattice or in continuum limit

➡ caveat: contour diagonal Lindblad terms need temporal regularisation to track operator ordering

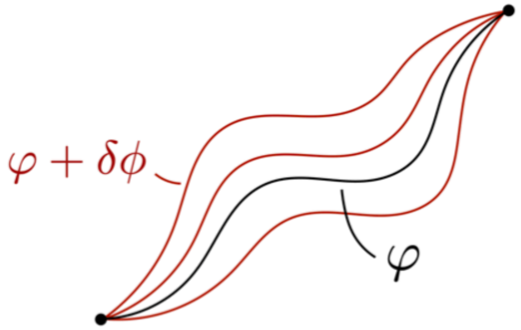
Keldysh functional integral: structural properties

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-)\mathrm{e}^{\mathrm{i}(S_{.} \quad [\Phi_+,\Phi_-]}$$



intuition for additional ± contour index of Keldysh field theory

1. Probability conservation (zero order)
2. Deterministic limit (first order)
3. Fluctuations (all orders)



1. Probability conservation / ”causality”

- trace / probability conservation:
  - Lindblad: $$\partial_t \mathrm{tr}\rho = \mathrm{tr}\big(-i(H\rho - \rho H) + \kappa \sum_i (L_i \rho L_i^\dagger - \tfrac{1}{2} L_i^\dagger L_i \rho - \tfrac{1}{2}\rho L_i^\dagger L_i)\big) = 0$$

    cyclicity
  - Keldysh: $$Z = \mathrm{tr}\rho(t) = 1 \qquad\qquad Z = \int \mathcal{D}(\Phi_+, \Phi_-)\mathrm{e}^{\mathrm{i}(S\quad[\Phi_+,\Phi_-]}$$

- will argue: reflected on the action as

$$\boxed{S[\Phi_+, \Phi_- = \Phi_+] = 0}$$

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  - redundancy

$$\langle \hat{O} \rangle (t) = \mathrm{tr}[\hat{O}\hat{\rho}(t)] = \mathrm{tr}[\hat{\rho}(t)\hat{O}]$$

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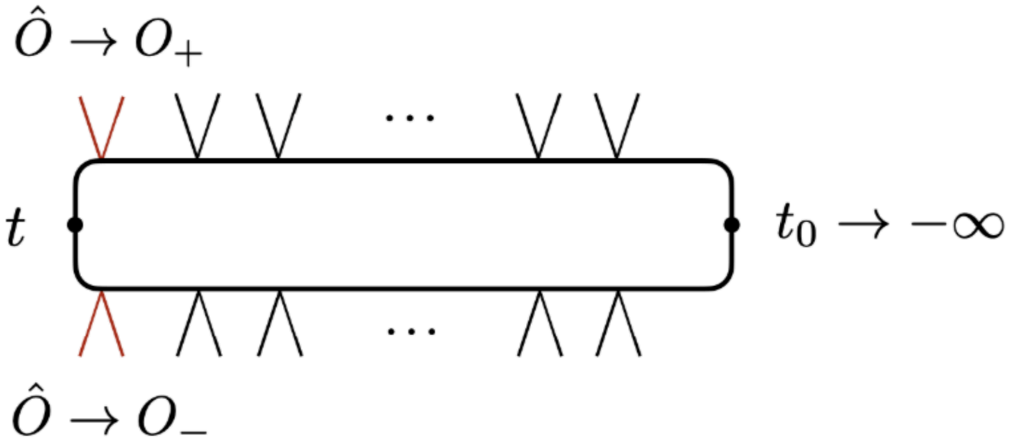
$$\langle \hat{O} \rangle (t) = \langle O_+(t) \rangle = \langle O_-(t) \rangle$$



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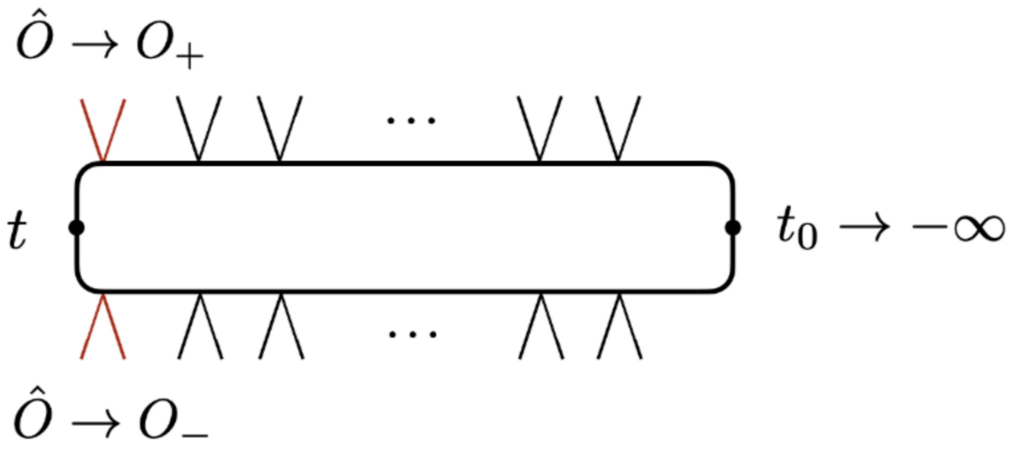

$$\Longrightarrow \; 0 = \partial_t Z = i\langle s(\Phi_+(t), \Phi_-(t)) \rangle = i\langle s(\Phi_+(t), \Phi_+(t)) \rangle \quad \forall t \qquad S = \int_{t_0}^{t} dt' \, s(\Phi_+(t'), \Phi_-(t'))$$

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- will argue: reflected on the action as

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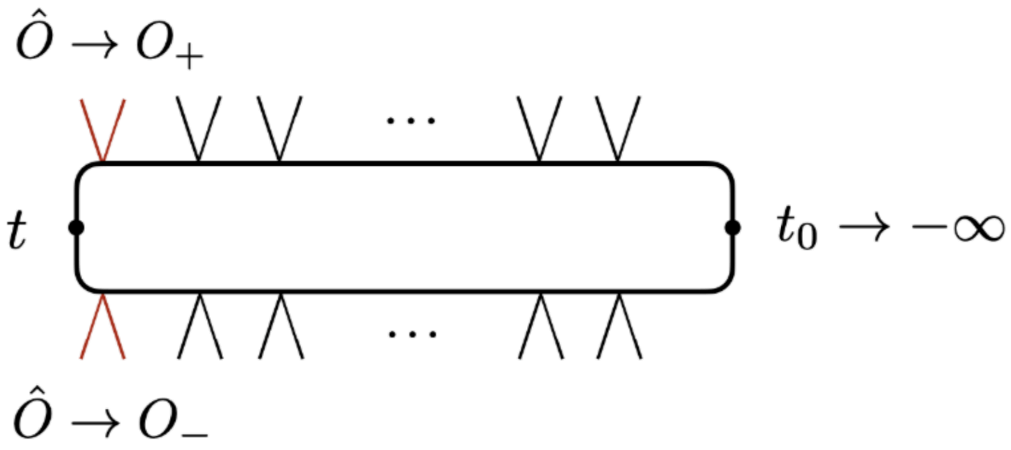

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- mnemonic: taking trace = ignoring contour order
- motivates Keldysh rotation

1. Probability conservation and Keldysh rotation

- trace / probability conservation:
  - Lindblad: $\partial_t \mathrm{tr}\rho = \mathrm{tr}\big(-i(H\rho - \rho H) + \kappa \sum_i (L_i \rho L_i^\dagger - \frac{1}{2} L_i^\dagger L_i \rho - \frac{1}{2}\rho L_i^\dagger L_i)\big) = 0$

    cyclicity
  - Keldysh: $Z = \mathrm{tr}\rho(t) = 1$ $\qquad Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$
- make probability conservation more handy:
  - starting point: contour basis: $S[\Phi_+, \Phi_-]$ with $S[\Phi_+, \Phi_- = \Phi_+] = 0$
  - Keldysh rotation $\begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_+ + \phi_- \\ \phi_+ - \phi_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}$

    center-of-mass

    relative

1. Probability conservation and Keldysh rotation

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relative
  - action in Keldysh/RAK basis $S[\Phi_c, \Phi_q]$
  - probability conservation $$\boxed{S[\Phi_c, \Phi_q = 0] = 0}$$

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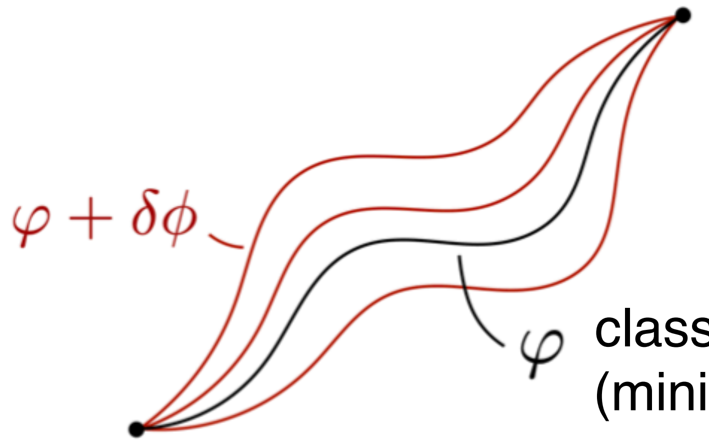
    - action in Keldysh/RAK basis $S[\Phi_c, \Phi_q]$
    - probability conservation $\boxed{S[\Phi_c, \Phi_q = 0] = 0}$
- interpretation of the fields: use $\langle \hat{\phi} \rangle = \langle \phi_+ \rangle = \langle \phi_- \rangle$
    - "classical" field can acquire expectation value (<—> condensation, spontaneous symmetry breaking) $\quad \langle \hat{\phi} \rangle = \langle \phi_c \rangle / \sqrt{2}, \quad \langle \phi_q \rangle = 0$
    - "quantum" field cannot

2. Deterministic limit



classical path
(minimizes S)

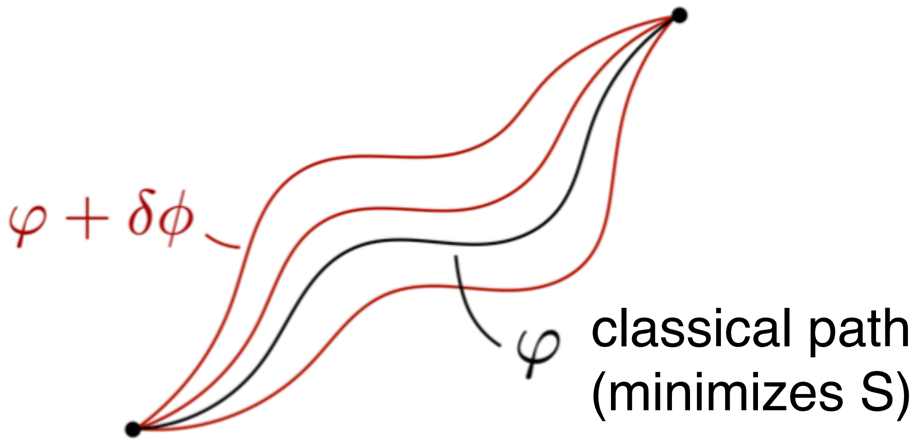
- Keldysh functional integral

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+, \Phi_-]}$$

- probability conservation: zero order in the quantum field

$$S[\Phi_c, \Phi_q = 0] = 0 \qquad Z = \mathrm{tr}\rho(t) = 1$$

2. Deterministic limit



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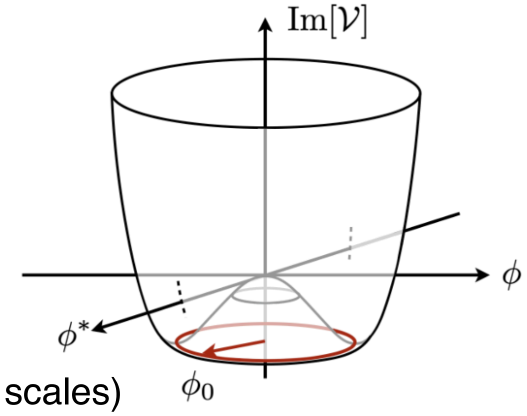


- first order in quantum field: ordering principle due to condensation
  - classical / occupation field: macroscopic occupation $N \to \infty$ (more precisely: only q=0 mode scales)

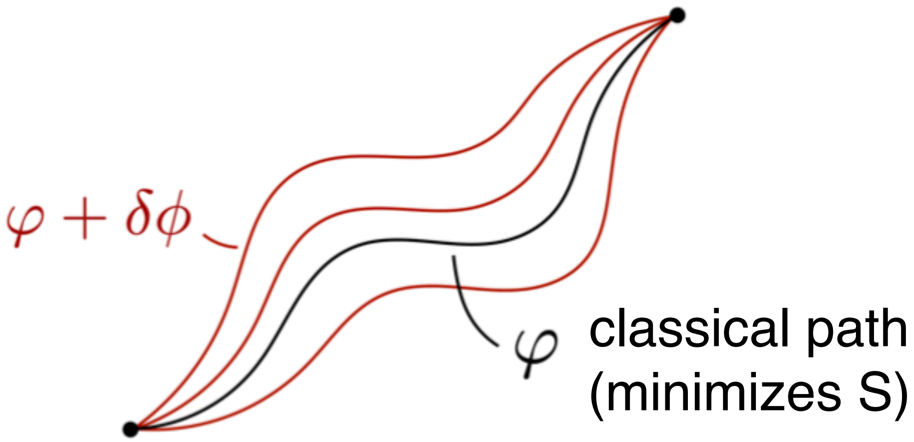
$$\phi_c(q) \sim N^{1/2} \implies \phi_c(x) \sim \frac{N^{1/2}}{V^{1/2}} \sim N^0$$

$$\phi_c(x) = V^{-1/2} \sum_q e^{-iqx} \phi_c(q)$$

  - quantum field:

$$\phi_q(q) \sim N^0 \implies \phi_q(x) \sim \frac{1}{V^{1/2}} \sim N^{-1/2}$$

2. Deterministic limit



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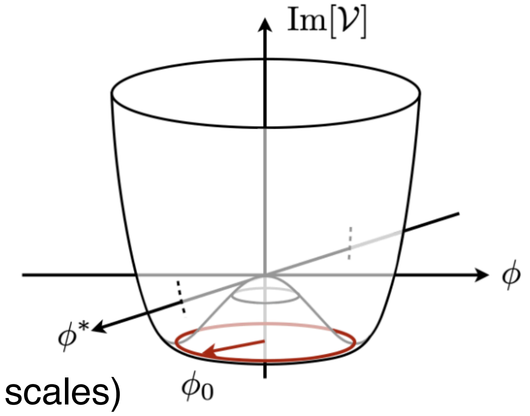


- first order in quantum field: ordering principle due to condensation
  - classical / occupation field: macroscopic occupation $N \to \infty$ (more precisely: only q=0 mode scales)

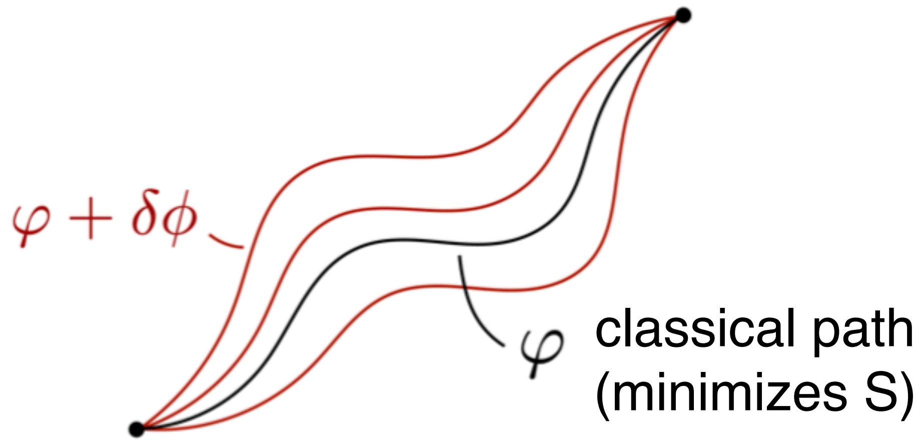
$$\phi_c(q) \sim N^{1/2} \implies \phi_c(x) \sim \frac{N^{1/2}}{V^{1/2}} \sim N^0 \qquad \phi_c(x) = V^{-1/2} \sum_q e^{-iqx} \phi_c(q)$$

  - quantum field:

$$\phi_q(q) \sim N^0 \implies \phi_q(x) \sim \frac{1}{V^{1/2}} \sim N^{-1/2}$$

- expand the action to leading order $S[\Phi_+ = (\Phi_c + \Phi_q)/\sqrt{2}, \Phi_- = (\Phi_c - \Phi_q)/\sqrt{2}] \approx \int_{\mathbf{x},t} [\phi_q \frac{\delta S}{\delta \phi_c} + \phi_q^* \frac{\delta S}{\delta \phi_c^*}]$
- can do integral over quantum field $Z = \int D[\phi_c, \phi_c] \delta \left[\frac{\delta S}{\delta \phi_c} \right] \delta \left[\frac{\delta S}{\delta \phi_c^*} \right]$ ◀

2. Deterministic limit



- Keldysh functional integral

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$$

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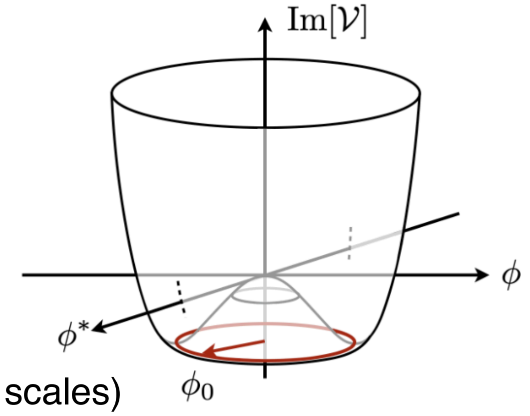

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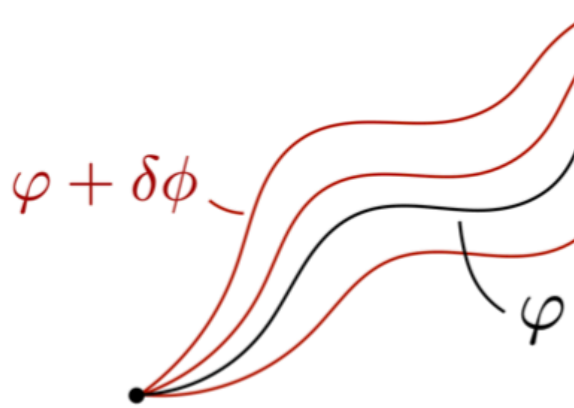
- can do integral over quantum field $Z = \int D[\phi_c, \phi_c] \delta \left[\frac{\delta S}{\delta \phi_c}\right] \delta \left[\frac{\delta S}{\delta \phi_c^*}\right]$

- only deterministic configuration contributes with $\frac{\delta S}{\delta \phi_c} = 0 = \frac{\delta S}{\delta \phi_c^*}$

➡ deterministic limit Lindblad phi^4: dissipative Gross-Pitaevskii mean field theory

exercise: make connection to
operator mean field theory
(find factor $\sqrt{2}$:))

2. Deterministic limit



classical path
(minimizes S)

- Keldysh functional integral

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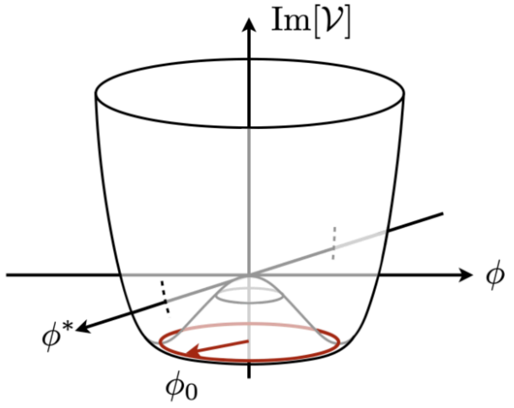


- Summary:

- deterministic limit of Keldysh functional integral dominated by single field configuration minimizing bare Keldysh action

$$\frac{\delta S}{\delta \phi_c} = 0 = \frac{\delta S}{\delta \phi_c^*}$$

- applications: systems in the presence of condensation/ collective degrees of freedom and low noise level, e.g.
  - Bose condensation: dynamics of zero temperature weakly interacting Bose gases (cold atoms)
  - light condensation: classical optics, waveguides, … (non-Hermitian physics)
  - non-linear dynamics Cross, Hohenberg, RMP (1993)
- higher orders in quantum field describe fluctuations

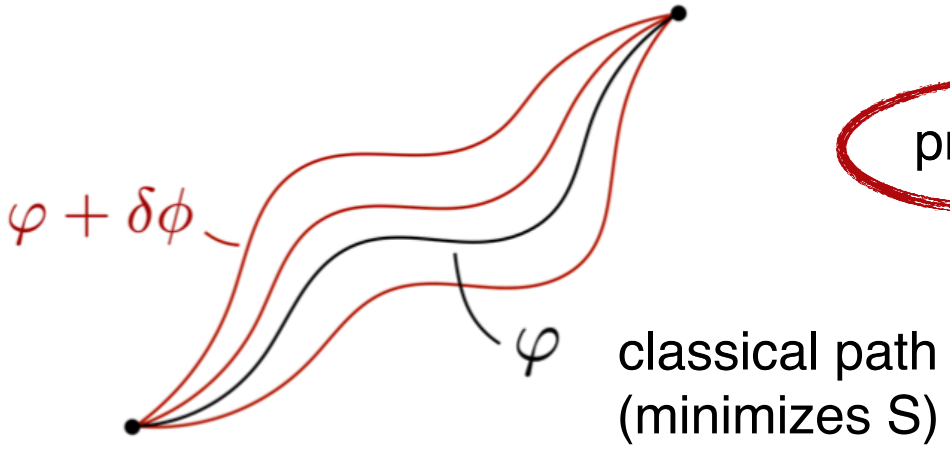
3. Fluctuations

- Classical field theories: single configuration has it all, the one minimising $S[\phi]$ i.e. $\frac{\delta S}{\delta \phi} = 0$

deterministic

- Quantum /statistical field theories: Summation over all possible field configurations

$$\int \mathcal{D}\phi e^{iS[\phi]}$$



probabilistic

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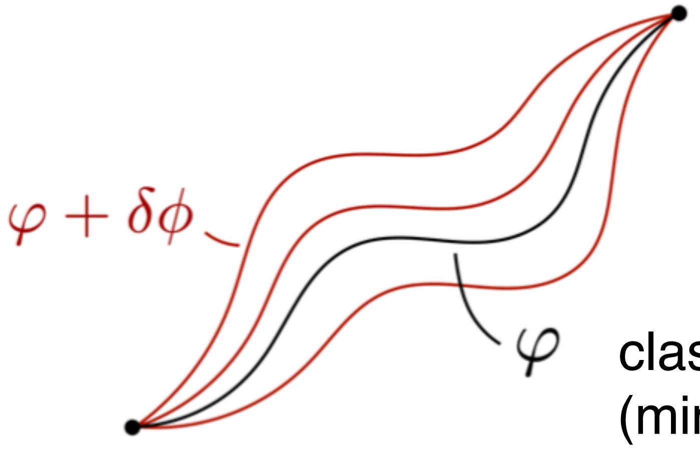

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- how to quantify deviations from the deterministic limit?

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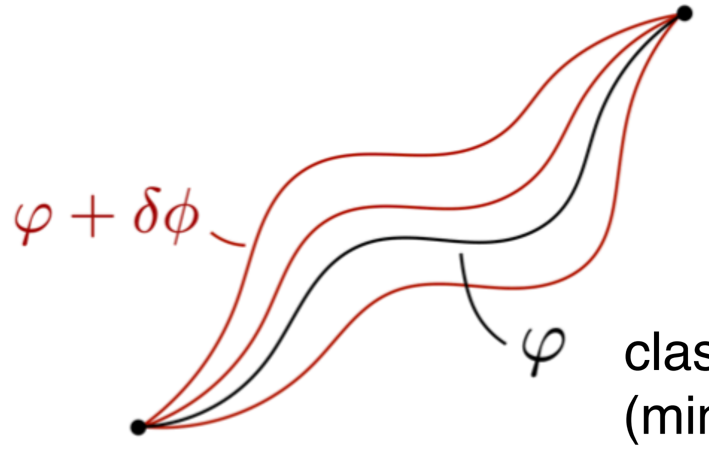

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- how to quantify deviations from the deterministic limit?

➡ correlation functions: fluctuations around deterministic configuration $\sim \langle \phi_c^*(\vec{x}, t) \phi_c(\vec{x}', t') \rangle$

det.: $= \langle \phi_c^*(\vec{x}, t) \rangle \langle \phi_c(\vec{x}', t') \rangle$

➡ response functions: impact of an external perturbation $\sim \langle \phi_q^*(\vec{x}, t) \phi_c(\vec{x}', t') \rangle$

exercise: verify relation to
operator formalism!

3. Fluctuations: Correlation vs. response functions

more details: L. Sieberer, M. Buchhold, SD,
Reports on Progress in Physics (2016)

- introduce complex contour dependent sources (cf. StatMech)

$$Z[j] = \langle e^{\mathrm{i}\int (j_+\phi_+^* - j_-\phi_-^* + c.c.)}\rangle = \langle e^{\mathrm{i}\int (j_c\phi_q^* + j_q\phi_c^* + c.c.)}\rangle \qquad Z[j=0] = Z = 1$$

normalization

- order parameter / occupation field:

$$\langle \phi_c(t,\mathbf{x})\rangle = -i\frac{\delta Z[j]}{\delta j_q^*(t,\mathbf{x})}\Big|_{j=0}$$

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$$\langle\phi_c(t,\mathbf{x})\rangle = -i\frac{\delta Z[j]}{\delta j_q^*(t,\mathbf{x})}\Big|_{j=0}$$

- single particle response: how does the field react to external perturbations?

relation to operator formalism (once and for all)

t = t', x = x'

$$G^R(t-t',\mathbf{x}-\mathbf{x}') = i\frac{\delta^2 Z}{\delta j_q^*(t,\mathbf{x})\delta j_c(t',\mathbf{x}')}\Big|_{j=0} = -i\langle\phi_c(t,\mathbf{x})\phi_q^*(t',\mathbf{x}')\rangle \overset{\downarrow}{=} -i\theta(t-t')\langle[\hat{\phi}(t,\mathbf{x}),\hat{\phi}^\dagger(t',\mathbf{x}')]\rangle \overset{\downarrow}{=} 1$$

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- single particle correlations: how strong are fluctuations?

t = t’, x=x’

$$G^K(t-t',\mathbf{x}-\mathbf{x}') = i\frac{\delta^2 Z}{\delta j_q^*(t,\mathbf{x})\delta j_q(t',\mathbf{x}')}\Big|_{j=0} = -i\langle \phi_c(t,\mathbf{x})\phi_c^*(t',\mathbf{x}')\rangle = -i\langle\{\hat\phi(t,\mathbf{x}),\hat\phi^\dagger(t',\mathbf{x}')\}\rangle \overset{\downarrow}{=} 2\langle\hat n(\mathbf{x})\rangle + 1$$

time and space translation invariance assumed

extra: how are states occupied?

exercise: verify relation to
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relation to operator formalism
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time and space translation
invariance assumed

extra: how are states occupied?

- total Green's function

$$G = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix} \qquad G^A = (G^R)^\dagger, \qquad (G^K)^\dagger = -G^K$$

prob. conservation

Hermitian conjugates

anti-Hermitian

exercise: prove formulas
on this page!

3. Fluctuations: Correlation vs. response functions & Keldysh action

more details: L. Sieberer, M. Buchhold, SD,
Reports on Progress in Physics (2016)

- by example: Lindblad equation for decaying cavity

$$\partial_t \rho = -i[\omega_0 \hat{a}^\dagger \hat{a}, \rho] + \kappa(2\hat{a}\rho\hat{a}^\dagger - \{\hat{a}^\dagger \hat{a}, \rho\})$$

- action:

$$S = \int dt(\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & i\partial_t - \omega_0 - i\kappa \\ i\partial_t - \omega_0 + i\kappa & 2i\kappa \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix}$$

$$\equiv P^A(\omega)$$

time domain

$\phi_\nu(t)$

$$= \int dt(\phi_c^*, \phi_q^*) \underbrace{\begin{pmatrix} 0 & \omega - \omega_0 - i\kappa \\ \underset{\equiv P^R(\omega)}{\omega - \omega_0 + i\kappa} & \underset{\equiv P^K}{2i\kappa} \end{pmatrix}}_{\text{claim: } G^{-1}(\omega)} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix}$$

frequency domain

$\phi_\nu(\omega)$

exercise: prove formulas
on this page!

3. Fluctuations: Correlation vs. response functions & Keldysh action

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- partition function: Gaussian integration

$$Z[j_c, j_q] = \langle e^{i \int \frac{d\omega}{2\pi} (j_c^* \phi_q + j_q^* \phi_c + \text{h.c.})} \rangle = e^{i \int \frac{d\omega}{2\pi} (j_q^*, j_c^*) G(\omega) \begin{pmatrix} j_q \\ j_c \end{pmatrix}}$$

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- single particle Green's function

$$\begin{pmatrix} \langle \phi_c(\omega)\phi_c^*(\omega')\rangle & \langle \phi_c(\omega)\phi_q^*(\omega')\rangle \\ \langle \phi_q(\omega)\phi_c^*(\omega')\rangle & \langle \phi_q(\omega)\phi_q^*(\omega')\rangle \end{pmatrix} = - \begin{pmatrix} \frac{\delta^2 Z}{\delta j_q^*(\omega)\delta j_q(\omega')} & \frac{\delta^2 Z}{\delta j_q^*(\omega)\delta j_c(\omega')} \\ \frac{\delta^2 Z}{\delta j_c^*(\omega)\delta j_q(\omega')} & \frac{\delta^2 Z}{\delta j_c^*(\omega)\delta j_c(\omega')} \end{pmatrix}\Bigg|_{j=0} = iG(\omega)\delta(\omega - \omega')$$

exercise: prove formulas
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- action:

time domain

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- summary in matrix components (generally valid — beyond Gaussian single mode example):

$$\boxed{G^{-1} = \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \implies G = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix} \qquad G^{R/A} = [P^{R/A}]^{-1} \qquad G^K = -G^R P^K G^A}$$

action matrix kernel

single particle Green's function

exercise: prove formulas
on this page!

3. Fluctuations: Correlation vs. response functions & Keldysh action

more details: L. Sieberer, M. Buchhold, SD,
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$$S = \int dt (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & i\partial_t - \omega_0 - i\kappa \\ i\partial_t - \omega_0 + i\kappa & 2i\kappa \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix}$$

time domain

$$\phi_\nu(t)$$

$$= \int dt (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & \omega - \omega_0 - i\kappa \\ \omega - \omega_0 + i\kappa & 2i\kappa \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix}$$

frequency domain

$$\phi_\nu(\omega)$$

- observables from the Green's functions:

response

  - decay of single-particle response:

$$G^R(t-t') = \int_\omega e^{i\omega(t-t')} G^R(\omega) = \theta(t-t') e^{i\omega_0(t-t')} e^{-\kappa(t-t')}$$

  - Lorentzian spectral density:

$$A(\omega) = \mathrm{Im} G^R(\omega) = \frac{2\kappa}{(\omega-\omega_0)^2 + \kappa^2}$$

exercise: prove formulas
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  - Lorentzian spectral density: $A(\omega) = \mathrm{Im} G^R(\omega) = \dfrac{2\kappa}{(\omega-\omega_0)^2 + \kappa^2}$

correlation

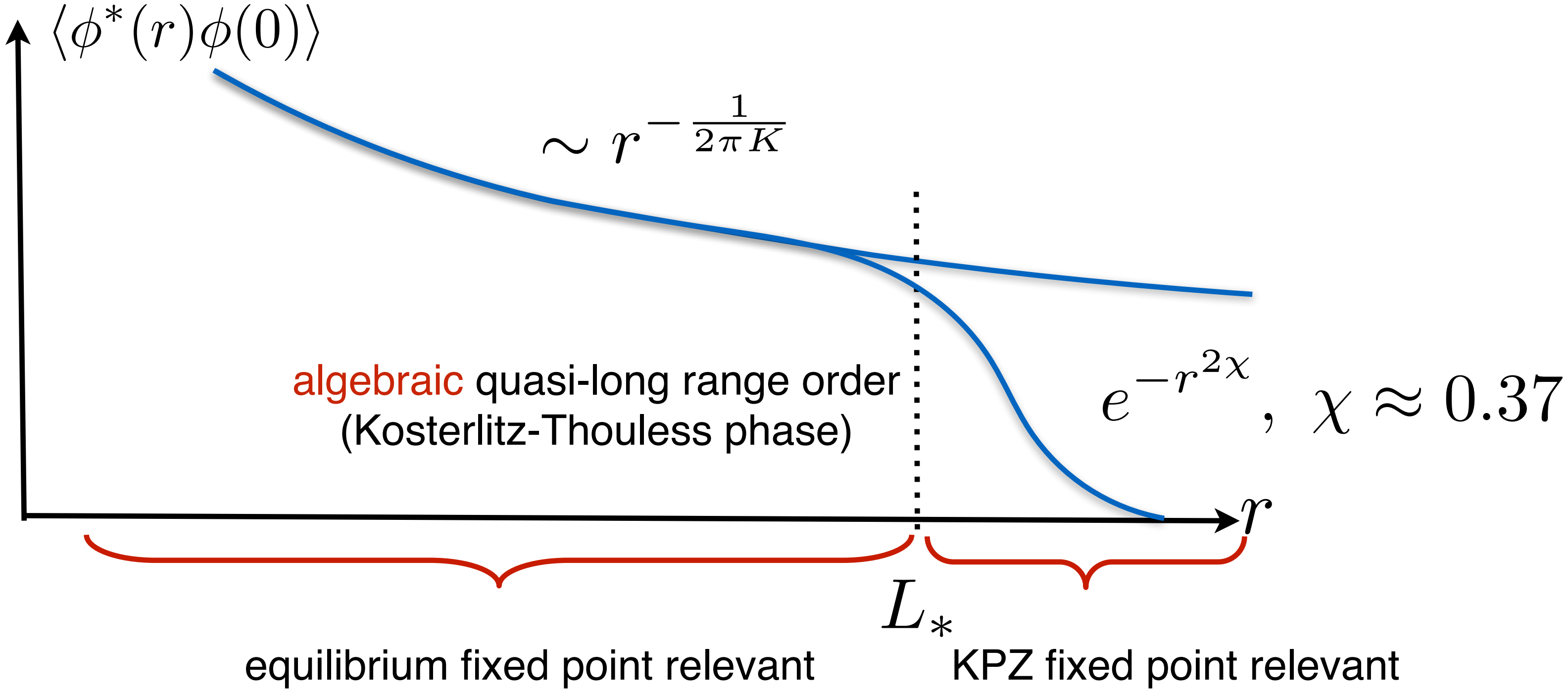
  - cavity mode occupation in stationary state $2\langle \hat{n}(t) \rangle + 1 = \langle \hat{a}^\dagger(t)\hat{a}(t) + \hat{a}(t)\hat{a}^\dagger(t) \rangle = iG^K(t-t) = i\int_\omega e^{i\omega(t-t)} G^K(\omega) = 1$

$\langle \hat{n}(t \to \infty) \rangle = 0$ $(t \to \infty)$

- ➡ correlation / statistical properties: G^K
- ➡ response / spectral properties: G^R

2. KPZ in exciton-polariton systems

- background: semiclassical limit of Lindblad-Keldysh action
- background: classifying equilibrium vs. non-equilibrium
- from XP to KPZ: absence of algebraic order out of equilibrium
- compact KPZ and non-equilibrium, phase transition



$\langle \phi^*(r)\phi(0)\rangle$

$\sim r^{-\frac{1}{2\pi K}}$

$e^{-r^{2\chi}}, \ \chi \approx 0.37$

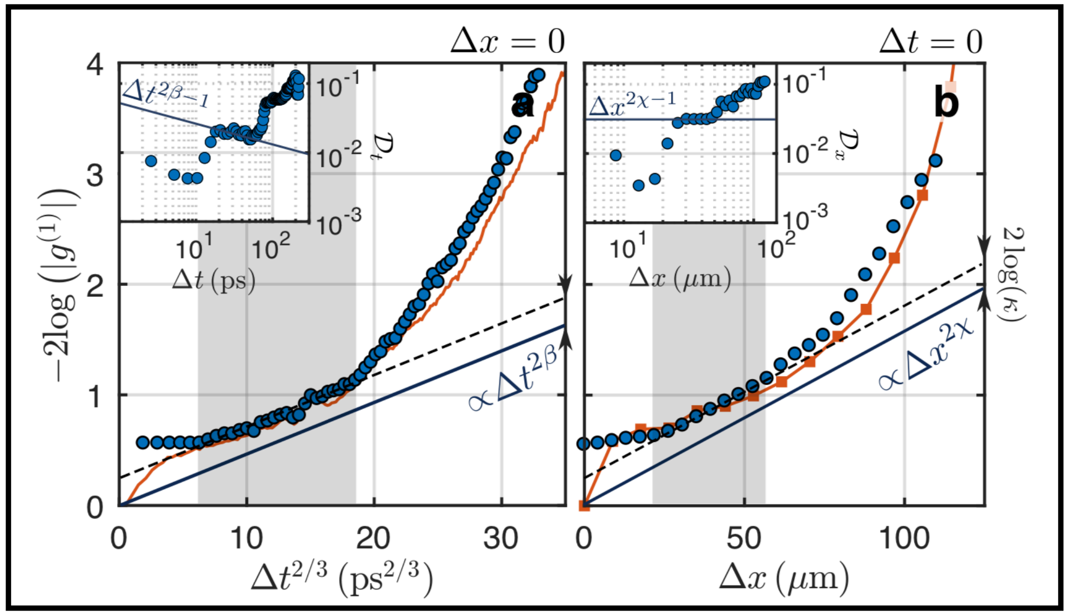
r

L_*

algebraic quasi-long range order (Kosterlitz-Thouless phase)

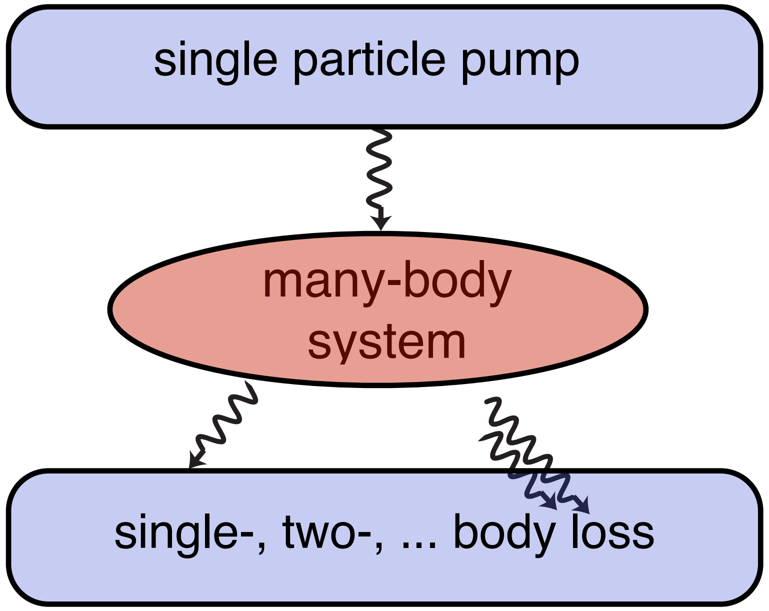
equilibrium fixed point relevant

KPZ fixed point relevant



Back to many-body model: ϕ^4-Lindbladian

- generic microscopic many-body model:



single particle pump

many-body system

single-, two-, ... body loss

$$\partial_t \rho = -i[H, \rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho]$$

$$H = \int_{\mathbf{x}} \hat{\phi}^\dagger_{\mathbf{x}} \left(\frac{\triangle}{2M} - \mu\right) \hat{\phi}_{\mathbf{x}} + \frac{\lambda}{2}(\hat{\phi}^\dagger_{\mathbf{x}} \hat{\phi}_{\mathbf{x}})^2$$

kinetic energy two-particle interaction

$$\mathcal{D}[\rho] = \gamma_p \int_{\mathbf{x}} [\hat{\phi}^\dagger_{\mathbf{x}}\, \rho\, \hat{\phi}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}_{\mathbf{x}} \hat{\phi}^\dagger_{\mathbf{x}}, \rho\}] \quad + \quad \gamma_l \int_{\mathbf{x}} [\hat{\phi}_{\mathbf{x}}\, \rho\, \hat{\phi}^\dagger_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^\dagger_{\mathbf{x}} \hat{\phi}_{\mathbf{x}}, \rho\}] \quad +$$

single particle pump single particle loss

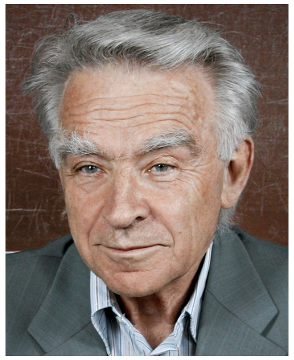
$$\kappa \int_{\mathbf{x}} [\hat{\phi}^2_{\mathbf{x}}\, \rho\, \hat{\phi}^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^{\dagger\,2}_{\mathbf{x}} \hat{\phi}^2_{\mathbf{x}}, \rho\}]$$

two-particle loss

Many-Body Master Equation

1-1 mapping

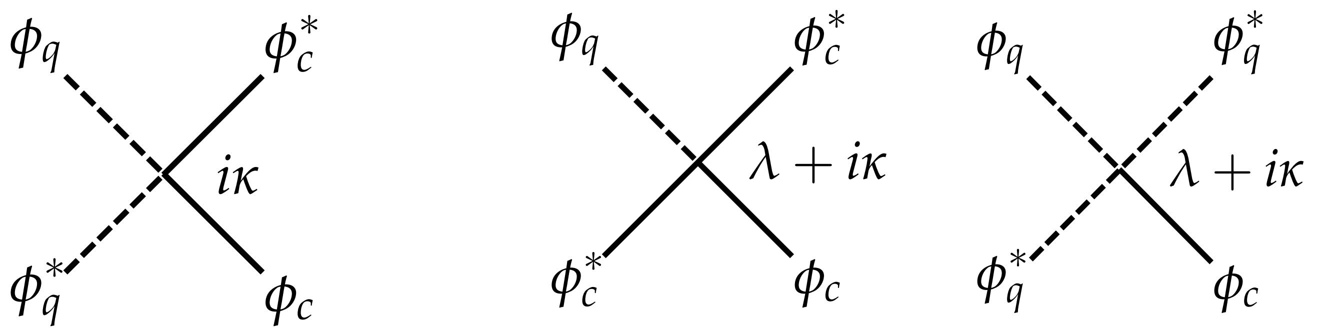
Keldysh functional integral



Many-body model: ϕ^4-Lindblad-Keldysh action

- after Keldysh rotation: $\phi_c = (\phi_+ + \phi_-)/\sqrt{2}, \quad \phi_q = (\phi_+ - \phi_-)/\sqrt{2}$

$$\mathcal{S} = \int_{t,\mathbf{x}} \left\{ (\phi_c^*, \phi_q^*) \underbrace{\begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix}}_{\text{after Fourier: } G^{-1}(\omega,\mathbf{q})} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + 2i\kappa \phi_c^* \phi_c \phi_q^* \phi_q - \tfrac{1}{2} \left[(\lambda + i\kappa) \left(\phi_c^{*2} \phi_c \phi_q + \phi_q^{*2} \phi_c \phi_q \right) + c.c. \right] \right\}$$



- Gaussian sector: inverse Green's function
  - retarded/advanced $\quad P^R(\omega, \mathbf{q}) = \omega - \left(\frac{\mathbf{q}^2}{2M} - \mu\right) + i\left(\gamma_l - \gamma_p\right)/2$
  - Keldysh component $\quad P^K = i\left(\gamma_l + \gamma_p\right)$

difference: distance from a phase transition (see above)

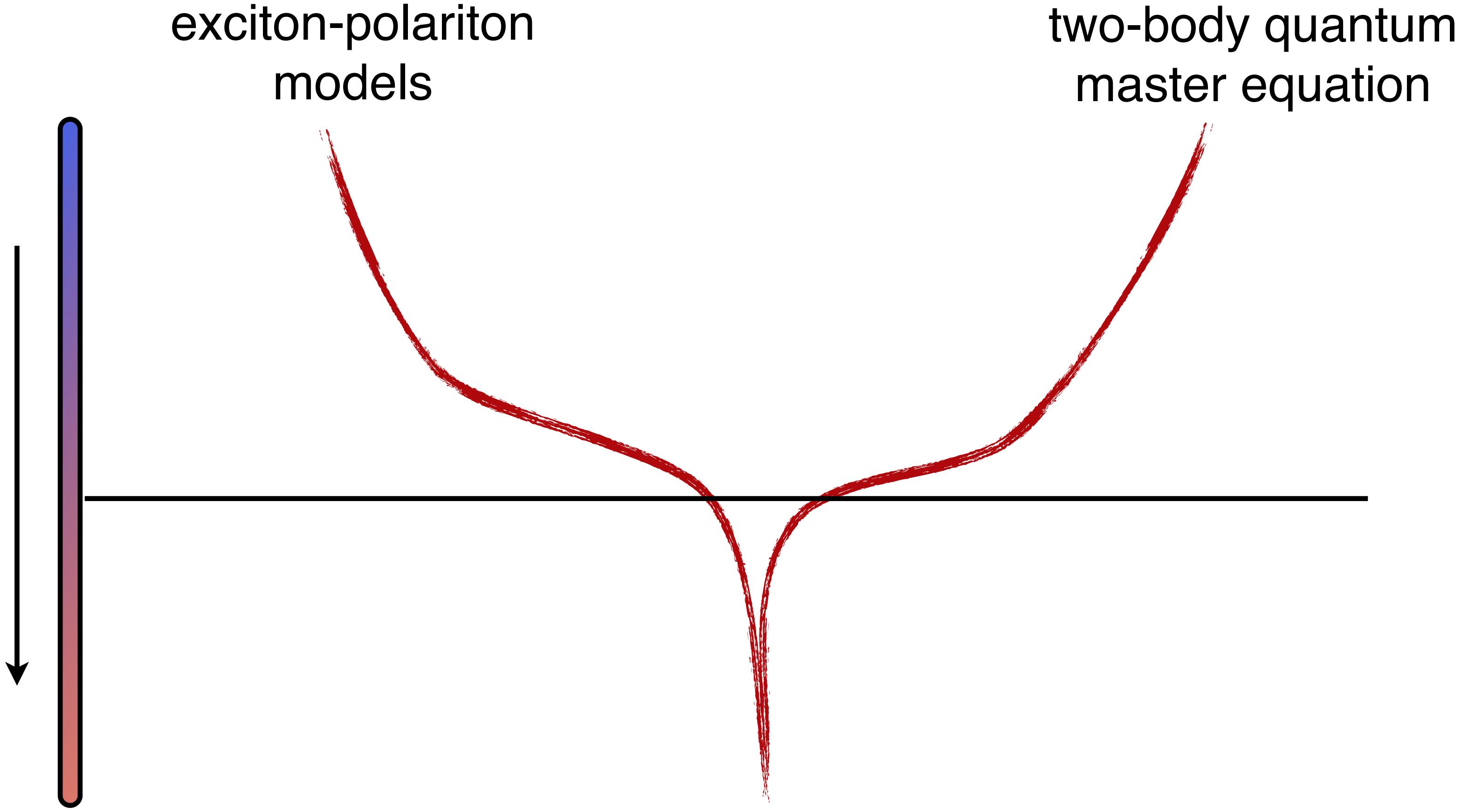
sum: noise of loss and pumping add up (substantiated below)

➡ now: simplifications in the semiclassical limit:

- sharp argument close to a critical point
- provides intuition for a frequency regime $\omega \ll \gamma = \gamma_l + \gamma_p$

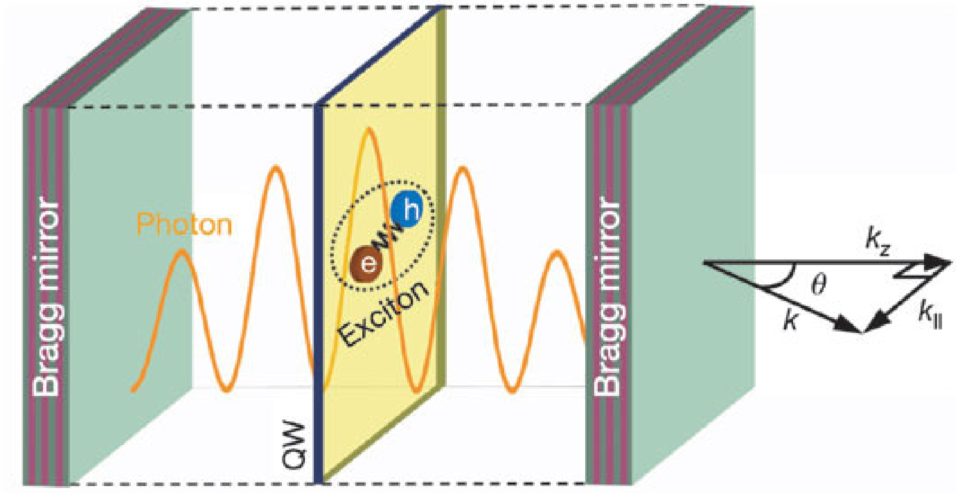
➡ now: "what is non-equilibrium about it?"

Semi-classical limit and Langevin equations

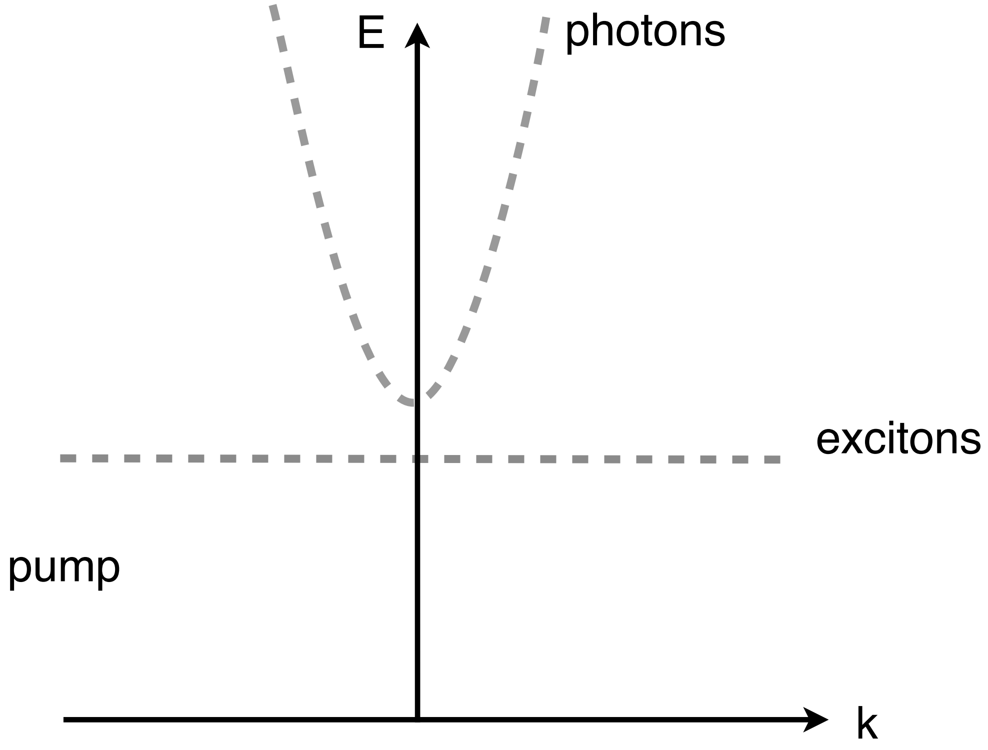


Intermezzo: Exciton-polariton systems

Kasprzak et al., Nature 2006



Imamoglu et al., PRA 1996



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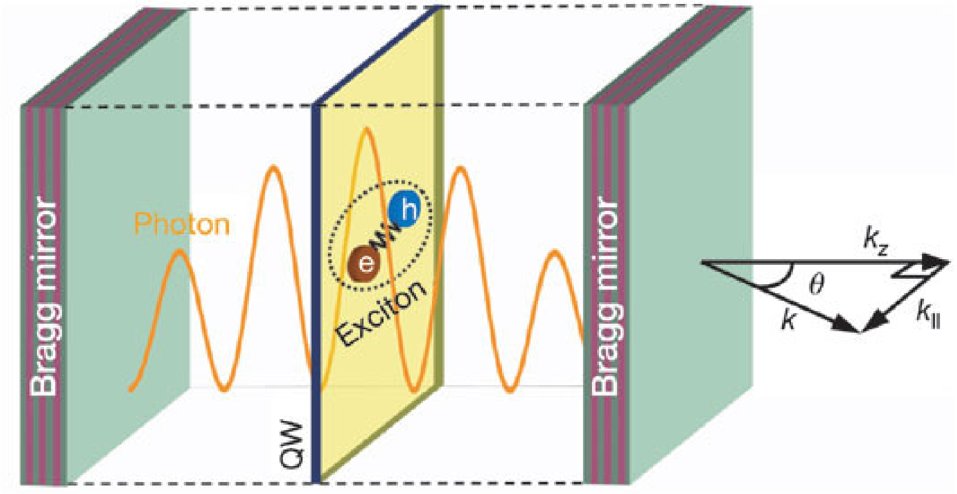

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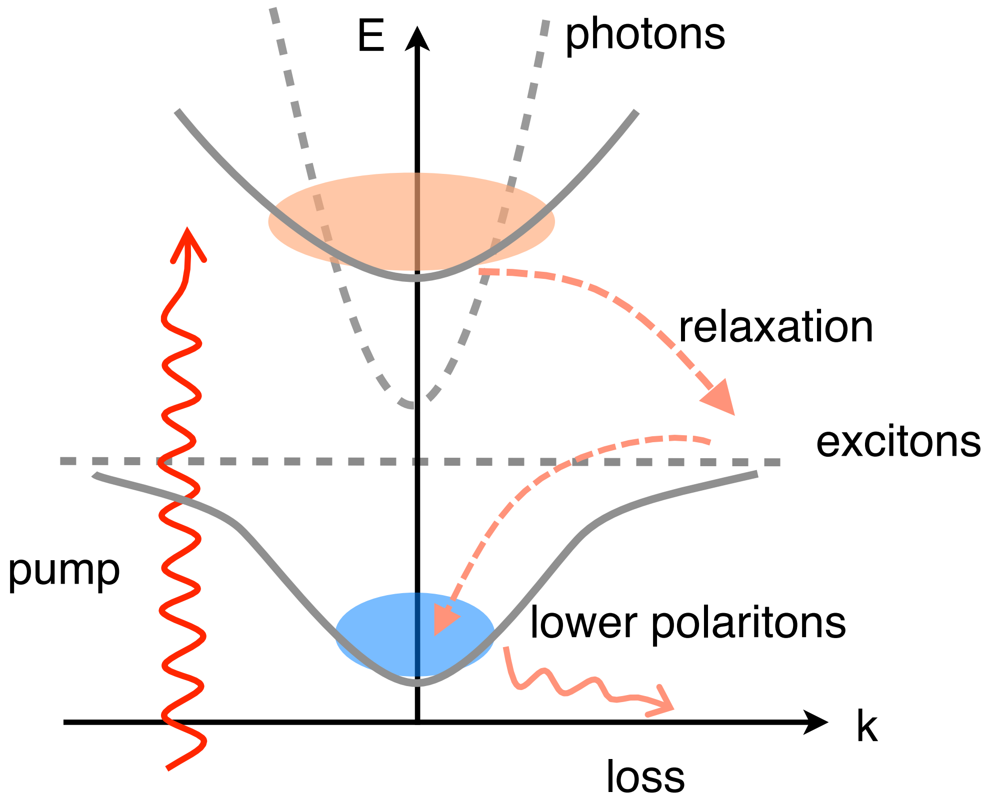

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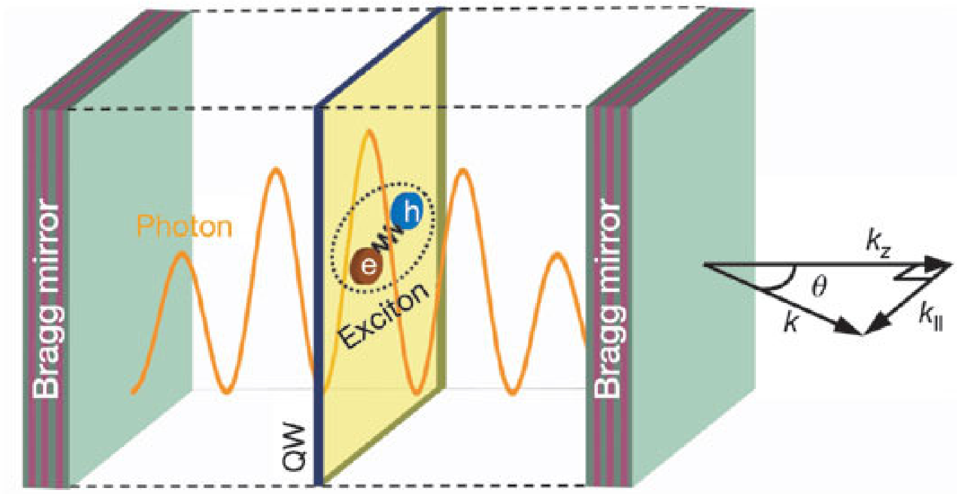

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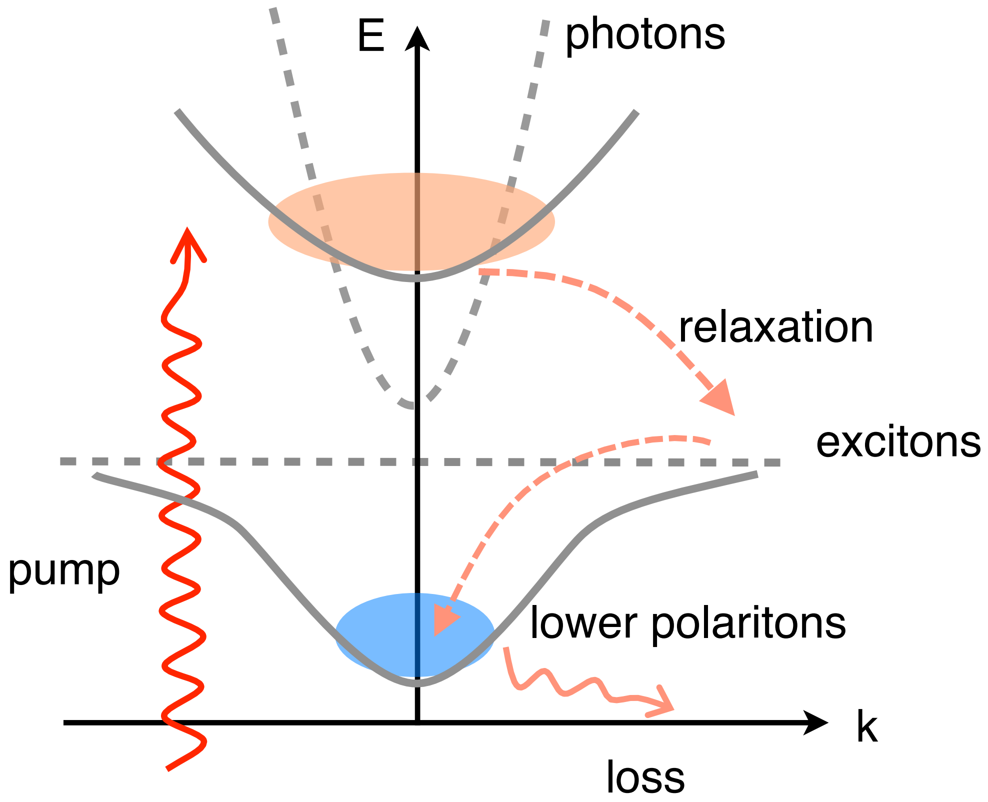

- phenomenological description: stochastic driven-dissipative Gross-Pitaevskii-Eq

$$i\partial_t \phi = \left[-\frac{\nabla^2}{2m} - \mu + i(\gamma_p - \gamma_l) + (\lambda - i\kappa) |\phi|^2 \right] \phi + \zeta$$

propagation

pump & loss rates

elastic collisions

two-body loss

Szymanska, Keeling, Littlewood PRL (04,06); PRB (07);
Wouters, Carusotto PRL (07,10)

$$\langle \zeta^*(t, \mathbf{x}) \zeta(t', \mathbf{x}') \rangle = \gamma \delta(t - t') \delta(\mathbf{x} - \mathbf{x}')$$

Intermezzo: Exciton-polariton systems

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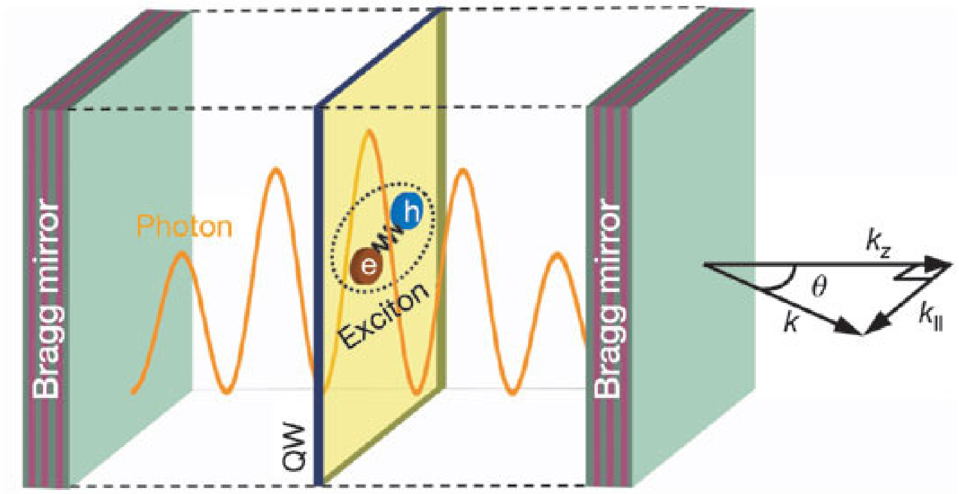

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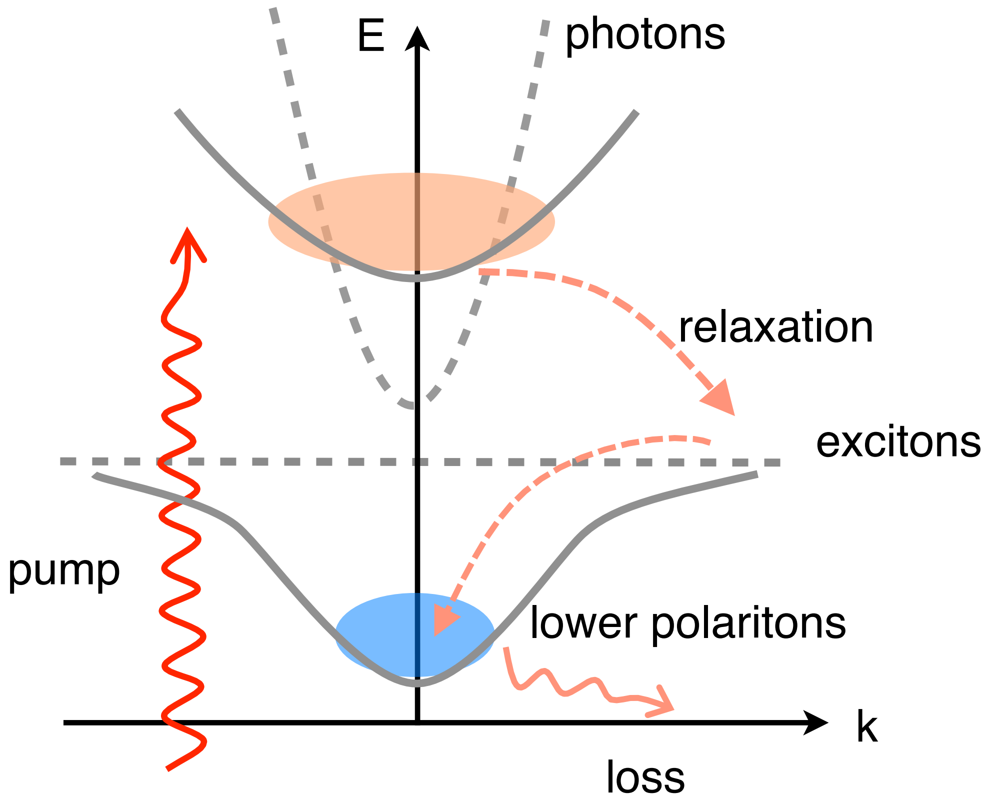

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$$\langle\zeta^*(t,\mathbf{x})\zeta(t',\mathbf{x}')\rangle = \gamma\delta(t-t')\delta(\mathbf{x}-\mathbf{x}')$$

- structure: $$i\partial_t\phi = \frac{\delta\mathcal{H}_c}{\delta\phi^*} + i\frac{\delta\mathcal{H}_d}{\delta\phi^*} + \zeta$$

coherent (reversible)

dissipative (irreversible)

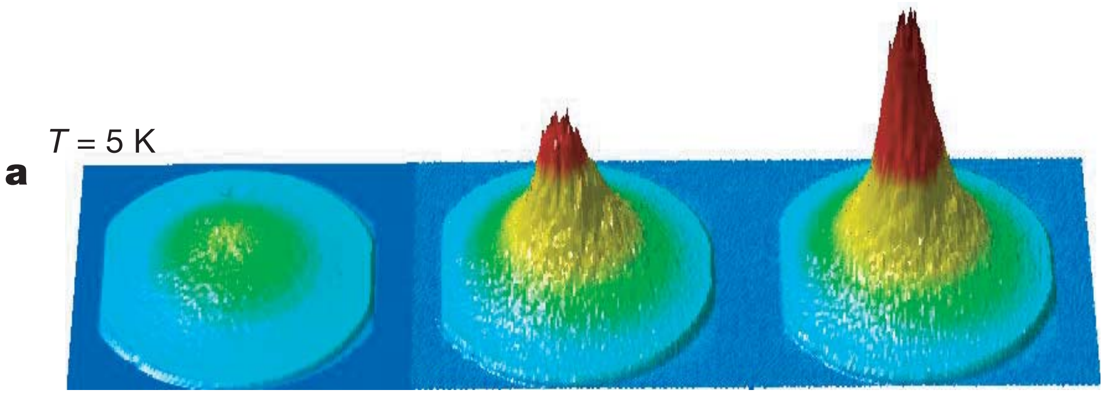
noise <-> random force allowing to explore various configurations

$$\mathcal{H}_\alpha = \int d^dx[r_\alpha|\phi_c|^2 + K_\alpha|\nabla\phi_c|^2 + g_\alpha|\phi_c|^4]$$

$$\alpha = c, d$$

Intermezzo: Exciton-polariton systems

- Bose condensation seen despite non-equilibrium conditions



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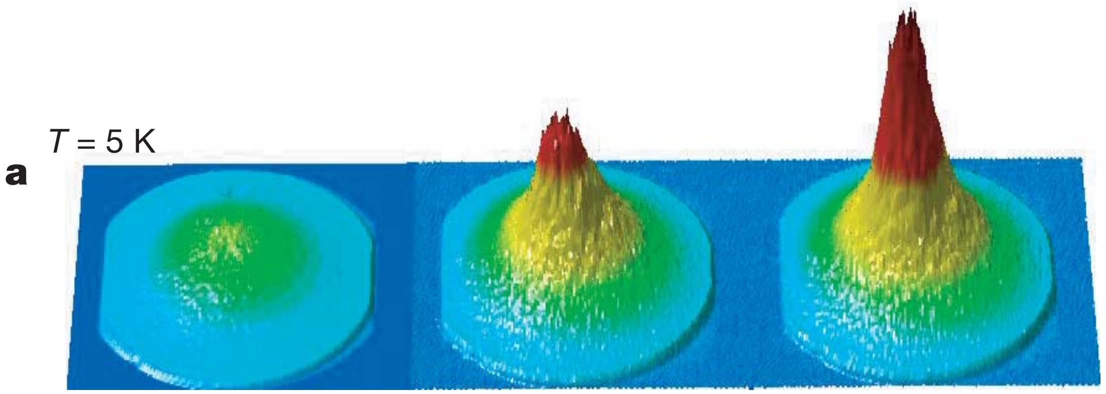

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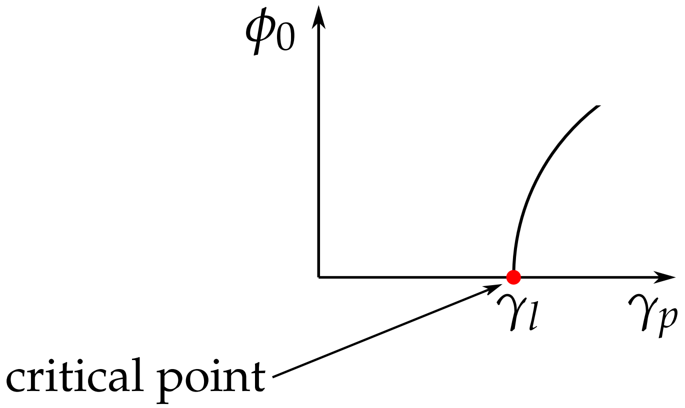
- stochastic driven-dissipative Gross-Pitaevskii-Eq

$$\cancel{i\partial_t}\phi = \left[-\cancel{\frac{\nabla^2}{2m}} - \mu + i(\gamma_p - \gamma_l) + (\lambda - i\kappa)\left|\phi\right|^2\right]\phi \cancel{+\zeta}$$

Szymanska, Keeling, Littlewood PRL (04, 06); PRB (07)); Wouters, Carusotto PRL (07,10)

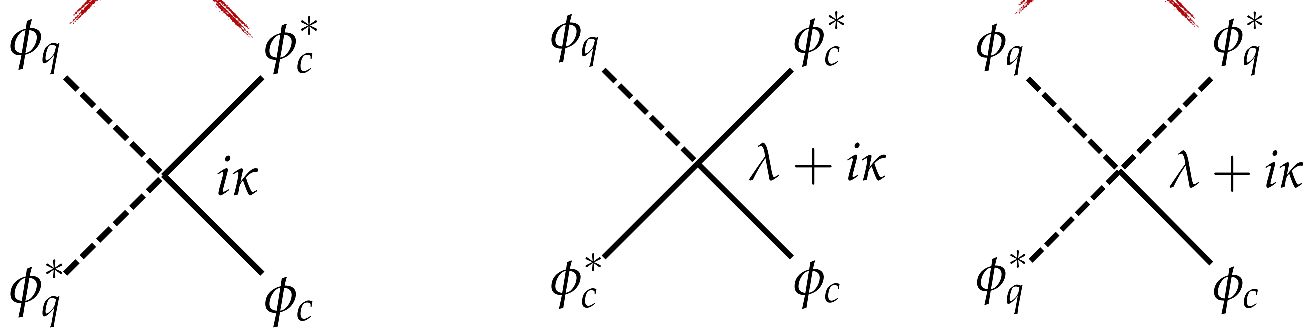
- mean field
  - neglect noise
  - homogeneous solution $\phi(\mathbf{x}, t) = \phi_0$



- naively, just as Bose condensation in equilibrium!
- Q1: How does this model relate to the Lindbladian and Lindblad-Keldysh field theory?
- Q2: What is “non-equilibrium” about it?

Semiclassical limit of Lindblad-Keldysh action: power counting

$$\mathcal{S} = \int_{t,\mathbf{x}} \left\{ \left(\phi_c^*, \phi_q^*\right) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + 2i\kappa \xcancel{\phi_c^* \phi_c} \phi_q^* \phi_q - \tfrac{1}{2} \left[\left(\lambda + i\kappa\right) \left(\phi_c^{*2} \phi_c \phi_q + \xcancel{\phi_q^{*2} \phi_c} \phi_q \right) + c.c. \right] \right\}$$



- Gaussian sector close to a critical point:
  - retarded/advanced $P^R(\omega, \mathbf{q}) = \omega - \mathbf{q}^2 \overbrace{- \mu + i\left(\gamma_l - \gamma_p\right)/2}^{\to 0} \boxed{\sim k^2}$
  - Keldysh component $P^K = i\left(\gamma_l + \gamma_p\right) \boxed{\sim k^0}$

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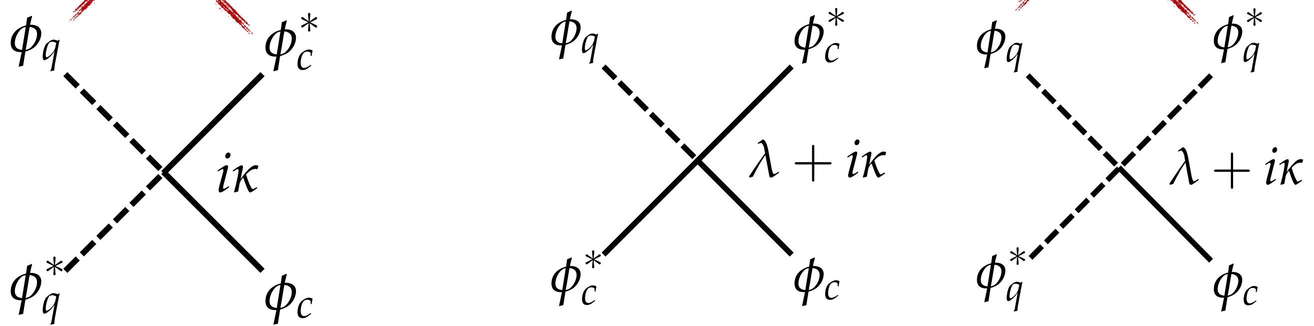

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  - d > 2: couplings with more than two quantum fields irrelevant in the RG sense

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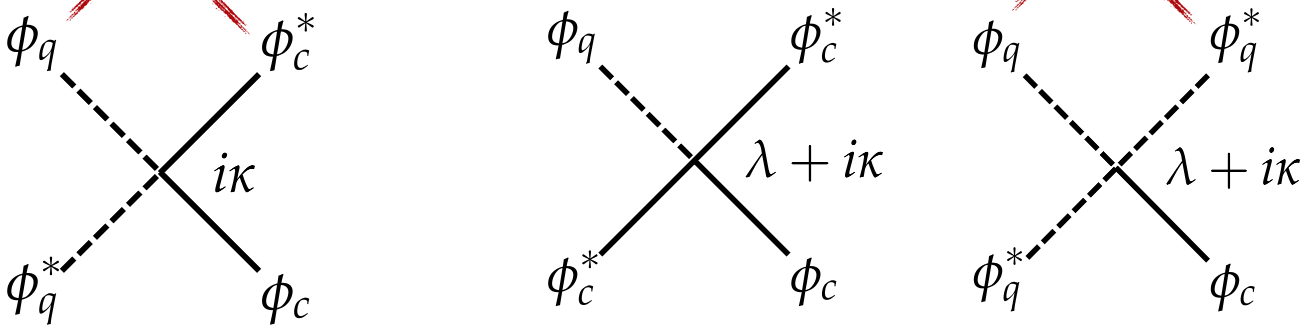

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- massive simplification: semi-classical Martin-Siggia-Rose-Janssen-de Dominicis action

$$S = \int_{t,\mathbf{x}} \left\{ \phi_q^* \frac{\delta \bar{S}[\phi_c]}{\delta \phi_c^*} + c.c. + i2\gamma\phi_q^*\phi_q \right\} \qquad \bar{S} = \int_{t,\mathbf{x}} \left\{ \phi_c^* i\partial_t \phi_c - \mathcal{H}_c + i\mathcal{H}_d \right\}$$

linear quadratic

Hermitian anti-Hermitian

Relation to driven-dissipative Gross-Pitaevskii-equation

- Equivalence of semiclassical theory and Langevin equations

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  - structure of action in semiclassical limit (Martin-Siggia-Rose-Janssen-de Dominicis functional integral)

$$Z = \int \mathcal{D}[\phi_c, \phi_c^*, \phi_q, \phi_q^*]\, e^{iS[\phi_c,\phi_c^*,\phi_q,\phi_q^*]}$$

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Relation to driven-dissipative Gross-Pitaevskii-equation

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- Decouple the quantum field (undoing Gaussian integration)

$$Z = \int \mathcal{D}[\xi, \xi^*] \, e^{-\frac{1}{2\gamma} \int_{t,\mathbf{x}} \xi^* \xi} \int \mathcal{D}[\phi_c, \phi_c^*, \phi_q, \phi_q^*] e^{i\left[\phi_q^* \left(\underbrace{i\partial_t \phi_c - \frac{\delta \mathcal{H}_c}{\delta \phi_c{}^*} + i \frac{\delta \mathcal{H}_d}{\delta \phi_c{}^*}}_{\frac{\delta \bar{S}}{\delta \phi_c^*}} - \xi \right) + c.c.\right]}$$

Relation to driven-dissipative Gross-Pitaevskii-equation

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- Recognize Fourier representation of delta-constraint: Langevin equation

$$Z = \int \mathcal{D}[\xi, \xi^*]\, e^{-\frac{1}{2\gamma}\int_{t,\mathbf{x}} \xi^* \xi} \int \mathcal{D}[\phi_c, \phi_c^*] \delta \left(\boxed{i\partial_t \phi_c - \frac{\delta \mathcal{H}_c}{\delta \phi_c^*} + i\frac{\delta \mathcal{H}_d}{\delta \phi_c^*} - \xi} \right) \delta\left(c.c.\right)$$

➡ noise averaging

➡ at each instant of time:

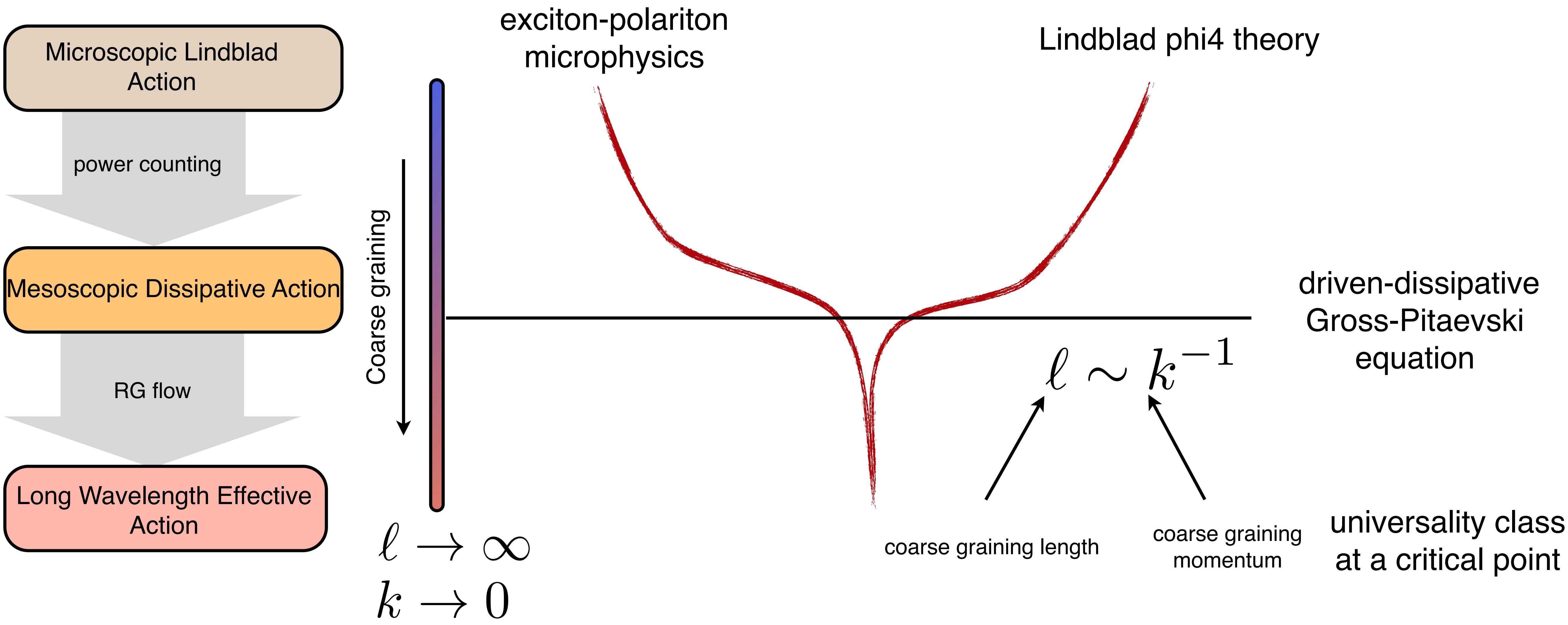
➡ driven-dissipative Gross-Pitaevski equation

➡ cf. deterministic limit: noise added, system explores many configurations

➡ interpretation: 2γ is the noise level

Semiclassical limit and exciton-polariton model

- example of “weak” universality (loss of memory of microscopic physics)



Microscopic Lindblad Action

power counting

Mesoscopic Dissipative Action

RG flow

Long Wavelength Effective Action

Coarse graining

$\ell \to \infty$

$k \to 0$

exciton-polariton microphysics

Lindblad phi4 theory

driven-dissipative Gross-Pitaevski equation

$\ell \sim k^{-1}$

coarse graining length

coarse graining momentum

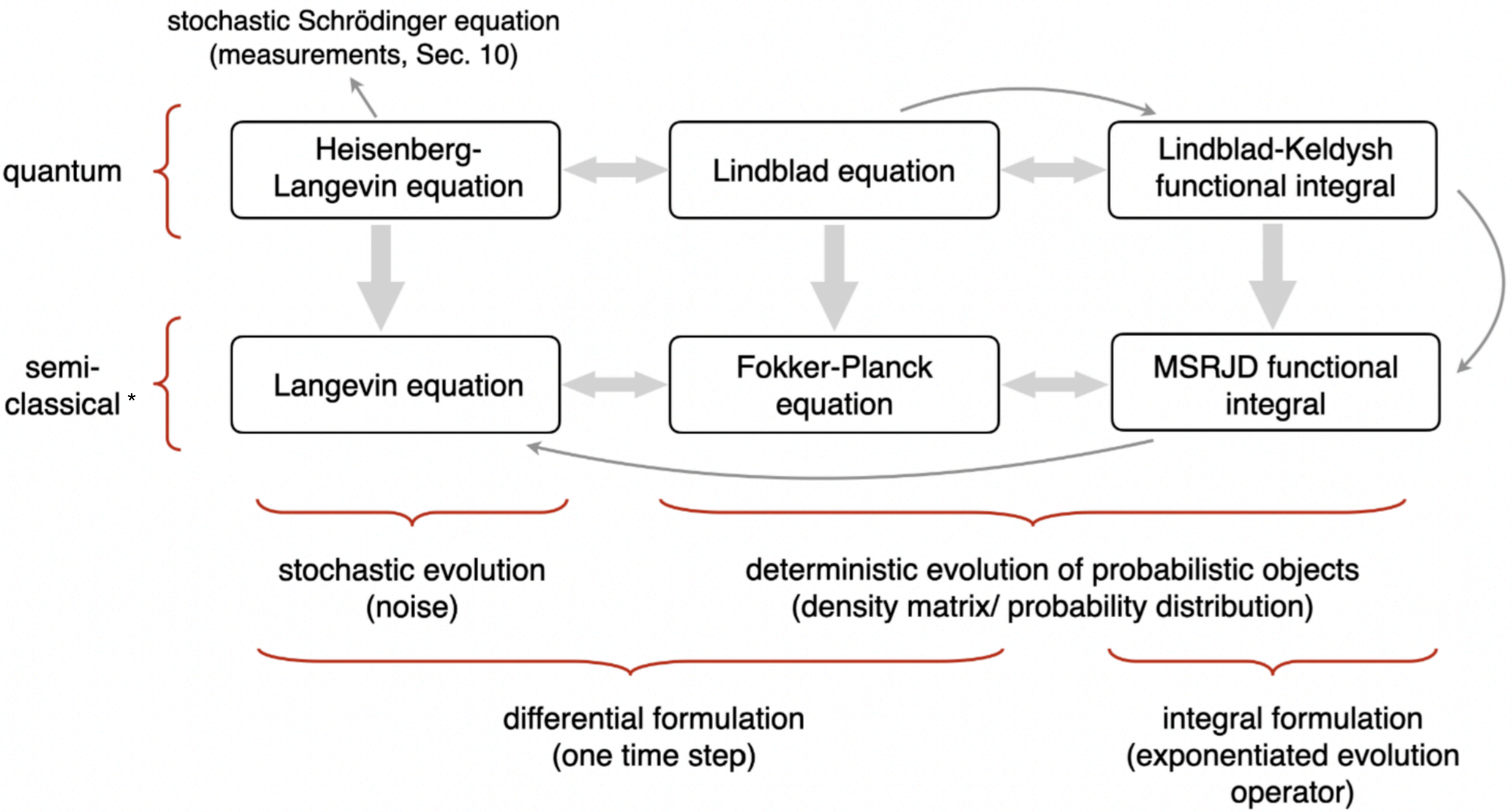
universality class at a critical point

➡ many microscopic models collapse to an effective low frequency model

➡ form dictated by microscopic symmetries

➡ longer wavelength behavior to be determined by calculation

Overview: Langevin equations, Lindblad equation, Keldysh integral



stochastic Schrödinger equation
(measurements, Sec. 10)

quantum

Heisenberg-Langevin equation

Lindblad equation

Lindblad-Keldysh functional integral

semi-classical *

Langevin equation

Fokker-Planck equation

MSRJD functional integral

stochastic evolution
(noise)

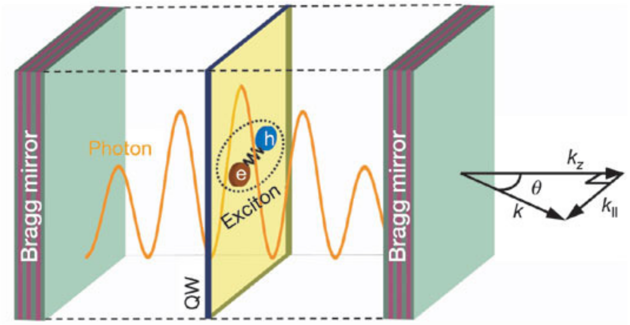
deterministic evolution of probabilistic objects
(density matrix/ probability distribution)

differential formulation
(one time step)

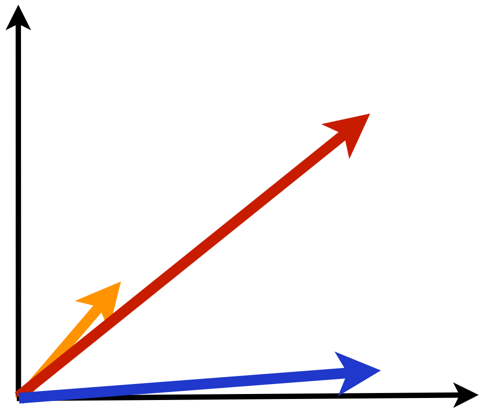
integral formulation
(exponentiated evolution operator)

* effects of phase coherence still present (cf. BEC as classical wave)

Equilibrium vs. non-equilibrium states



$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+, \Phi_-]}$$



Equilibrium vs. non-equilibrium stationary states

- different notions of ‘non-equilibrium’

Time evolution

➡ time translation invariance broken (e.g. thermalization, Floquet..)

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Stationary states (considered here)

➡ flux equilibria: equilibrium vs. non-equilibrium conditions

  - not visible in static observables:

$$\rho = e^{-\beta H}/\mathrm{tr} e^{-\beta H}$$

➡ any positive semidefinite Hermitian operator can be written like this

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- visible dynamical observables, e.g.:

$$\langle \psi^\dagger(t)\psi(0)\rangle \qquad \psi(t) = e^{iHt}\psi e^{-iHt}$$

➡ non-equilibrium conditions are encoded in the generator of dynamics

  ➡ thermal equilibrium realized if generator of dynamics coincides with the one in statistical weight

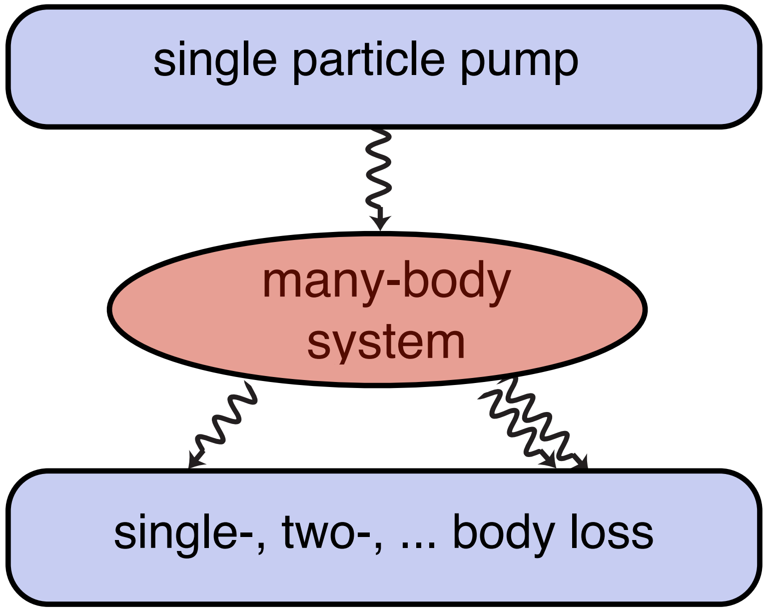
  ➡ otherwise must expect non-equilibrium conditions (Lindbladian)

Equilibrium vs. non-equilibrium stationary states

- non-equilibrium stationary states:
  - open system: is it the coupling to a bath —> irreversibility?
    - ➡ no, can be compatible with thermal equilibrium (Caldeira-Leggett Models)

Equilibrium vs. non-equilibrium stationary states

- non-equilibrium stationary states:
  - open system: is it the coupling to a bath —> irreversibility?
    - ➡ no, can be compatible with thermal equilibrium (Caldeira-Leggett Models)
  - driven & open system: coupling to multiple incompatible baths (e.g. different temperatures, chemical potential, driving strength…)



single particle pump

many-body system

single-, two-, ... body loss

- result from independent microscopic processes
- system is ‘confused’ which bath to thermalise to
- some fluxes through the system

➡ driven open nature incompatible with thermal equilibrium

➡ how to sharply quantify?

Equilibrium vs. non-equilibrium stationary states: symmetry

- more formally: quantum master equation

$$\underbrace{\partial_t \rho = -i[H,\rho]}_{\Longrightarrow S_H} + \underbrace{\mathcal{D}[\rho]}_{\Longrightarrow S_{\mathcal{D}}}$$

- equivalent Keldysh functional integral: $Z = \int \mathcal{D}\phi_{\pm} e^{i(S_H[\phi_{\pm}]+S_{\mathcal{D}}[\phi_{\pm}])}$

- equilibrium dynamics microscopically generated by a time-independent (undriven) Hamiltonian alone

$$S_{\mathcal{D}} = 0$$

Equilibrium vs. non-equilibrium stationary states: symmetry

- more formally: quantum master equation

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- equilibrium dynamics microscopically generated by a time-independent (undriven) Hamiltonian alone

$$S_{\mathcal{D}} = 0$$

➡ symmetry of Keldysh action under discrete transformation

L. Sieberer, A Chiocchetta, U. Täuber, A. Gambassi, SD PRB (2015); F. Haehl, R. Loganayagam, M. Rangamani, JHEP (2016); M. Crossley, P. Glorioso, H. Liu, JHEP (2016)

$$\boxed{\mathcal{T}_\beta : \quad \phi_\pm(t,\mathbf{x}) \to \phi_\pm(-t+i\beta/2,\mathbf{x}), \quad i \to -i} \qquad \mathcal{T}_\beta^2 = 1 \qquad \beta = 1/T$$

Equilibrium symmetry: implications

- symmetry of Schwinger-Keldysh action of undriven Hamiltonian under discrete transformation

$$\mathcal{T}_\beta : \quad \phi_\pm(t,\mathbf{x}) \to \phi_\pm(-t+i\beta/2,\mathbf{x}), \quad i \to -i \qquad \mathcal{T}_\beta^2 = 1 \qquad \beta = 1/T$$

- implications:
  - for correlation functions, ± basis:

$$\langle \mathcal{O}[\phi_\pm] \rangle = \langle \mathcal{T}_\beta(\mathcal{O}[\phi_\pm]) \rangle \qquad \langle \mathcal{O}[\phi_\pm] \rangle = \int \mathcal{D}\phi_\pm \mathcal{O}[\phi_\pm] e^{iS[\phi_\pm]}$$

  - physical consequence (Keldysh basis): fluctuation-dissipation relations, any order, e.g. single particle sector:

$$\boxed{G^K(\omega,\mathbf{q}) = (2n_B(\omega/T)+1)[G^R(\omega,\mathbf{q}) - G^A(\omega,\mathbf{q})]}$$

any order <=> detailed balance
<=> global thermal equilibrium

correlations Bose distribution responses

- connection to operator formalism: compact functional formulation of Kubo-Martin-Schwinger boundary condition: for any two operators A,B,

$$\langle A(t)B(t')\rangle = \langle B(t'-i\beta)A(t)\rangle. \qquad \langle \mathcal{O}\rangle = \mathrm{tr}(\mathcal{O}\rho)$$

  - reason: $A(t) = e^{iHt}Ae^{-iHt}, \rho = e^{-\beta H}/\mathrm{tr}e^{-\beta H}$
    $\Rightarrow A(t)\rho = \rho A(t-i\beta)$ & cyclic invariance

Equilibrium symmetry: Semiclassical limit

- Full symmetry:

$$\mathcal{T}_\beta : \quad \phi_\pm(t,\mathbf{x}) \to \phi_\pm(-t+i\beta/2,\mathbf{x}), \quad i \to -i \qquad\qquad \beta = 1/T$$

$$= e^{\pm i\frac{\beta}{2}\partial_t}\phi_\pm(-t,\mathbf{x})$$

- semiclassical limit: T large => $e^{\pm i\frac{\beta}{2}\partial_t} \approx 1 \pm i\frac{\beta}{2}\partial_t$

- action on the fields:

irrelevant by power counting

$$\mathcal{T}_\beta\phi_c(t,\mathbf{x}) = \phi_c^*(-t,\mathbf{x}) + \frac{i}{2T}\partial_t\cancel{\phi_q^*(-}t,\mathbf{x}),$$
$$\mathcal{T}_\beta\phi_q(t,\mathbf{x}) = \phi_q^*(-t,\mathbf{x}) + \frac{i}{2T}\partial_t\phi_c^*(-t,\mathbf{x})$$

reproduces classical result

H. K. Janssen (1976); C. Aron et al, J Stat. Mech (2011)

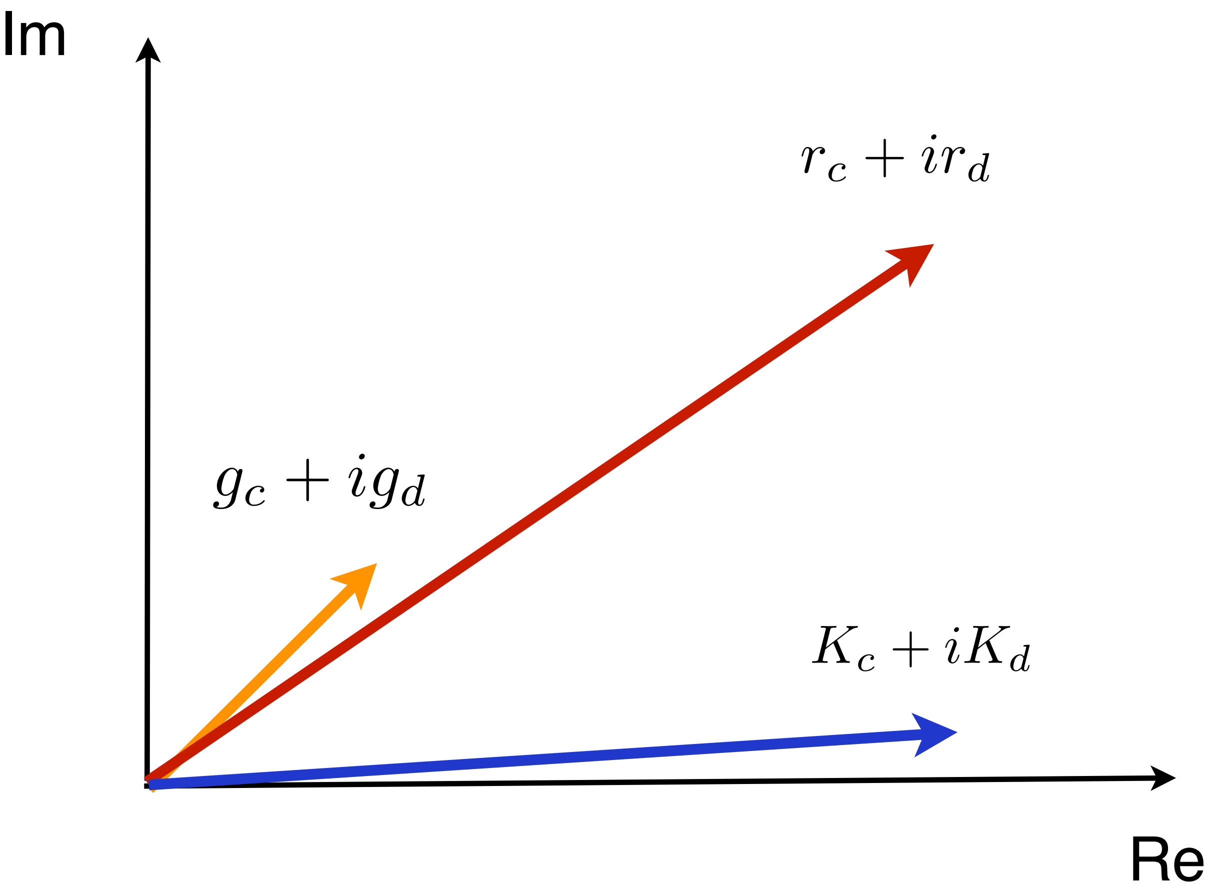
➡ obtain geometric interpretation of the equilibrium symmetry

Geometric interpretation: equilibrium vs. non-equilibrium dynamics

- in semi-classical Martin-Siggia-Rose-Janssen-de Dominicis action

$$S = \int_{t,\mathbf{x}} \left\{ \phi_q^* \frac{\delta \bar{S}[\phi_c]}{\delta \phi_c^*} + c.c. + i2\gamma \phi_q^* \phi_q \right\} \qquad \bar{S} = \int_{t,\mathbf{x}} \{ \phi_c^* i\partial_t \phi_c - \mathcal{H}_c + i\mathcal{H}_d \}$$

$$\mathcal{H}_\alpha = \int d^d x [r_\alpha |\phi_c|^2 + K_\alpha |\nabla \phi_c|^2 + g_\alpha |\phi_c|^4] \qquad \alpha = c, d$$

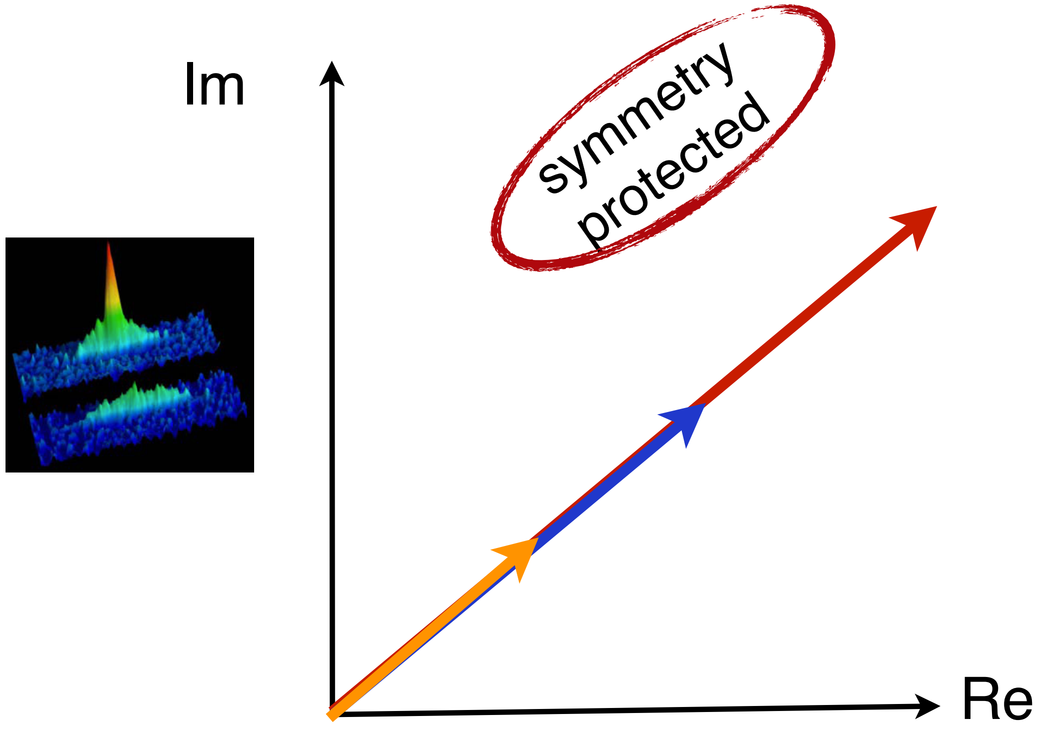


incoherent/ irrev. dynamics $\mathcal{H}_d$

coherent/ reversible dynamics $\mathcal{H}_c$

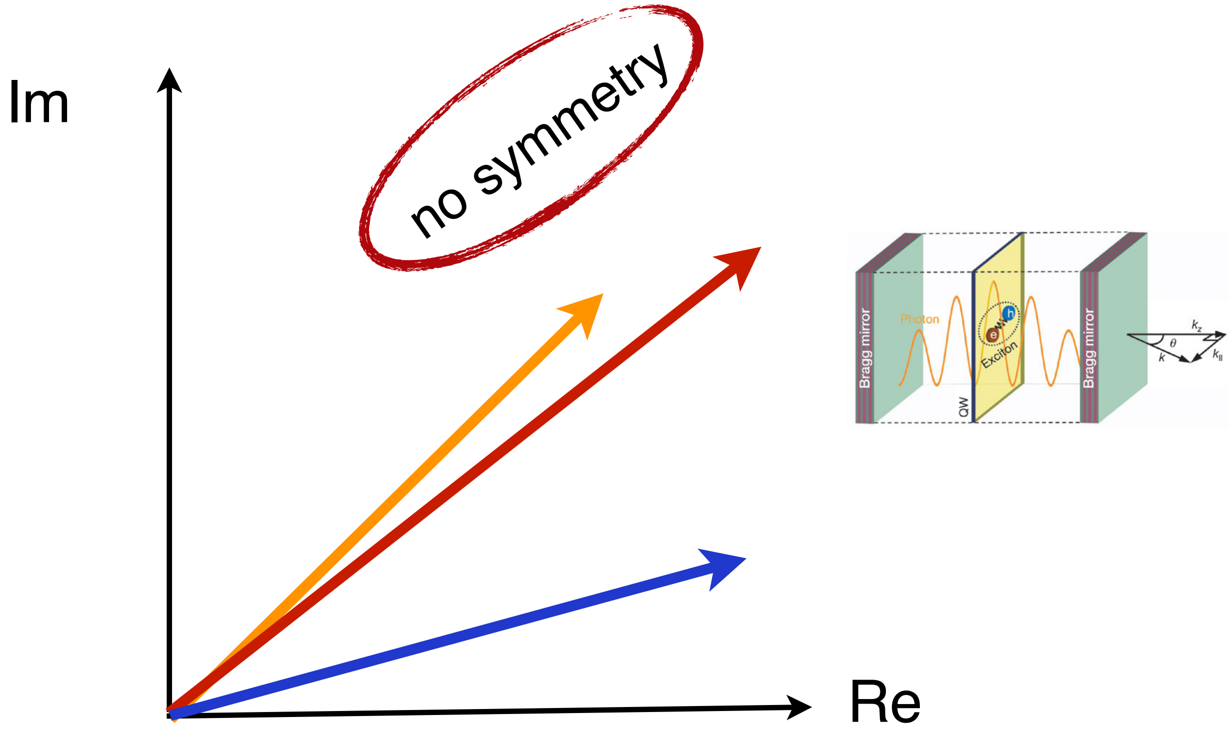
Geometric interpretation: equilibrium vs. non-equilibrium dynamics

equilibrium dynamics



- coherent and dissipative dynamics may occur simultaneously
- but they are not independent

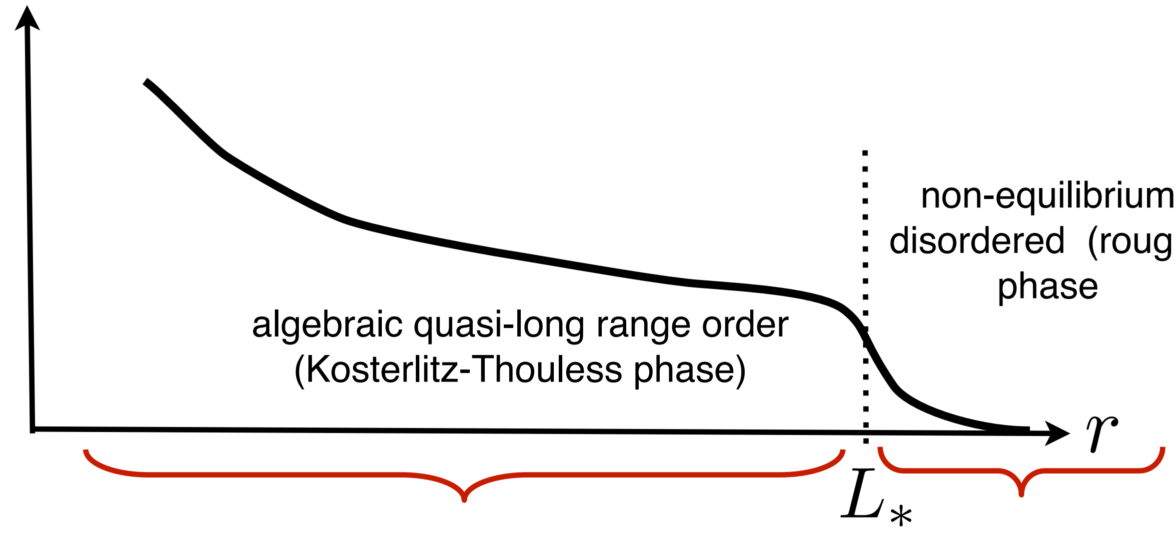
non-equilibrium dynamics



- coherent and dissipative dynamics do occur simultaneously
- they result from different dynamical resources

➡ what are the physical consequences of the spread in the complex plane?

Application:
Fate of BKT physics in driven open quantum systems



E. Altman, L. Sieberer, L. Chen, SD, J. Toner, PRX (2015)
G. Wachtel, L. Sieberer, SD, E. Altman, PRB (2016)
L. Sieberer, G. Wachtel, E. Altman, SD, PRB (2016)

L. He, L. Sieberer, E. Altman, SD, PRB (2015)
L. He, L. Sieberer, SD, PRL (2017)

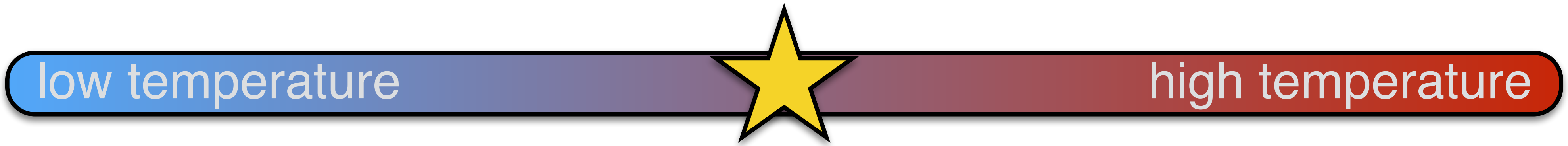
Microscopic
Quantum Optics

“Thermodynamic”
Many-body physics

Long wavelength
Statistical mechanics

Phase transitions in two dimensions

- continuous symmetry U(1): no spontaneous symmetry breaking, but a phase transition



- correlations

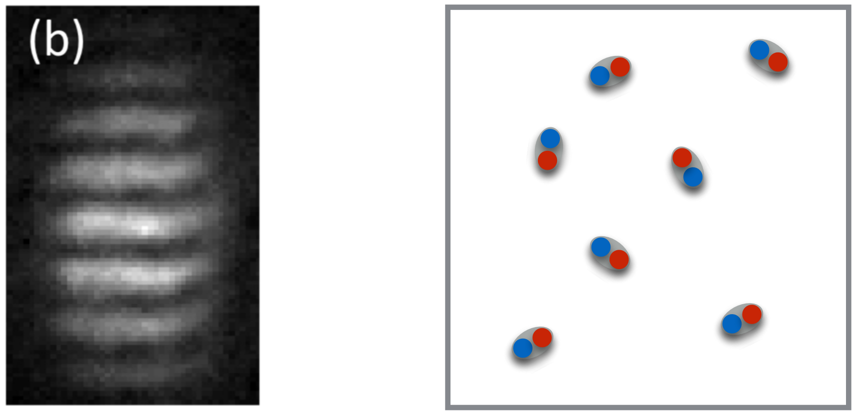
$$\langle \phi^*(\mathbf{x})\phi(0)\rangle \sim r^{-\frac{1}{2\pi K}} \qquad\qquad \sim e^{-r/\xi}$$

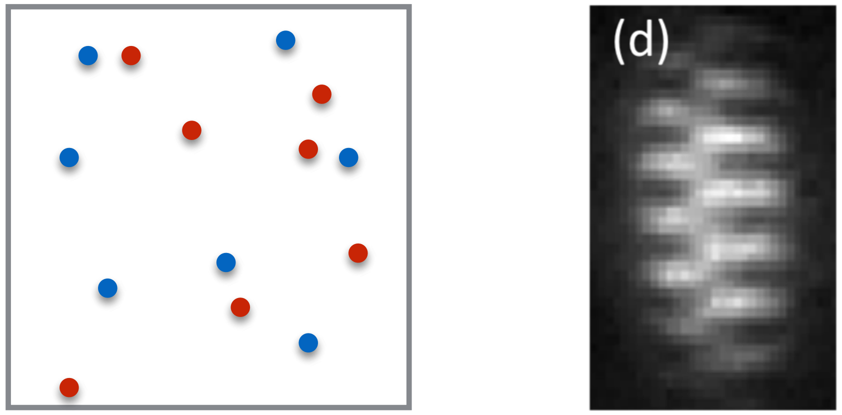
- responses: superfluidity

$$\rho_s \neq 0 \qquad\qquad \rho_s = 0$$

- BKT (Berezhinskii-Kosterlitz-Thouless) transition: unbinding of vortex-antivortex pairs

J. M. Kosterlitz, D. J. Thouless J. Phys. C (1973)

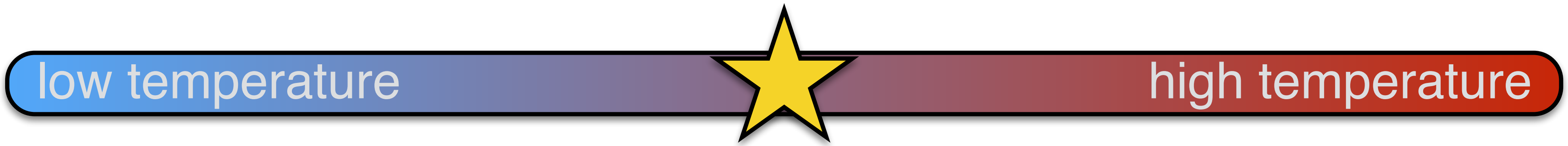




matter wave interferometry:
Z. Hadzibabic et al. Nature (2006)

… fate in driven open condensates?

Short reminder: Algebraic correlations



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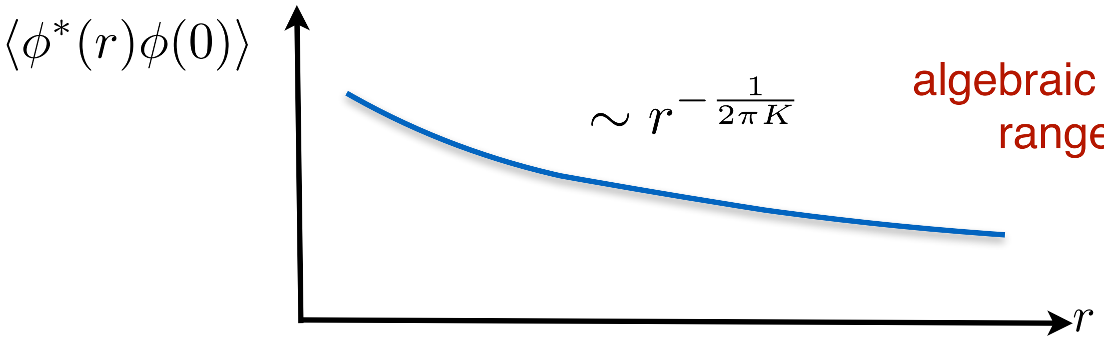
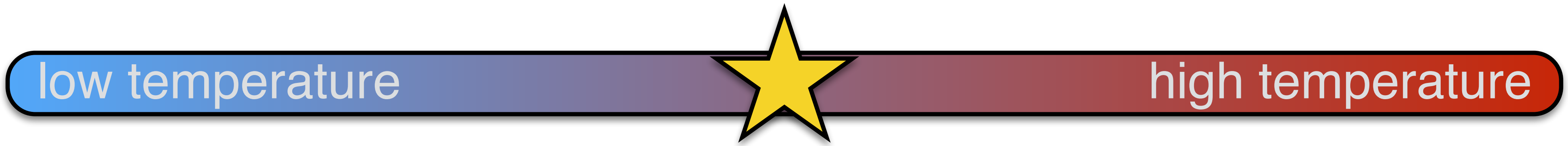
- physical reason: gapless spin wave fluctuations



algebraic quasi-long range order

Short reminder: Algebraic correlations



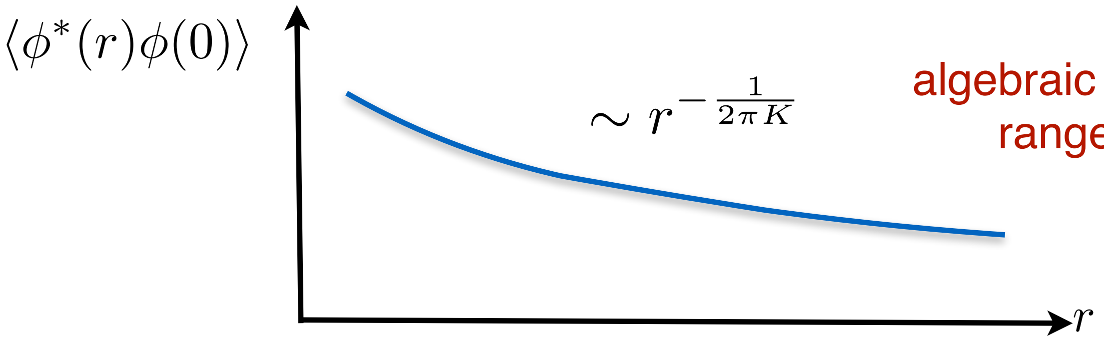
- correlations

$$\langle \phi^*(\mathbf{x})\phi(0)\rangle \sim r^{-\frac{1}{2\pi K}} \qquad\qquad \sim e^{-r/\xi}$$

- physical reason: gapless spin wave fluctuations
  - phase-amplitude decomposition $\phi(\mathbf{x}) = \rho(\mathbf{x})^{1/2} e^{i\theta(\mathbf{x})} \approx \sqrt{n_0}e^{i\theta(\mathbf{x})}$

$$\langle \phi^*(\mathbf{x})\phi(0)\rangle \approx n_0 \langle e^{i(\theta(\mathbf{x})-\theta(0))}\rangle = n_0 e^{-\frac{1}{2}\langle(\theta(\mathbf{x})-\theta(0))^2\rangle}$$

  - spin wave action $S = \frac{K}{2}\int d^2x (\nabla\theta)^2 = \frac{K}{2}\int \frac{d^2q}{(2\pi)^2}\, q^2\, \theta(-\mathbf{q})\theta(\mathbf{q})$
  - phase correlator $\frac{1}{2}\langle(\theta(\mathbf{x})-\theta(0))^2\rangle = \int^{1/a} \frac{d^2q}{(2\pi)^2}\frac{(e^{iqr}-1)}{Kq^2} = \frac{1}{2\pi K}\log(r/a)$



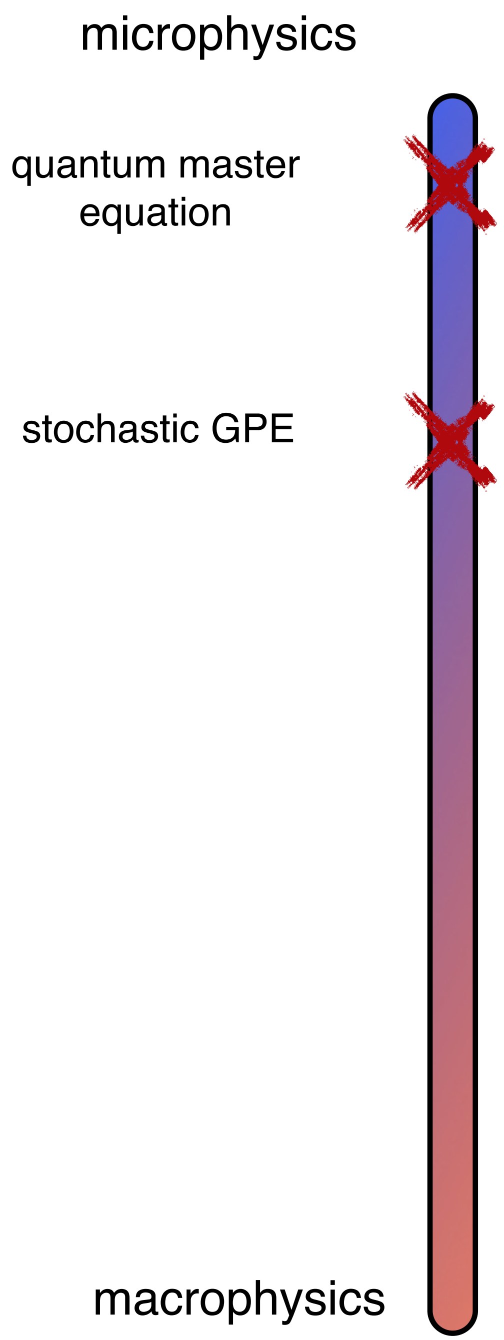
algebraic quasi-long range order

Description: Effective model

- mesoscopic starting point: driven-dissipative stochastic Gross-Pitaevski equation

$$i\partial_t \phi = \left[-\frac{\nabla^2}{2m} - \mu + i(\gamma_p - \gamma_l) + (g - i\kappa)\left|\phi\right|^2 \right] \phi + \zeta$$

$$\phi(t, \mathbf{x}) = \sqrt{\rho(t, \mathbf{x})} e^{i\theta(t, \mathbf{x})}$$



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$$\phi(t,\mathbf{x}) = \sqrt{\rho(t,\mathbf{x})}e^{i\theta(t,\mathbf{x})}$$

- effective low frequency dynamics

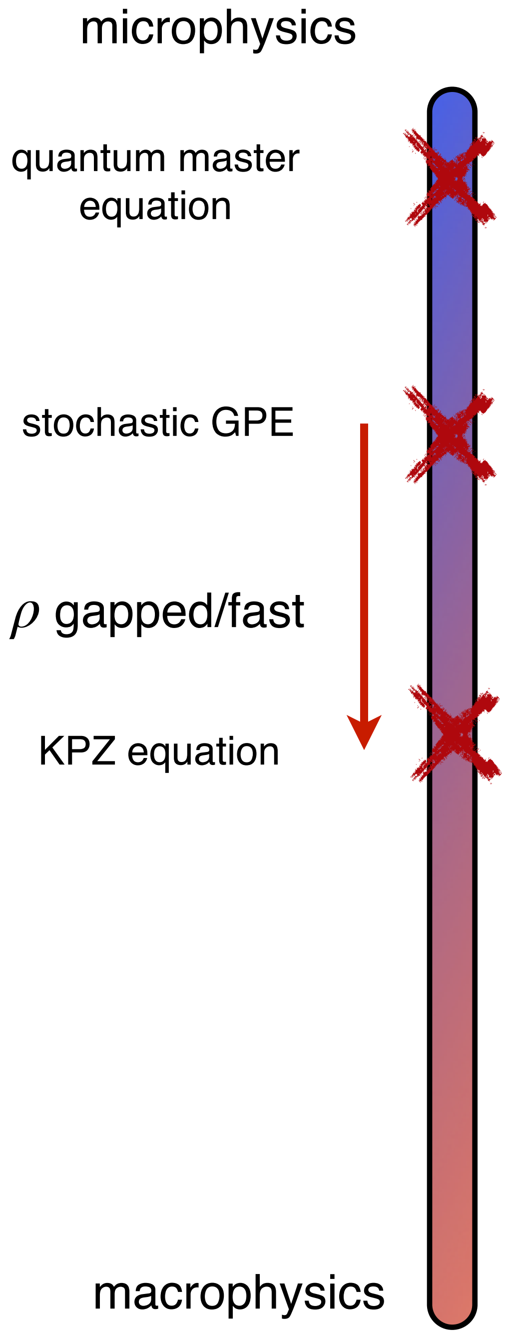
see also: G. Grinstein et al., PRL (1993)

$$\partial_t \theta = D\nabla^2\theta + \lambda(\nabla\theta)^2 + \xi$$

phase diffusion phase nonlinearity Markov noise

form of the KPZ equation

Kardar, Parisi, Zhang, PRL (1986)



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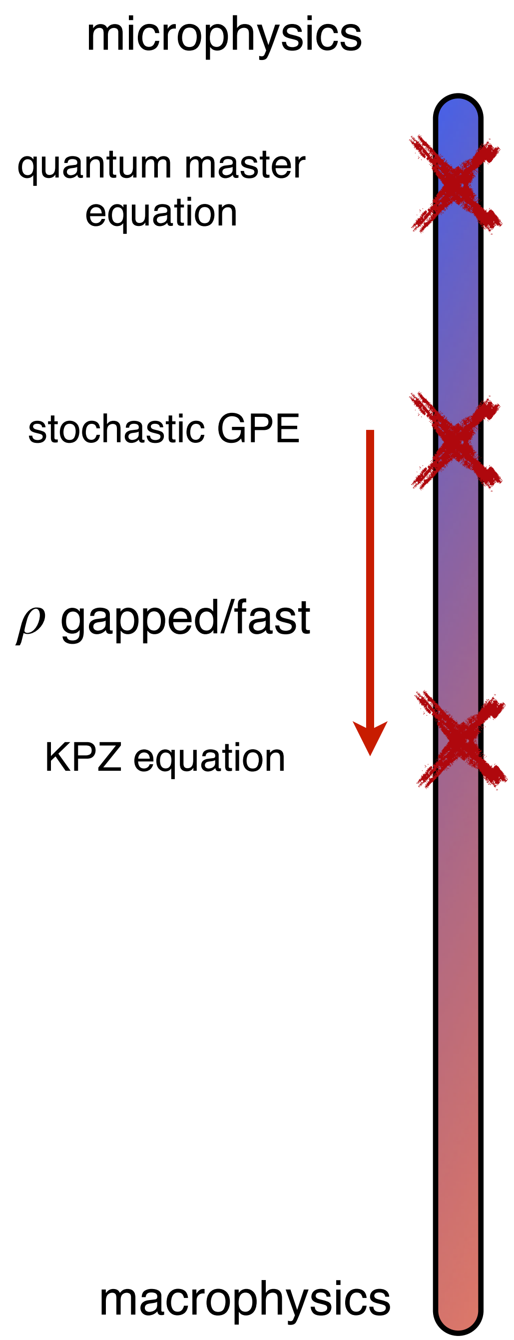
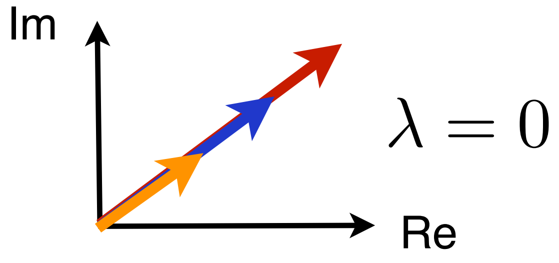
phase diffusion   phase nonlinearity   Markov noise

form of the KPZ equation

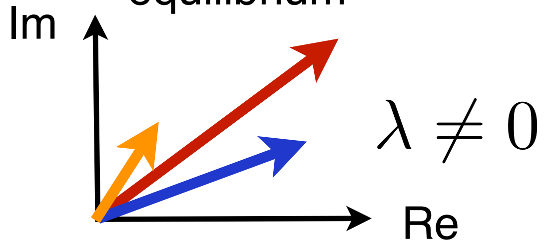
Kardar, Parisi, Zhang, PRL (1986)

- meaning: non-linear spin wave mode
- nonlinearity: single-parameter measure of non-equilibrium strength (ruled out in equilibrium by symmetry)





equilibrium



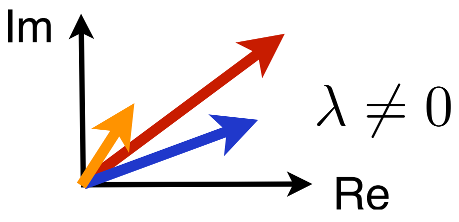
non-equilibrium

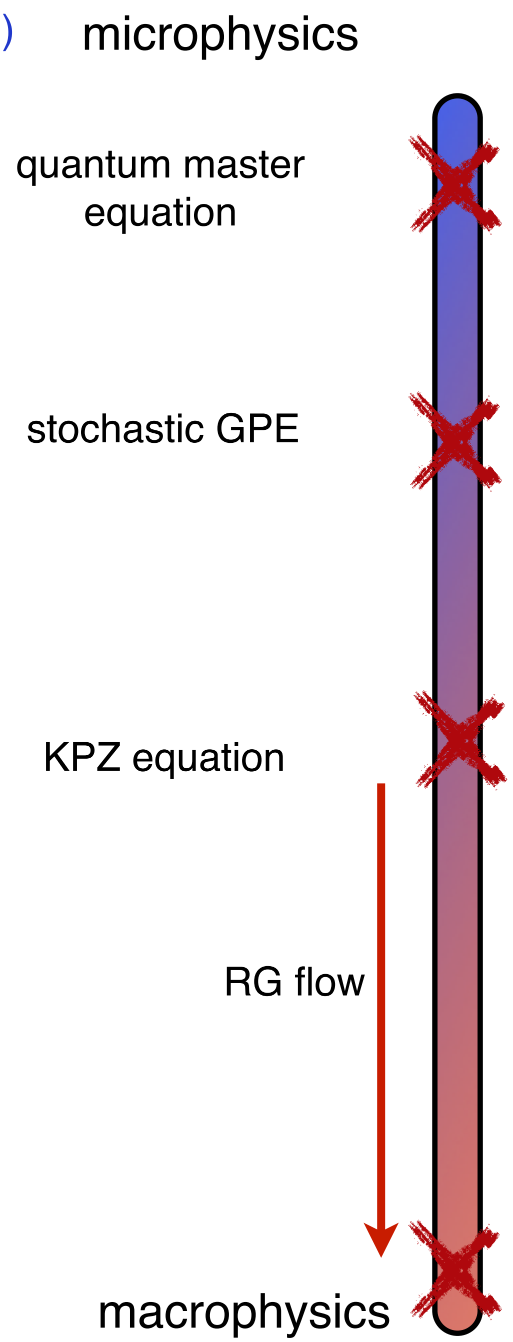
E. Altman, L. Sieberer, L. Chen, SD, J. Toner, PRX (2015)

Physical implication I: Smooth KPZ fluctuations

- How important are non-equilibrium conditions at large distance?

- behavior of non-equilibrium strength under coarse graining (RG flow)





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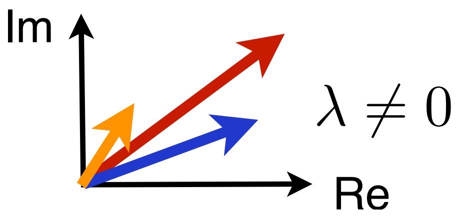
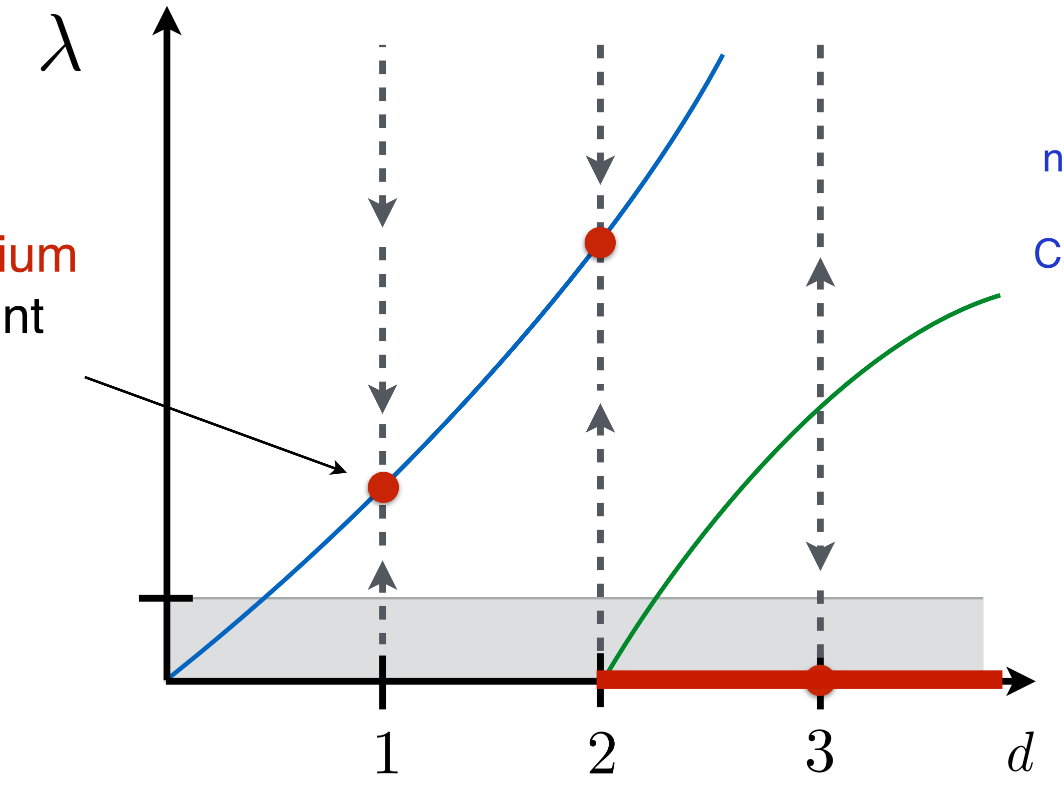
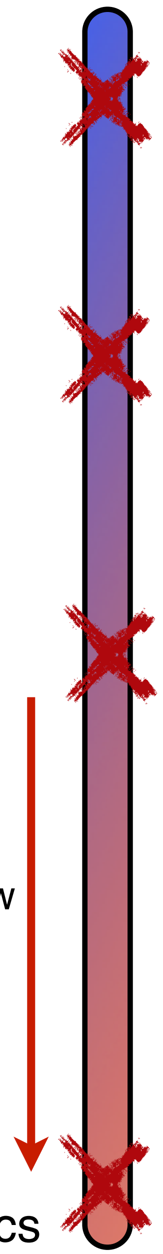




strong non-equilibrium
<-> KPZ fixed point

non-perturbative characterisation of
strong coupling fixed point:
Canet et al, PRL (2010); PRE (2011)

weak non-equilibrium
<-> equilibrium fixed point



microphysics

quantum master equation

stochastic GPE

KPZ equation

RG flow

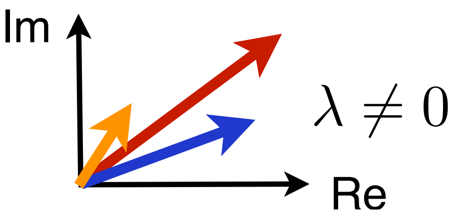
macrophysics

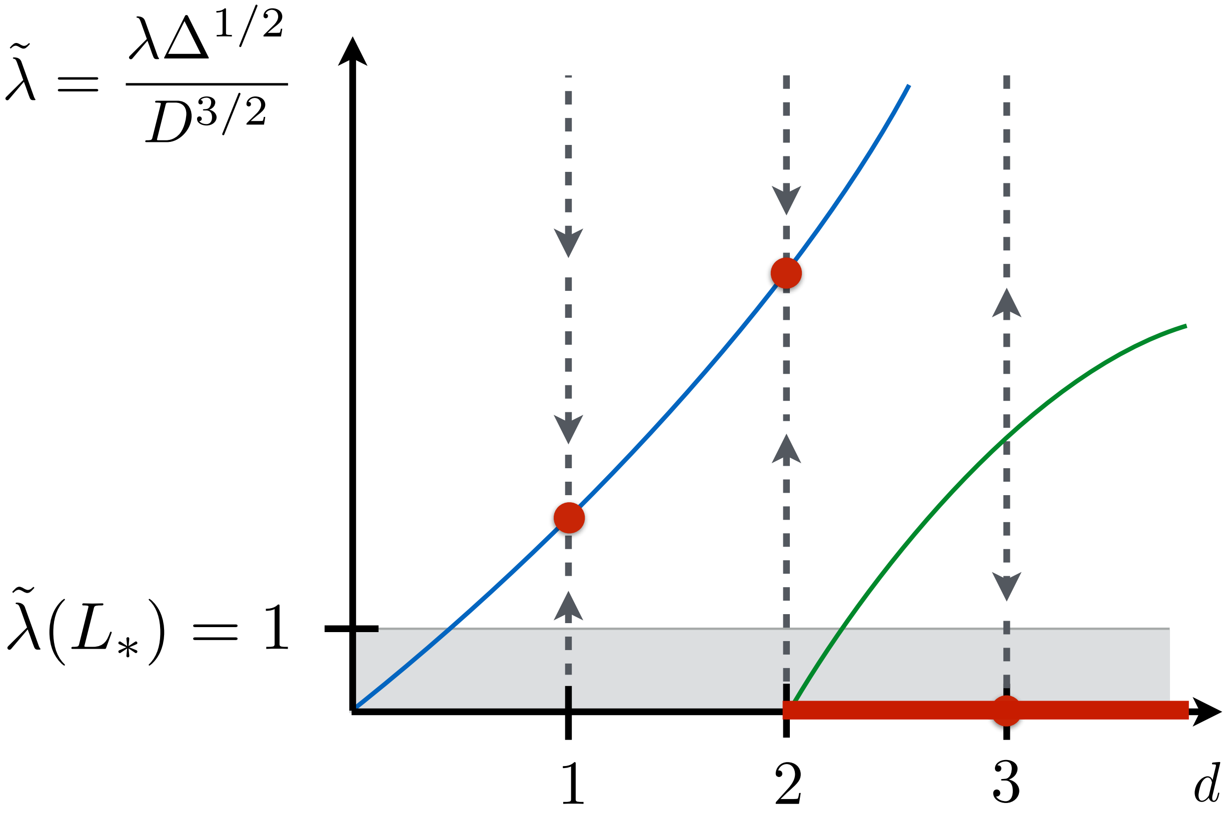
- general trend: non-equilibrium effects in systems with gapless mode are
  - enhanced in d = 1,2
  - softened in d = 3 (below a threshold)

Physical implication I: Smooth KPZ fluctuations

- How important are non-equilibrium conditions at large distance?

- behavior of non-equilibrium strength under coarse graining (RG flow)

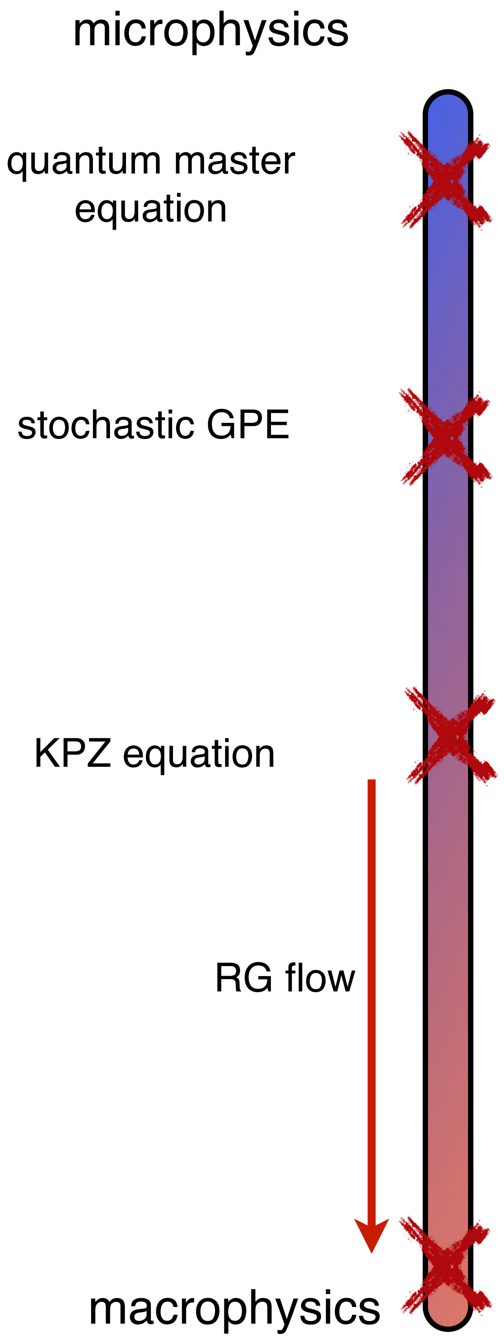




- 2D: implication: a length scale is generated

$$L_* = a_0 e^{\frac{16\pi D^3}{\lambda^2 \Delta}}$$

microscopic (healing) length



microphysics

quantum master equation

stochastic GPE

KPZ equation

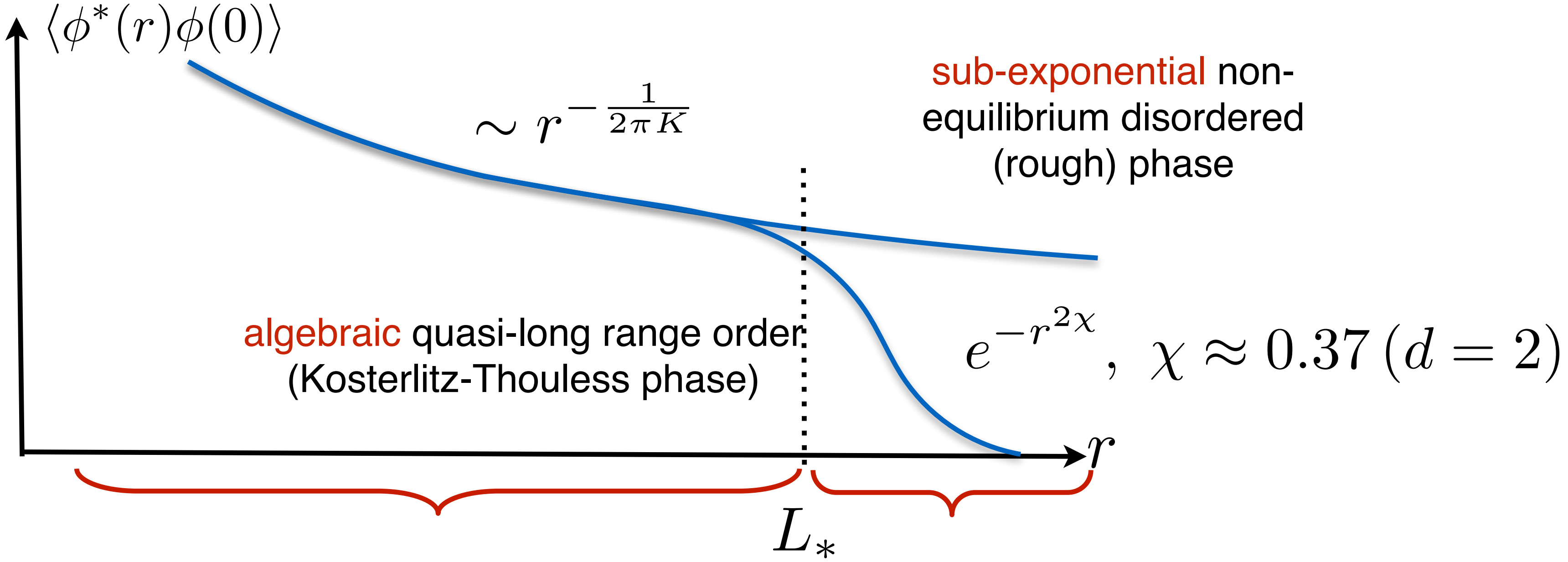
RG flow

macrophysics

E. Altman, L. Sieberer, L. Chen, SD, J. Toner, PRX (2015)

2D: Absence of algebraic order out of equilibrium

- generated length scale distinguishes two regimes: $L_* = a_0 e^{\frac{16\pi D^3}{\lambda^2 \Delta}}$



sub-exponential non-equilibrium disordered (rough) phase

algebraic quasi-long range order (Kosterlitz-Thouless phase)

equilibrium fixed point relevant

KPZ fixed point relevant

E. Altman, L. Sieberer, L. Chen, SD, J. Toner, PRX (2015)

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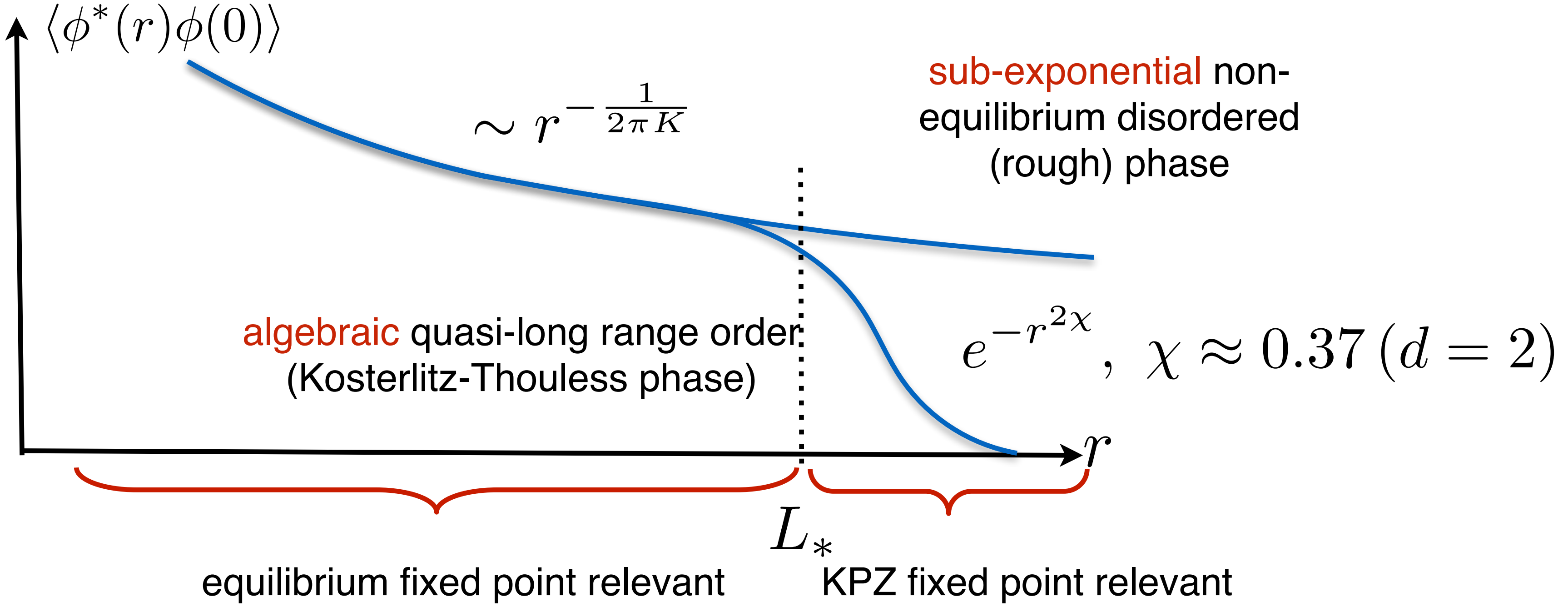

➡ algebraic order absent in any two-dimensional driven open system at the largest distances
➡ but crossover scale exponentially large for small deviations from equilibrium
➡ observation in 1D systems (KPZ scaling exponents found)

E. Altman, L. Sieberer, L. Chen, SD, J. Toner, PRX (2015)

2D: Absence of algebraic order out of equilibrium

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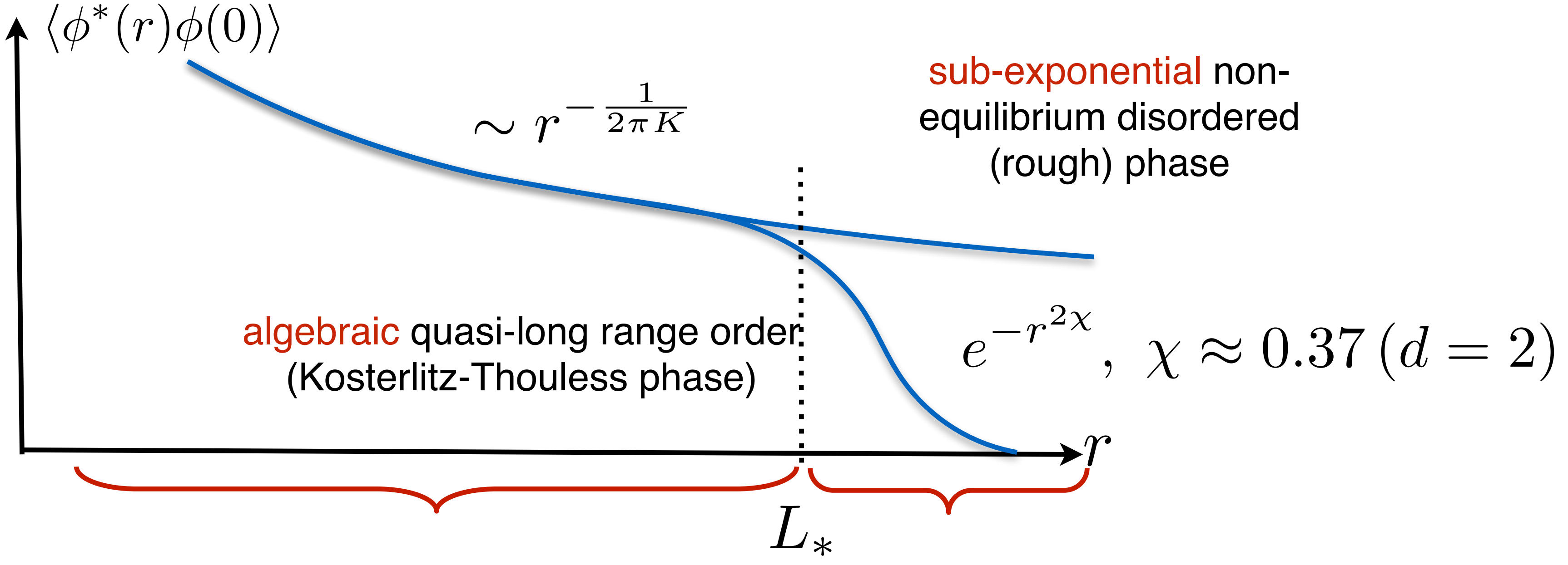


sub-exponential non-equilibrium disordered (rough) phase

algebraic quasi-long range order (Kosterlitz-Thouless phase)

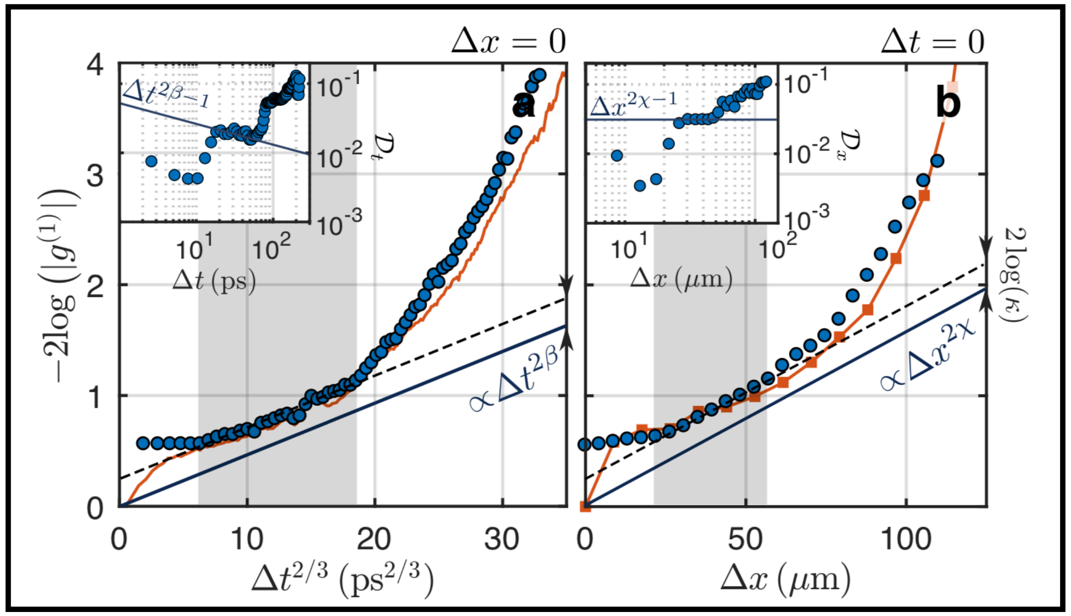
$e^{-r^{2\chi}}, \ \chi \approx 0.37 \, (d = 2)$

equilibrium fixed point relevant

KPZ fixed point relevant

- ➡ algebraic order absent in any two-dimensional driven open system at the largest distances
- ➡ but crossover scale exponentially large for small deviations from equilibrium
- ➡ observation in 1D systems (KPZ scaling exponents found)

$\beta = 1/3$ temporal scaling



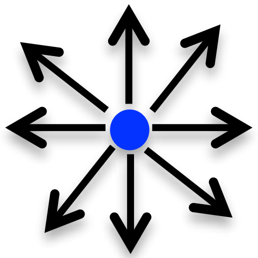
spatial scaling $\chi = 1/2$

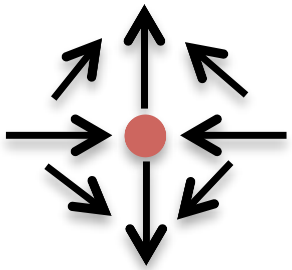
Q. Fontaine et al., Nature (2022)

Physical implications II: Non-equilibrium Kosterlitz-Thouless

- KPZ equation for phase variable

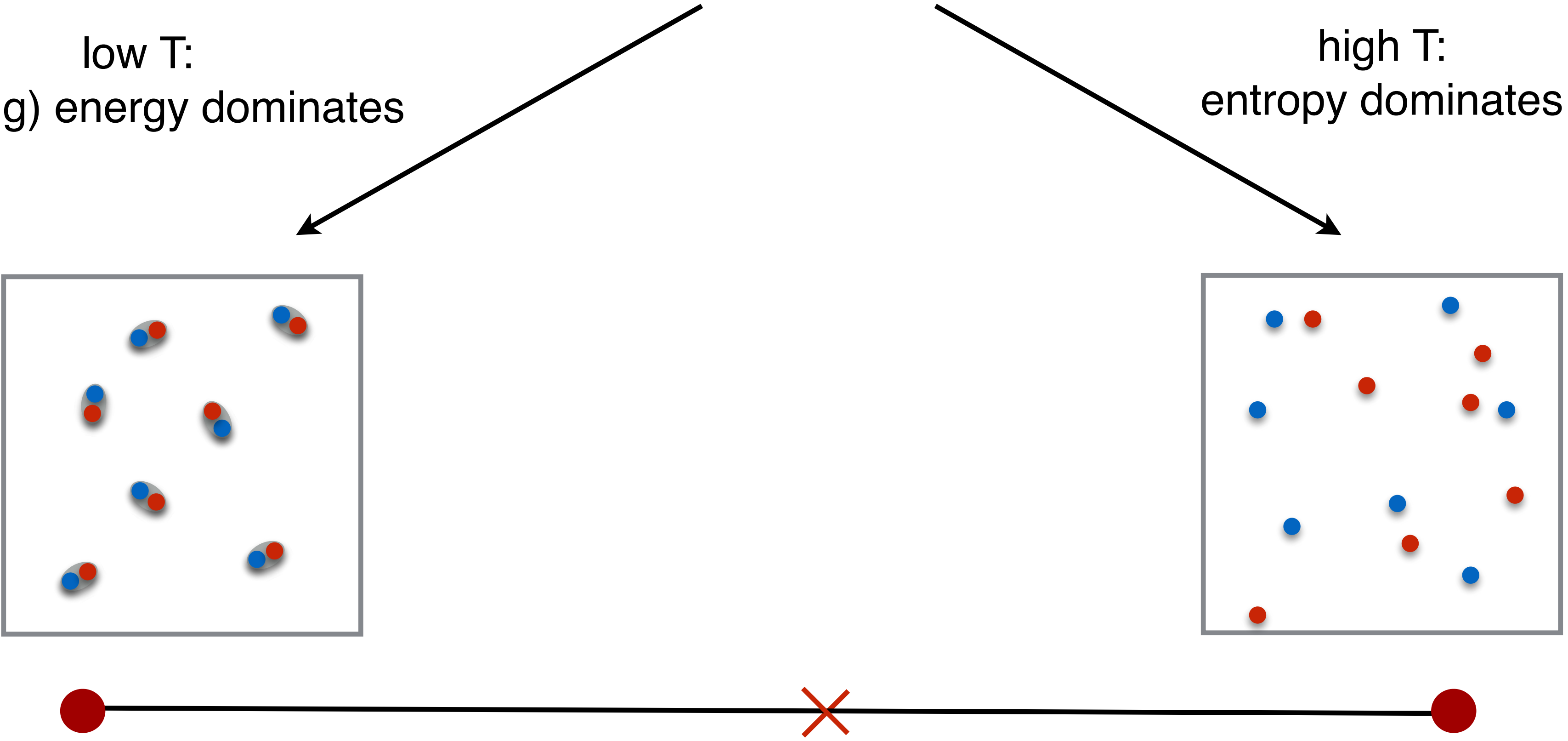
$$\partial_t \theta = D\nabla^2\theta + \lambda(\nabla\theta)^2 + \xi$$


vortex


anti-vortex

- compact nature of phase allows for vortex defects in 2D!
- key ingredient of Kosterlitz-Thouless transition

$$F = E - TS$$

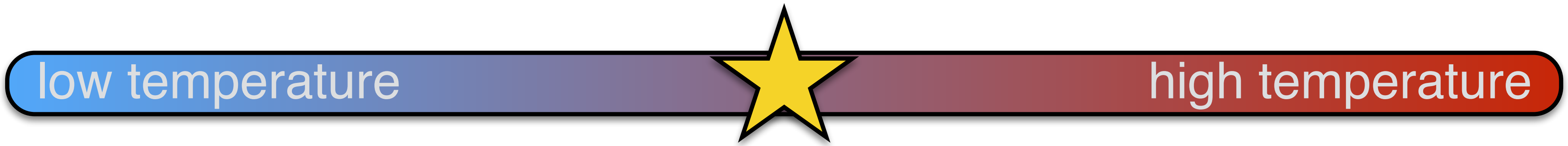


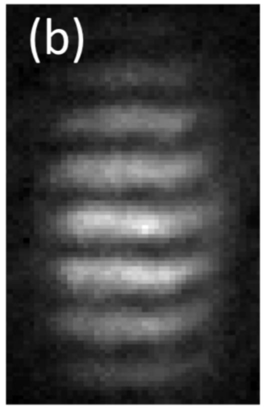
low T:
(binding) energy dominates

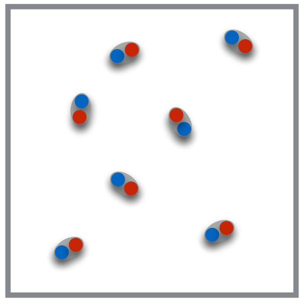
high T:
entropy dominates

➡ how is this scenario modified in the driven system?

Mini-review: BKT transition

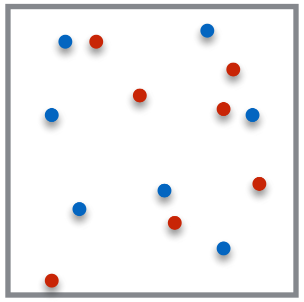


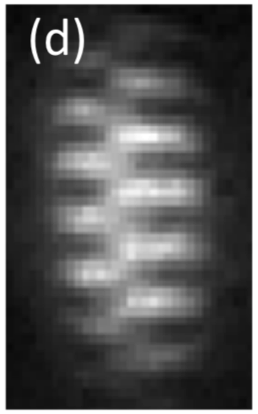




- BKT transition: unbinding of vortex-antivortex pairs

  J. M. Kosterlitz, D. J. Thouless J. Phys. C (1973)



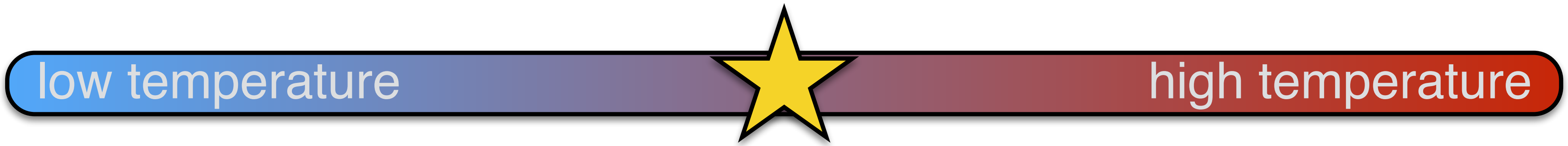


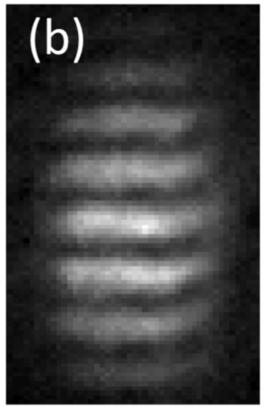
- Single vortex picture: balance of energy (deterministic) and entropy (statistic)
  - Low T: vortices and antivortices bound in neutral pairs (irrelevant at long distance)
  - Q: when is it favorable (free energy minimum) to have unbound vortices?

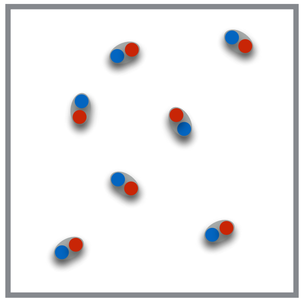
- more accurate approach via electrostatic duality: J. M. Kosterlitz, J. Phys. C (1974)

  electric field <=> smooth phase fluct. (KPZ) $\mathbf{E} = -\hat{e}_z \times \nabla\theta$ charges <=> vortices

Mini-review: BKT transition

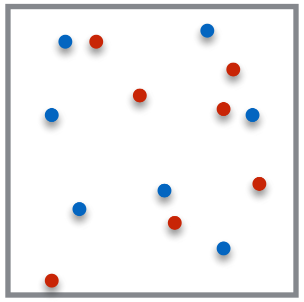


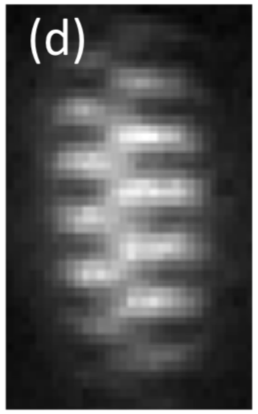




- BKT transition: unbinding of vortex-antivortex pairs

  J. M. Kosterlitz, D. J. Thouless J. Phys. C (1973)





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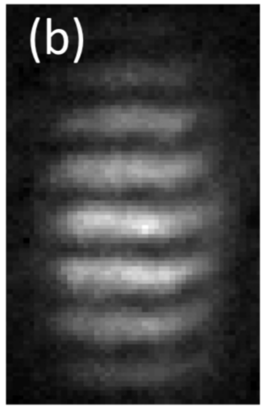

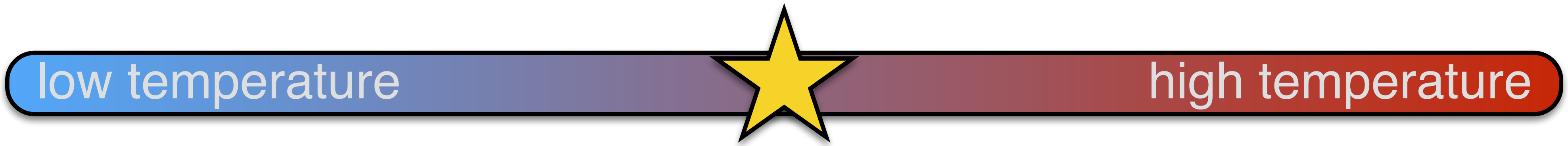

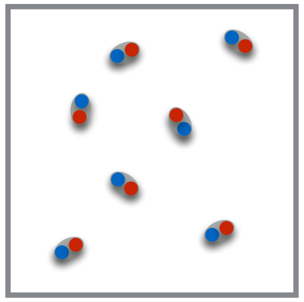

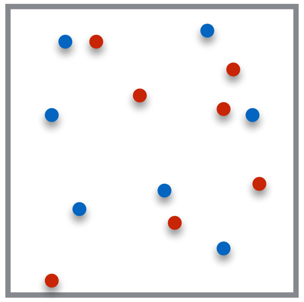

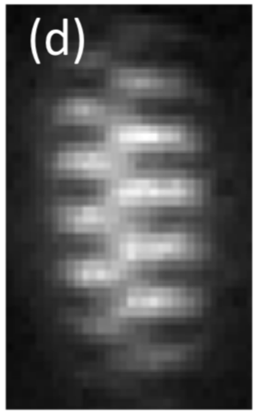

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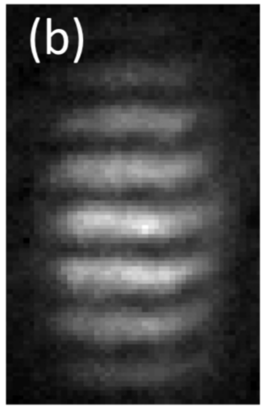

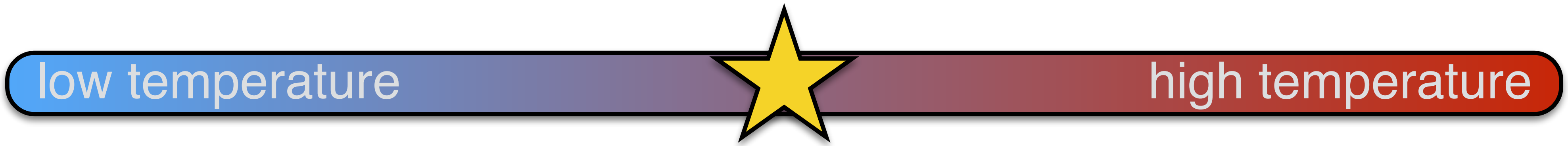

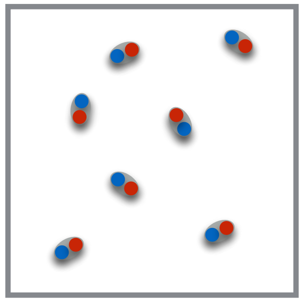

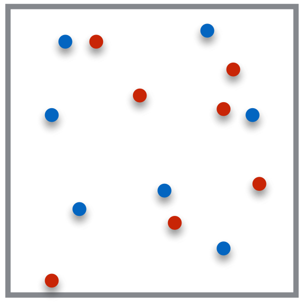

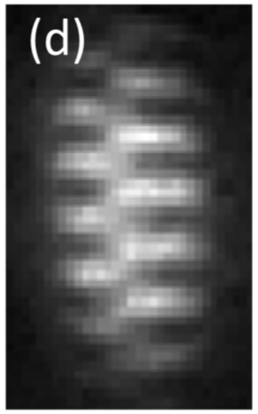

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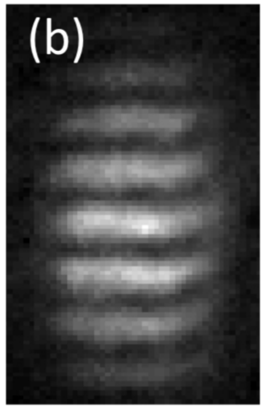

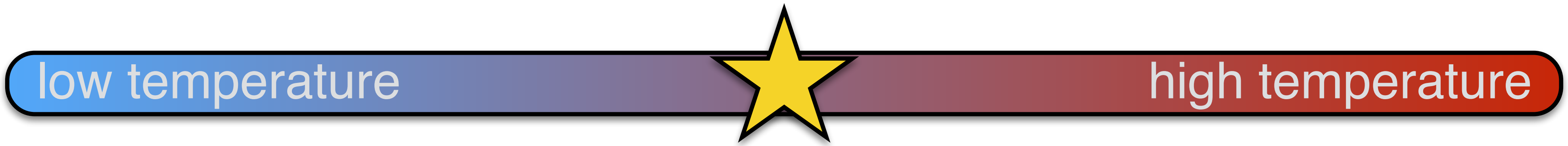

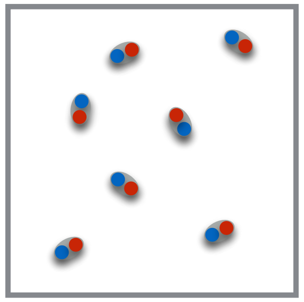

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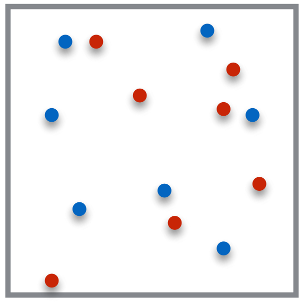

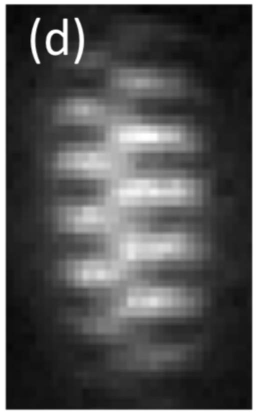

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Outline

Keldysh theory general: A. Kamenev, *Field theory or non-equilibrium systems*, Cambridge University Press

Review: L. Sieberer, M. Buchhold, SD, *Keldysh Field Theory for Driven Open Quantum Systems*, Reports on Progress in Physics (2016);

L. Sieberer, M. Buchhold, J. Marino, SD, *Universality in Driven Open Quantum Matter*, arxiv (2023)

1. From the Lindblad equation to the Lindblad-Keldysh functional integral

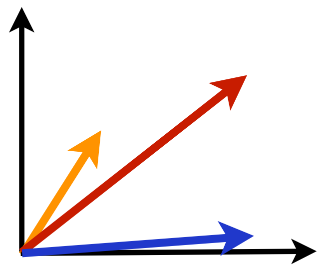
- Lindblad equation for driven open quantum matter
- construction of Lindblad-Keldysh functional integral

$$\partial_t \rho = -i[H, \rho] + \mathcal{L}[\rho]$$

2. KPZ equation in exciton-polariton condensates

$$\mathrm{e}^{\mathrm{i}\Gamma[\Phi]} = \int \mathcal{D}\delta\Phi \mathrm{e}^{\mathrm{i}S_M[\Phi+\delta\Phi]}$$

- background: semiclassical limit, classifying eq. vs. non-eq. states
- from XP to KPZ: absence of algebraic order out of equilibrium
- compact KPZ and non-equilibrium phase transition

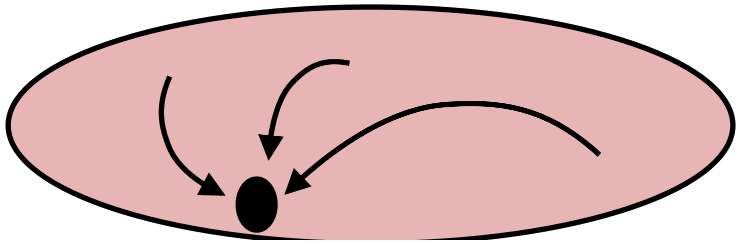


3. Principles of universality in driven open quantum matter

- the principles: eq. vs. non-eq.; pure vs. mixed states; weak vs. strong symmetries
- application: 1D KPZ in open vs. closed systems

4. Macroscopic non-equilibrium phenomena from weak non-equilibrium drive

- non-equilibrium O(N) models: phase structure, limit cycles
- novel non-equilibrium criticality at onset of a limit cycle
- route towards KPZ via breaking of time translation symmetry



5. Quantum aspects: topology in driven open quantum matter

$$\frac{\theta}{4\pi} \int A_+ \, dA_+ - A_- \, dA_-$$

- fermion topological dark states in Lindblad evolution
- universality of topological response: pure states, mixed states

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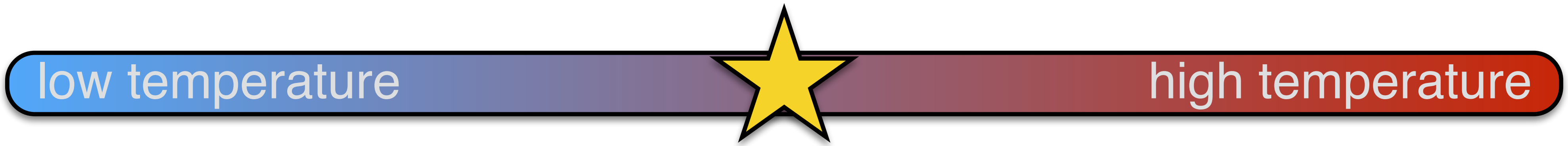

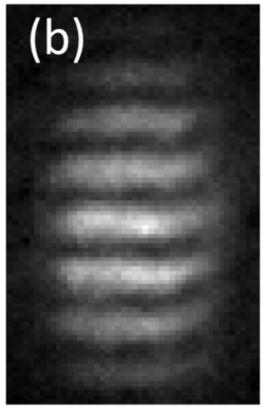

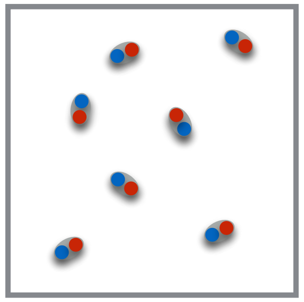

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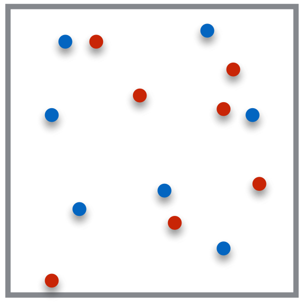

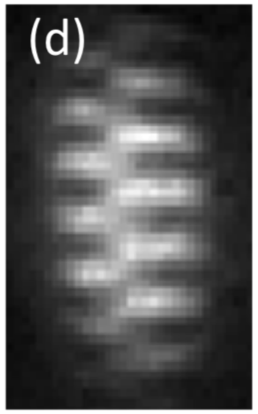

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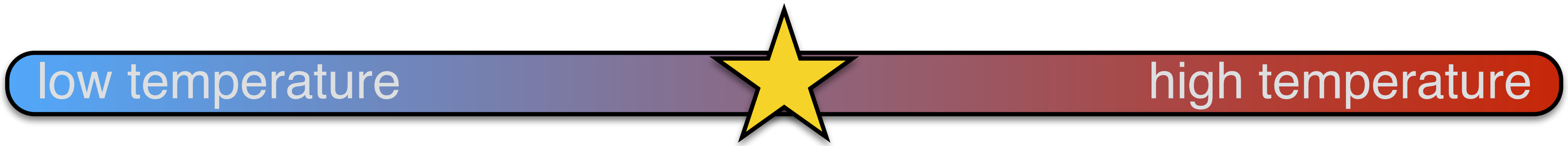

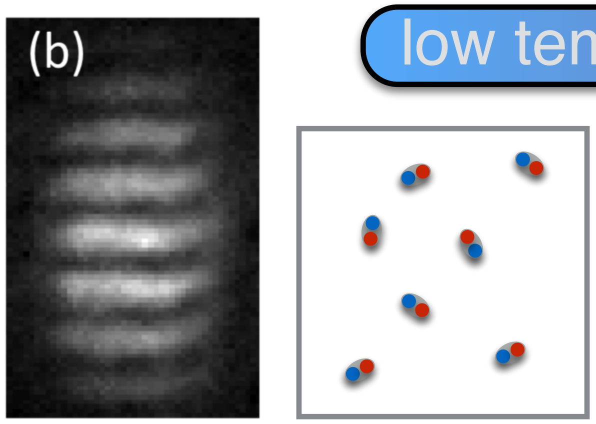

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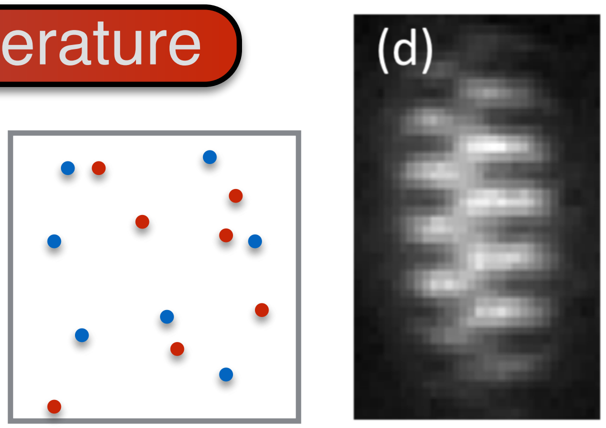

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L. Sieberer, G. Wachtel, E. Altman, SD, PRB (2016)
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- wait a second — we ignored a fundamental symmetry of polaritons so far: local discrete gauge invariance

$$\phi(t,\mathbf{x}) = \rho(t,\mathbf{x})e^{i\theta(t,\mathbf{x})} \qquad \theta_{t,\mathbf{x}} \mapsto \theta_{t,\mathbf{x}} + 2\pi n_{t,\mathbf{x}}$$

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$$\theta_{t+\epsilon,\mathbf{x}} = \theta_{t,\mathbf{x}} + \epsilon\left(\mathcal{L}[\theta]_{t,\mathbf{x}} + \eta_{t,\mathbf{x}}\right) + 2\pi n_{t,\mathbf{x}}$$

$$\mathcal{L}[\theta]_{t,\mathbf{x}} = -\sum_{\hat{\mathbf{a}}}\left[D\sin(\theta_{t,\mathbf{x}} - \theta_{t,\mathbf{x}+\hat{\mathbf{a}}}) + \frac{\lambda}{2}(\cos(\theta_{t,\mathbf{x}} - \theta_{t,\mathbf{x}+\hat{\mathbf{a}}}) - 1)\right]$$

stochastic difference equation

$$Z = \sum_{\{\tilde{n}_{t,\mathbf{x}}\}}\int \mathcal{D}[\theta]e^{iS[\theta,\tilde{n}]}$$

discrete noise MSRJD functional integral

$$Z \propto \sum_{\substack{\{n_{vX},\tilde{n}_{vX},\\ \mathbf{J}_{vX},\tilde{\mathbf{J}}_{vX}\}}}\int \mathcal{D}[\phi,\tilde{\phi},\mathbf{A},\tilde{\mathbf{A}}]e^{iS[\phi,\tilde{\phi},\mathbf{A},\tilde{\mathbf{A}},n_v,\tilde{n}_v,\mathbf{J}_v,\tilde{\mathbf{J}}_v]}$$

non-equilibrium electrodynamic theory

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Charges <=> vortices

Dynamical & non-equilibrium analog of Kosterlitz-Thouless construction

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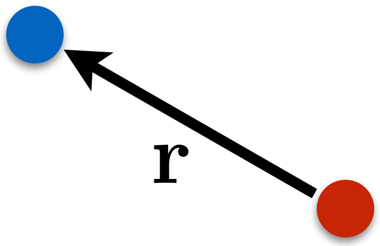
non-equilibrium electrodynamic theory

➡ Q: What is the effective theory for vortices out of equilibrium?

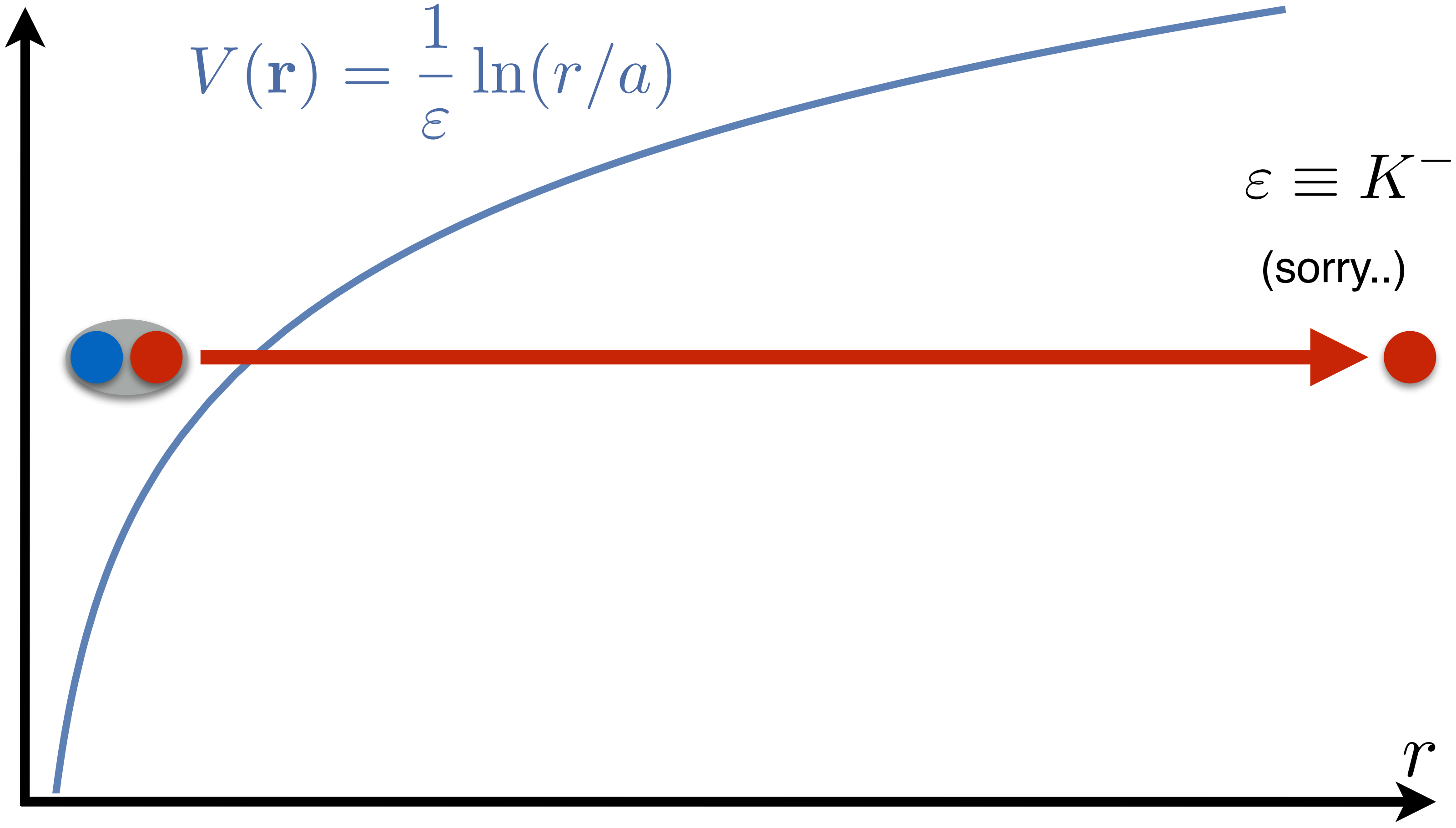
Effective theory for a single vortex-antivortex pair

- equation of motion for a single vortex-antivortex pair

$$\frac{d\mathbf{r}}{dt} = -\mu \nabla V(r) + \boldsymbol{\xi}$$



equilibrium: Coulomb potential (2D)



$$V(\mathbf{r}) = \frac{1}{\varepsilon} \ln(r/a)$$

$$\varepsilon \equiv K^{-1} \equiv D^{-1}$$

(sorry..)

(diffusion cst. / spin stiffness)

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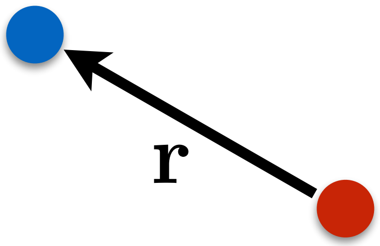
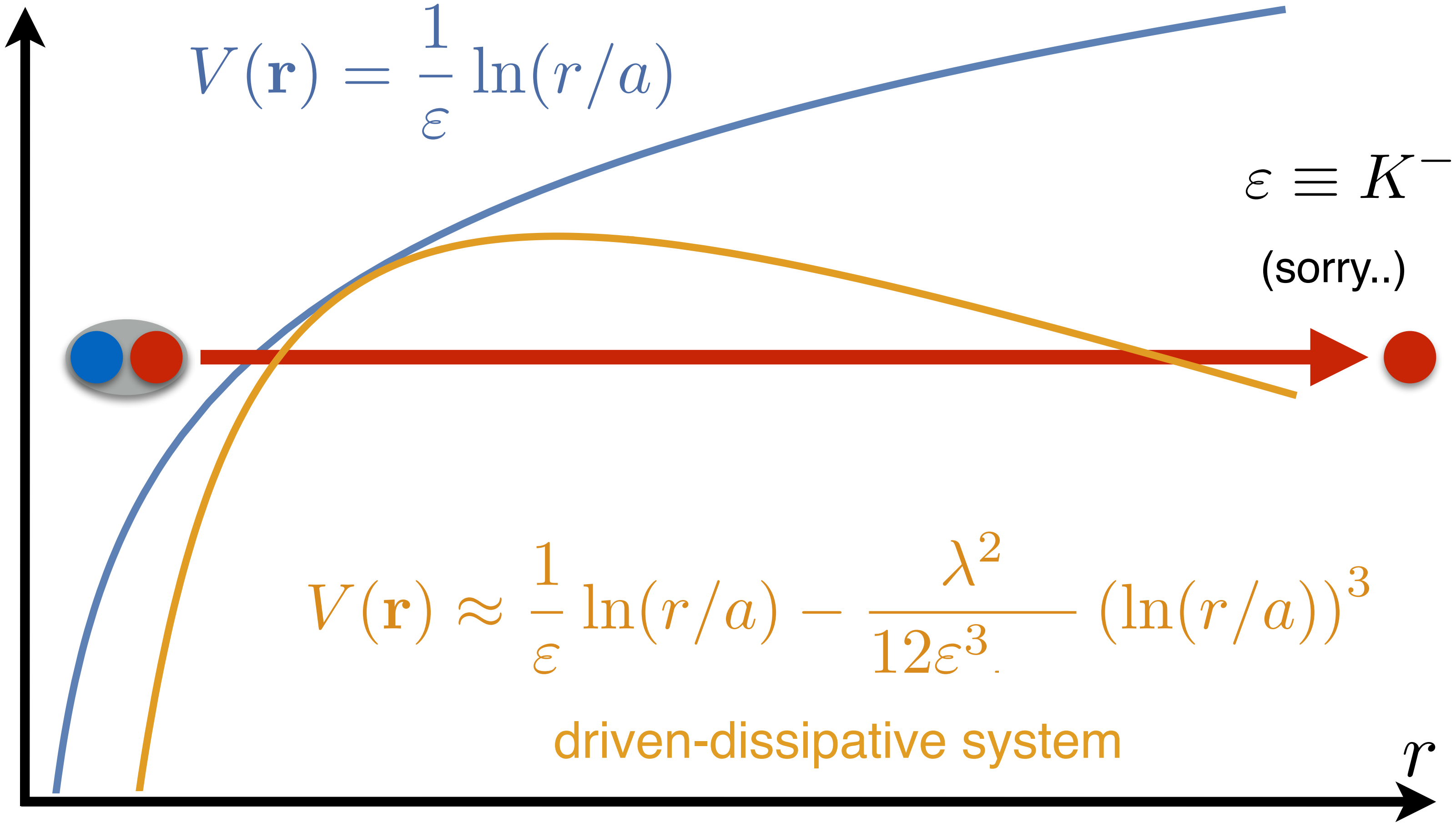




equilibrium: Coulomb potential (2D)

$$V(\mathbf{r}) = \frac{1}{\varepsilon} \ln(r/a)$$

$$\varepsilon \equiv K^{-1} \equiv D^{-1}$$

(sorry..)

(diffusion cst. / spin stiffness)

$$V(\mathbf{r}) \approx \frac{1}{\varepsilon} \ln(r/a) - \frac{\lambda^2}{12\varepsilon^3} (\ln(r/a))^3$$

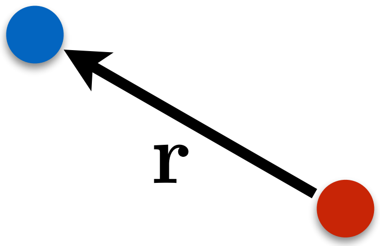
driven-dissipative system

➡ noise-activated unbinding for a single pair (at exp small rate)

Effective theory for a single vortex-antivortex pair

- equation of motion for a single vortex-antivortex pair

$$\frac{d\mathbf{r}}{dt} = -\mu \nabla V(r) + \boldsymbol{\xi}$$



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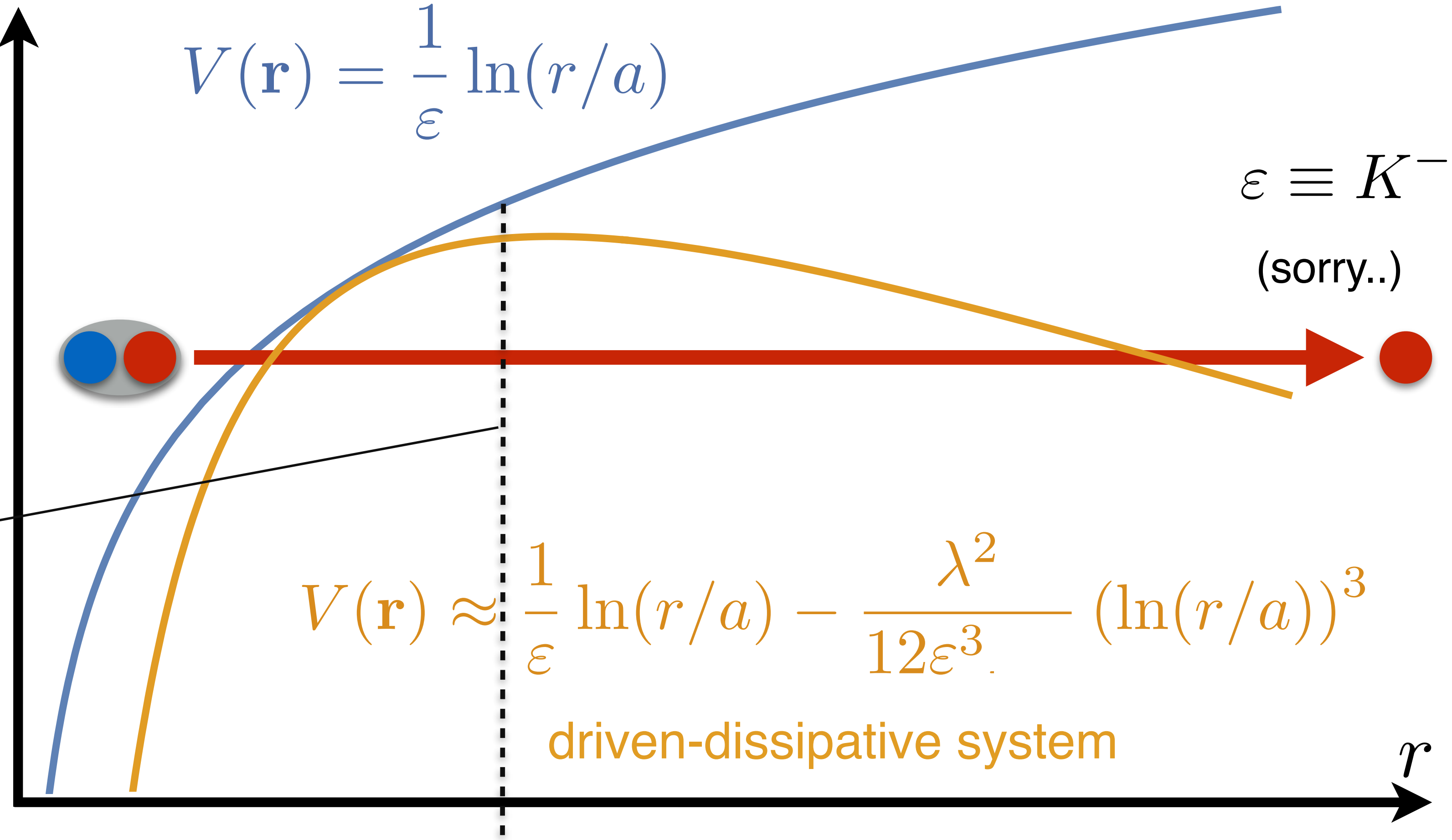

$$\varepsilon \equiv K^{-1} \equiv D^{-1}$$

(sorry..)

(diffusion cst. / spin stiffness)

length scale:

$$L_v = a_0 e^{\frac{2D}{\lambda}}$$

see also: I Aranson et al., PRB (1998) two-vortex problem

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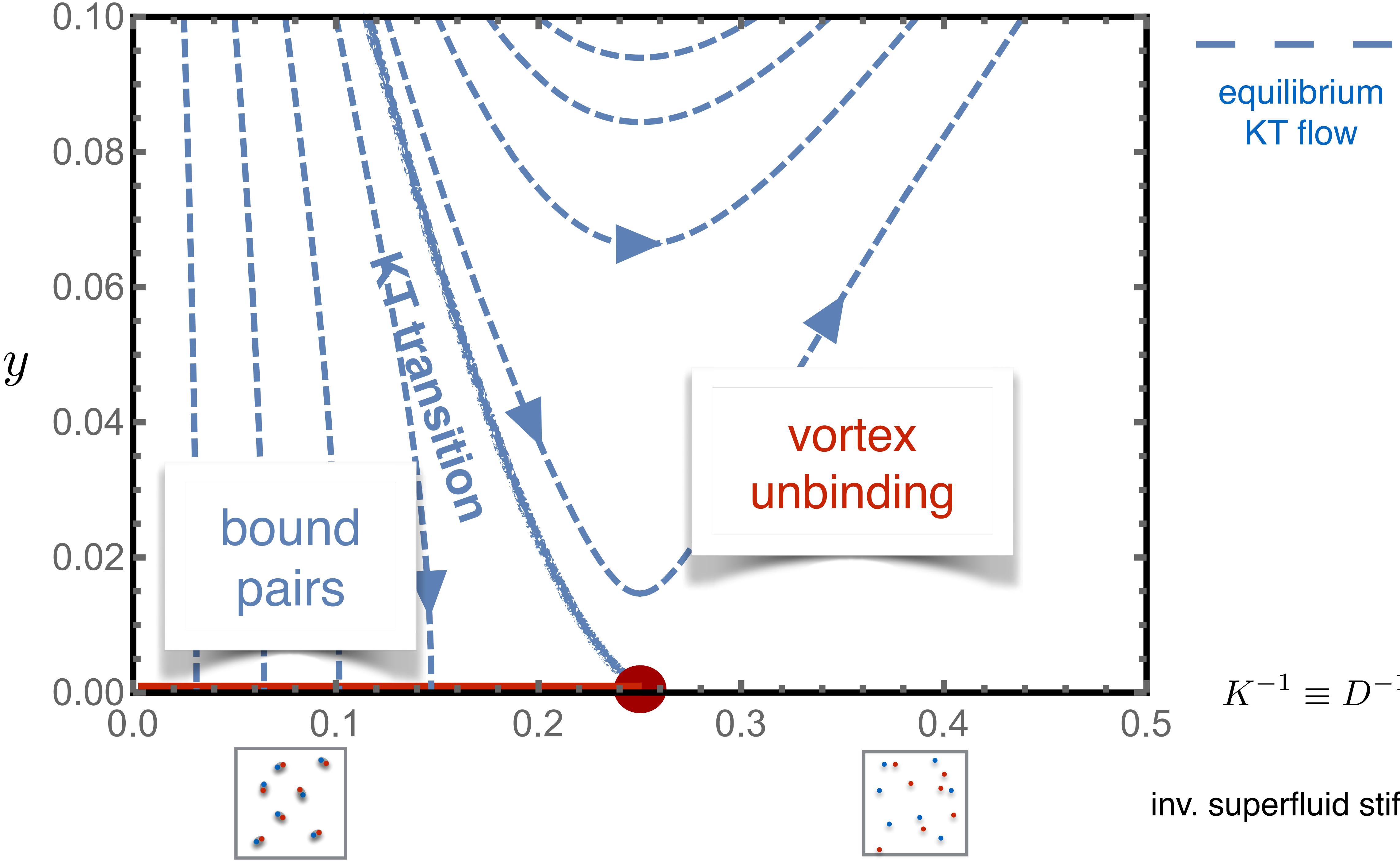
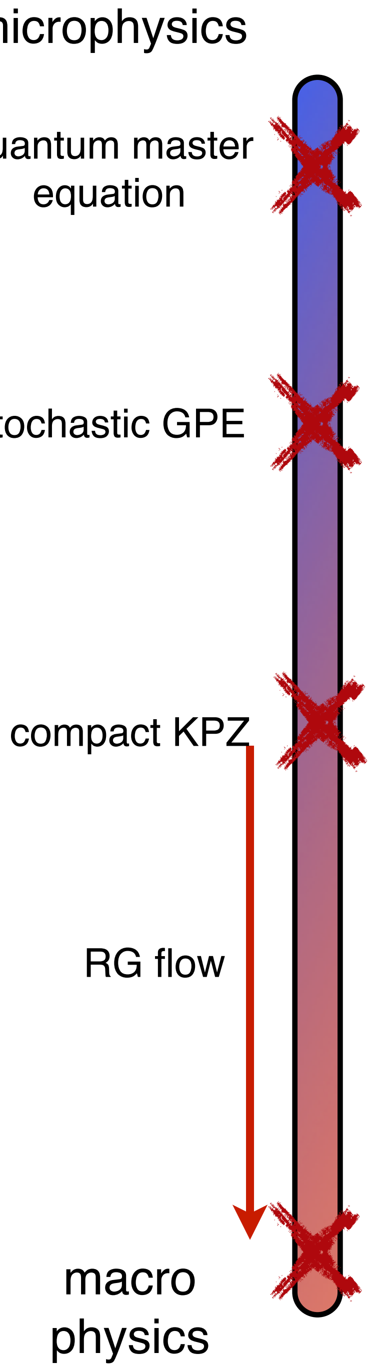
Many pairs: Corrections to Kosterlitz-Thouless flow

$$\frac{d\varepsilon}{d\ell} = \frac{2\pi^2 y^2}{T} \qquad \frac{dy}{d\ell} = \left[2 - \frac{\pi}{\epsilon T} + \cancel{\frac{\lambda^2}{4K^2D^2}}\left(\frac{1}{4} + \ell\right)\right] y \qquad \frac{dT}{d\ell} = \cancel{\frac{\lambda^2 T}{2K^2D^2}}\left(\frac{1}{4} + \ell\right)$$

- parameter y: ~ probability to create single vortex at distance l (flow parameter)





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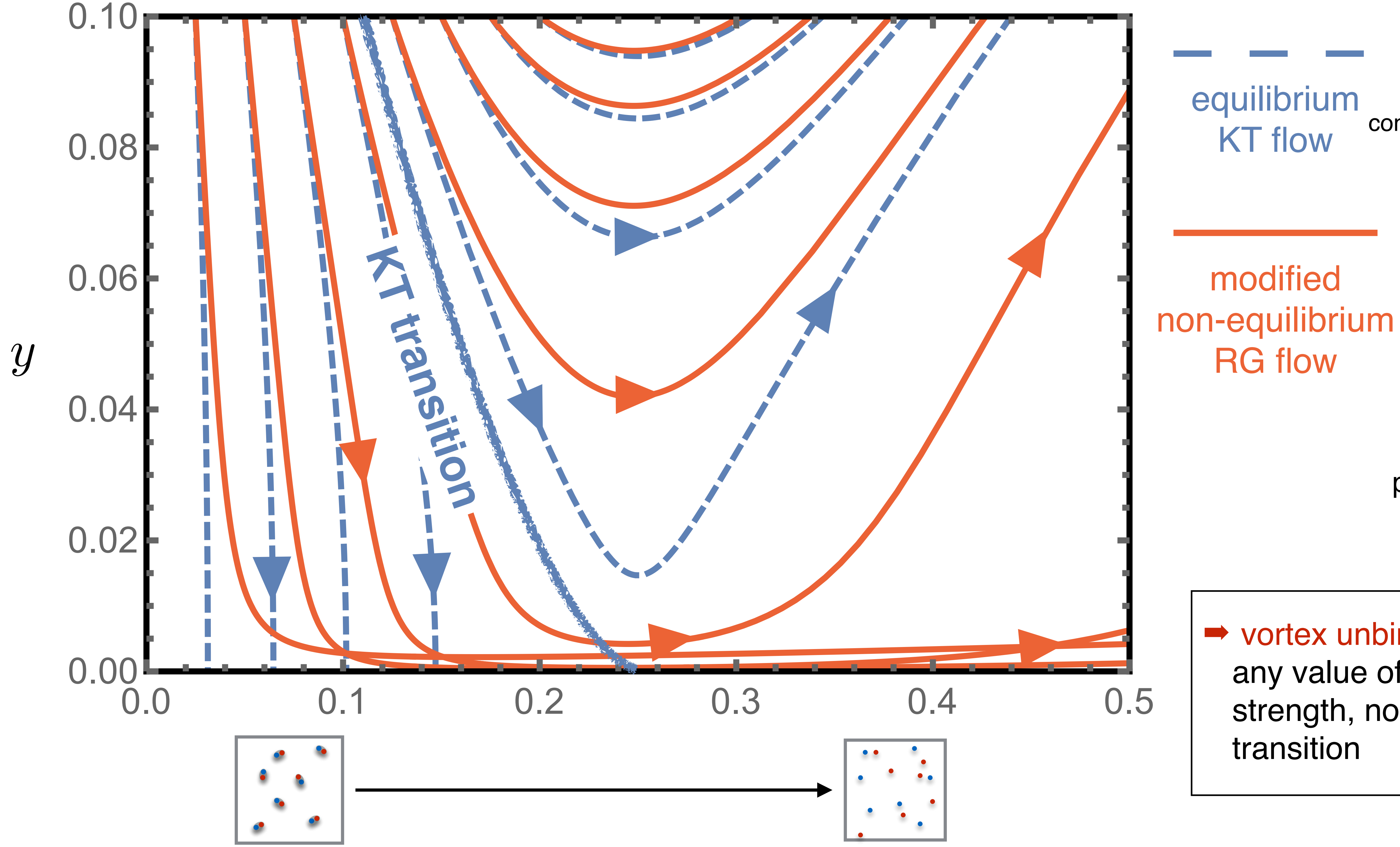
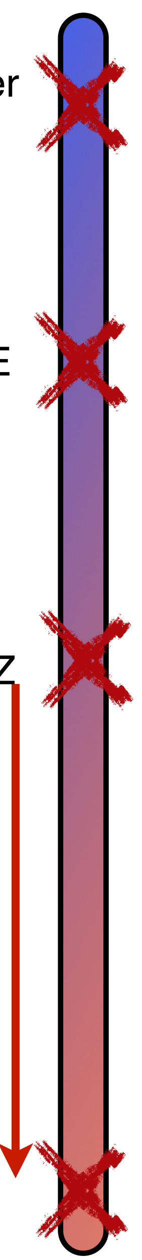




➡ vortex unbinding for any value of the noise strength, no phase transition

Competing length scales and suppression of KT

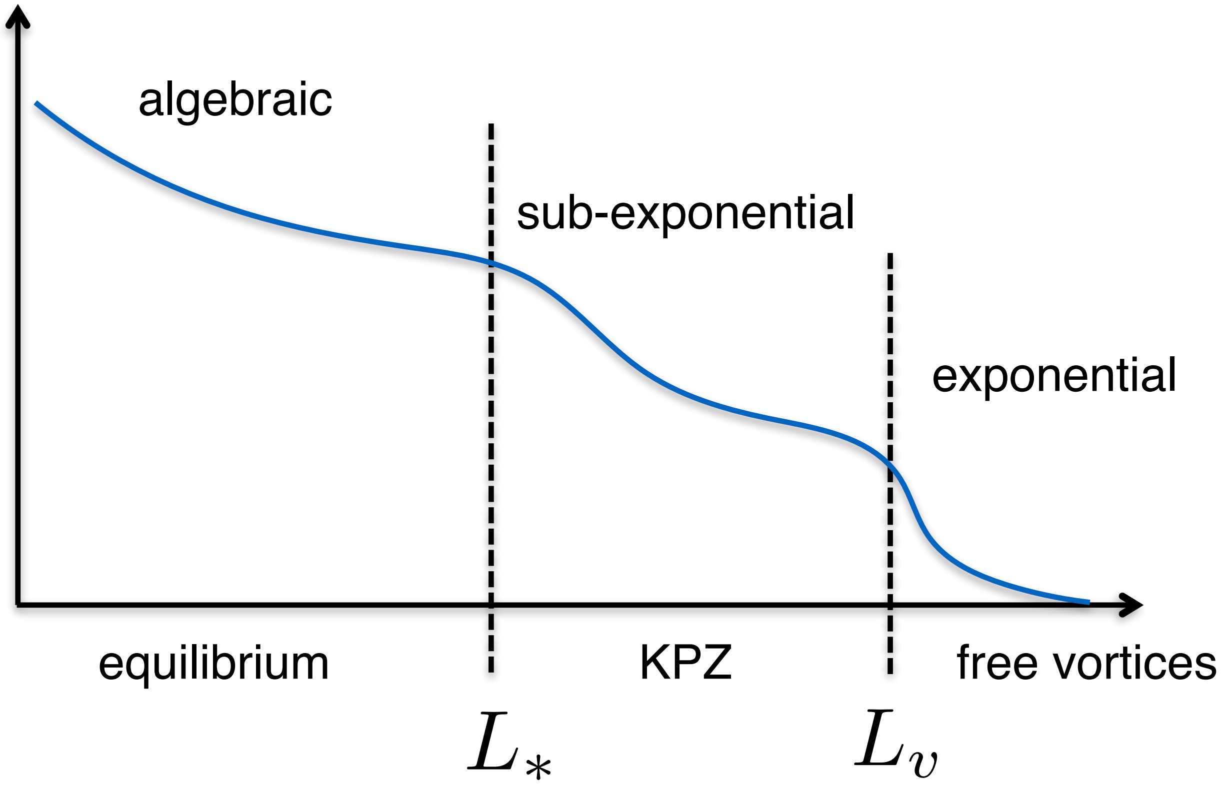
- two emergent length scales in complementary approaches:

$$L_* = a_0 e^{\frac{16\pi D^3}{\lambda^2 \Delta}}$$

KPZ length

$$L_v = a_0 e^{\frac{2D}{\lambda}}$$

vortex length



- numerical confirmation of two-scale scenario in 1D (defects: vortices in (1+1)D space-time)

L. He, L. Sieberer, SD PRL (2017)

- 2D lattice simulations encouraging to see KPZ

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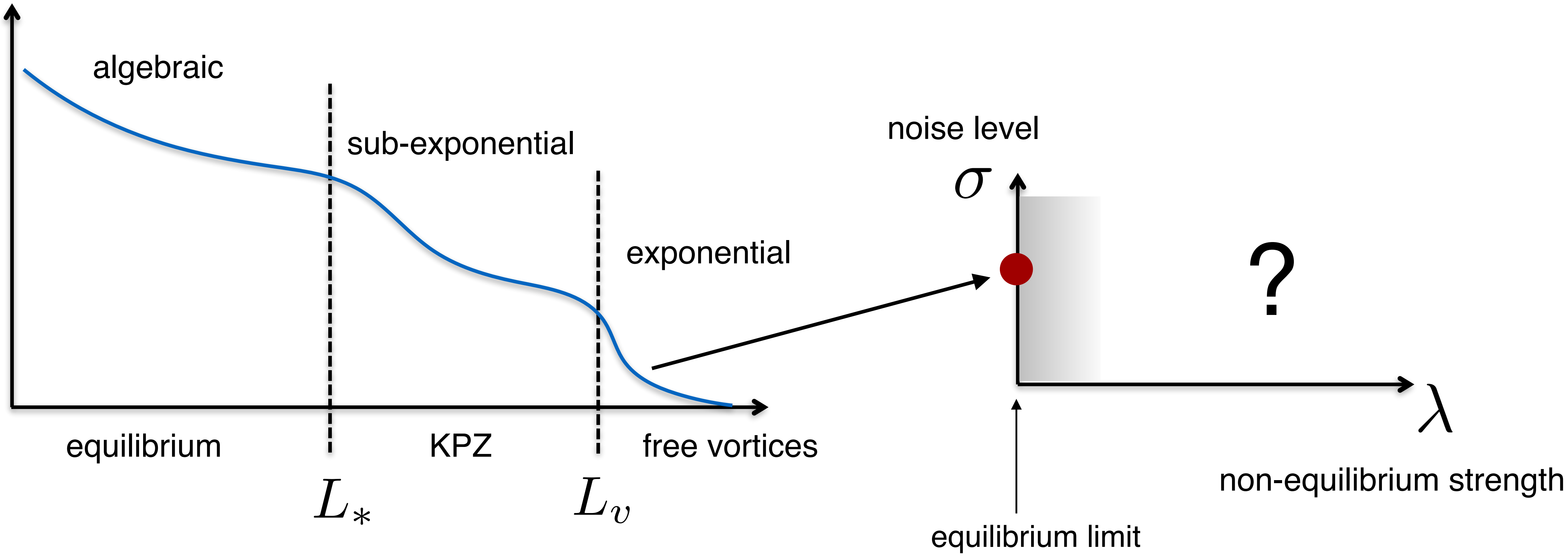

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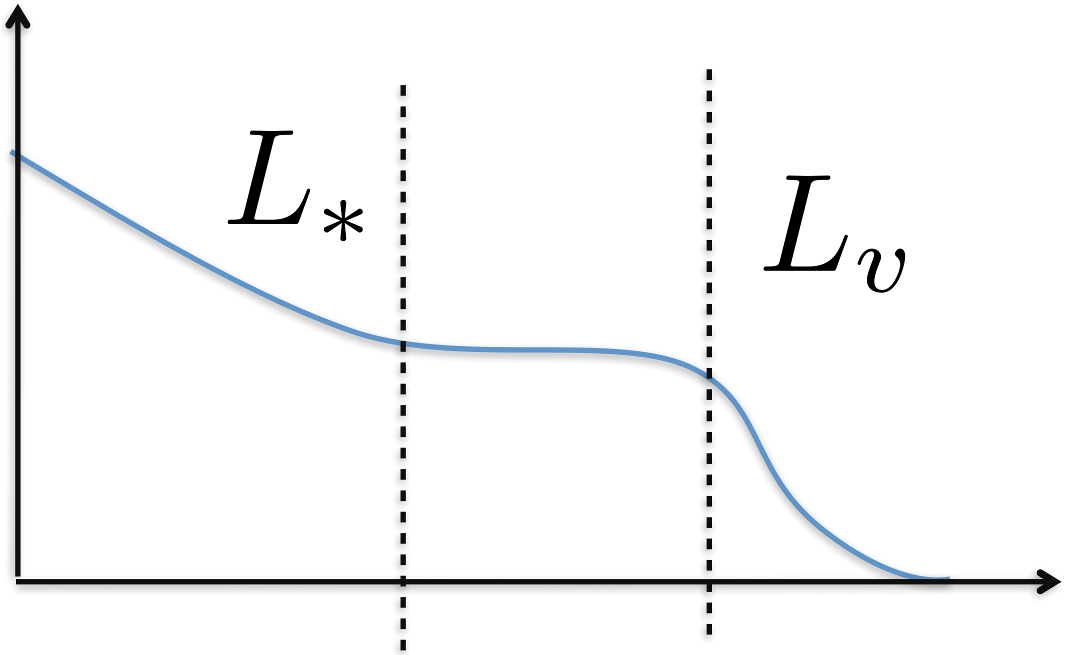
Kosterlitz-Thouless physics fragile to non-equilibrium perturbation

A phase transition driven by non-equilibrium strength



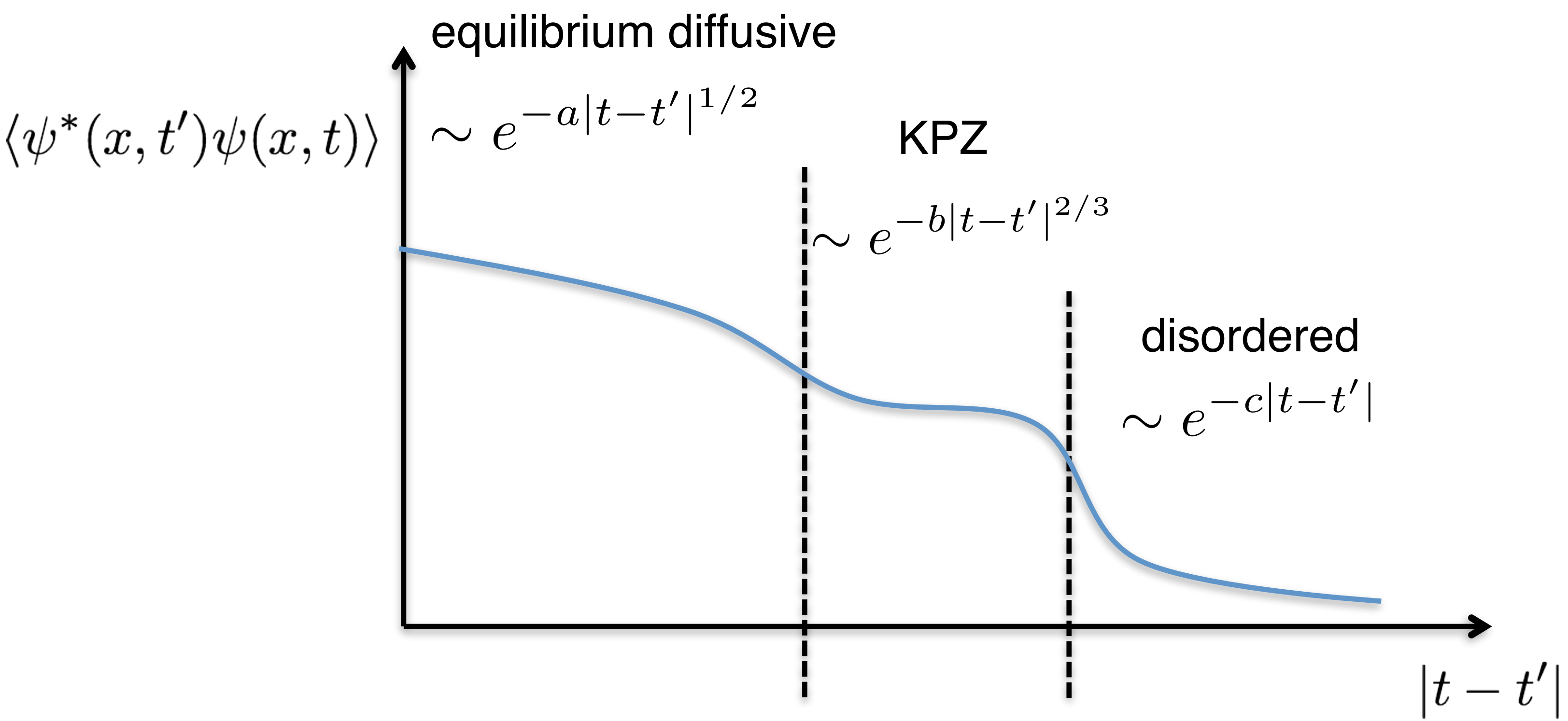
L. He, L. Sieberer, E. Altman, SD, PRB (2015)
L. He, L. Sieberer, SD, PRL (2017)

L. He, L. Sieberer, E. Altman, SD, PRB (2015)
see also: K. Yi, V. Gladilin, M. Wouters, PRB (2015)

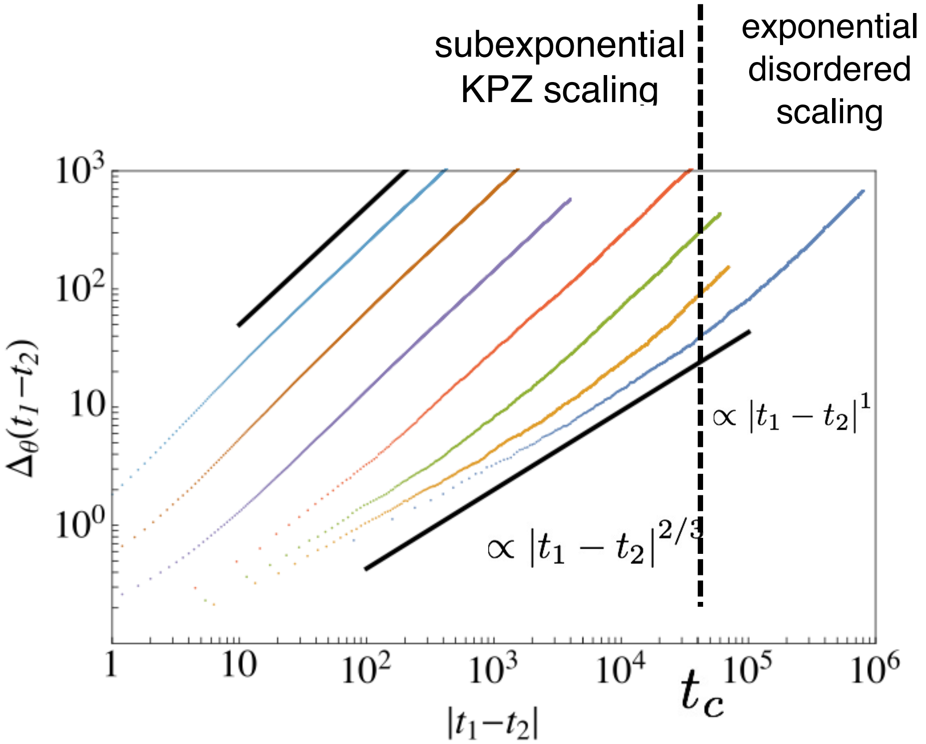
L. He, L. Sieberer, SD, PRL (2017)

Sequence of Scales

- direct numerical solution of driven-dissipative GPE in one dimension
- Study temporal instead of spatial coherence function (near equilibrium)



numerical evidence



- Crossover scale $T_* \sim \sigma^{-2}$ $T_v \sim e^{E_c/\sigma}$

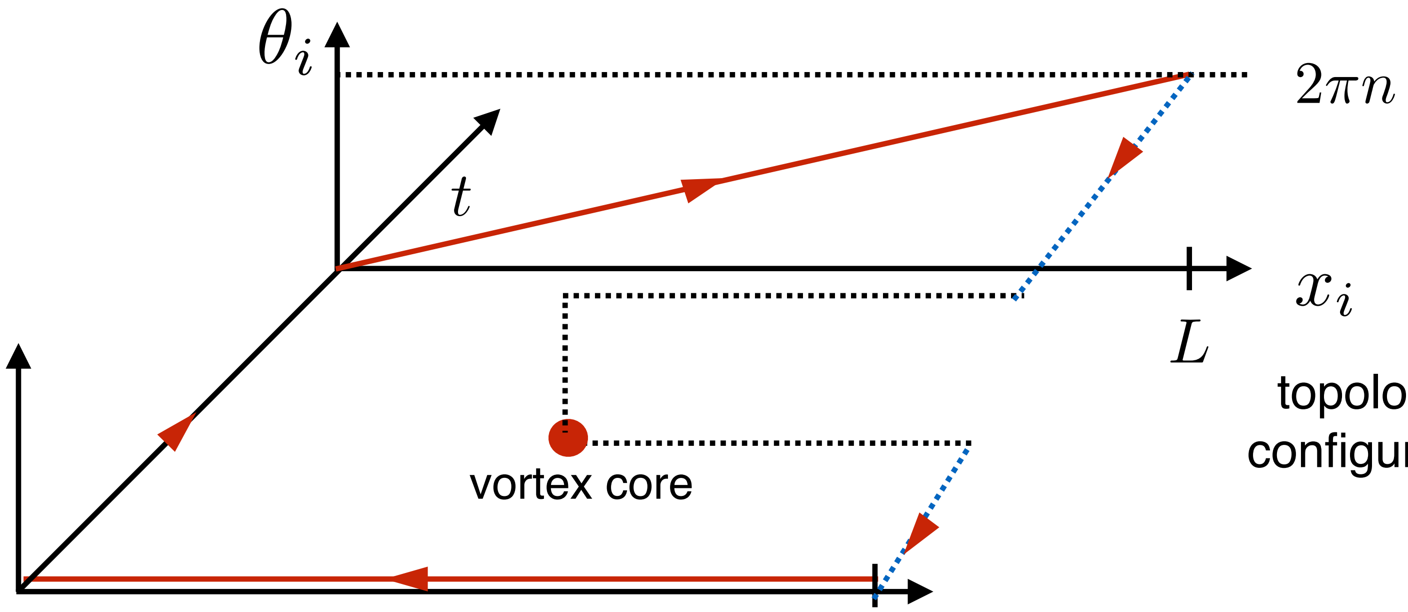
noise level

algebraic exponential

➡ no spatial vortices in 1D —> what causes the emergent length scale beyond KPZ?

Space-time vortices in 1D: dynamics of phase slips

- Physical origin: compactness of phase field



periodic boundary cond.

topologically nontrivial phase field configurations on (1+1)D space-time plane

- Vorticity condition $$\oint_{\mathcal{C}} ds \nabla\theta = 2\pi n$$

any closed path enclosing vortex core

- unbound at infinitesimal noise level (weak non-equilibrium)
  - interaction potential: $(\partial_t + D\partial_x^2)^{-1} \sim (Dt)^{-1/2} e^{-x^2/(4Dt)}$ cf. 2D static equilibrium: $\nabla^{-2} \sim \log(|\mathbf{x}|)$

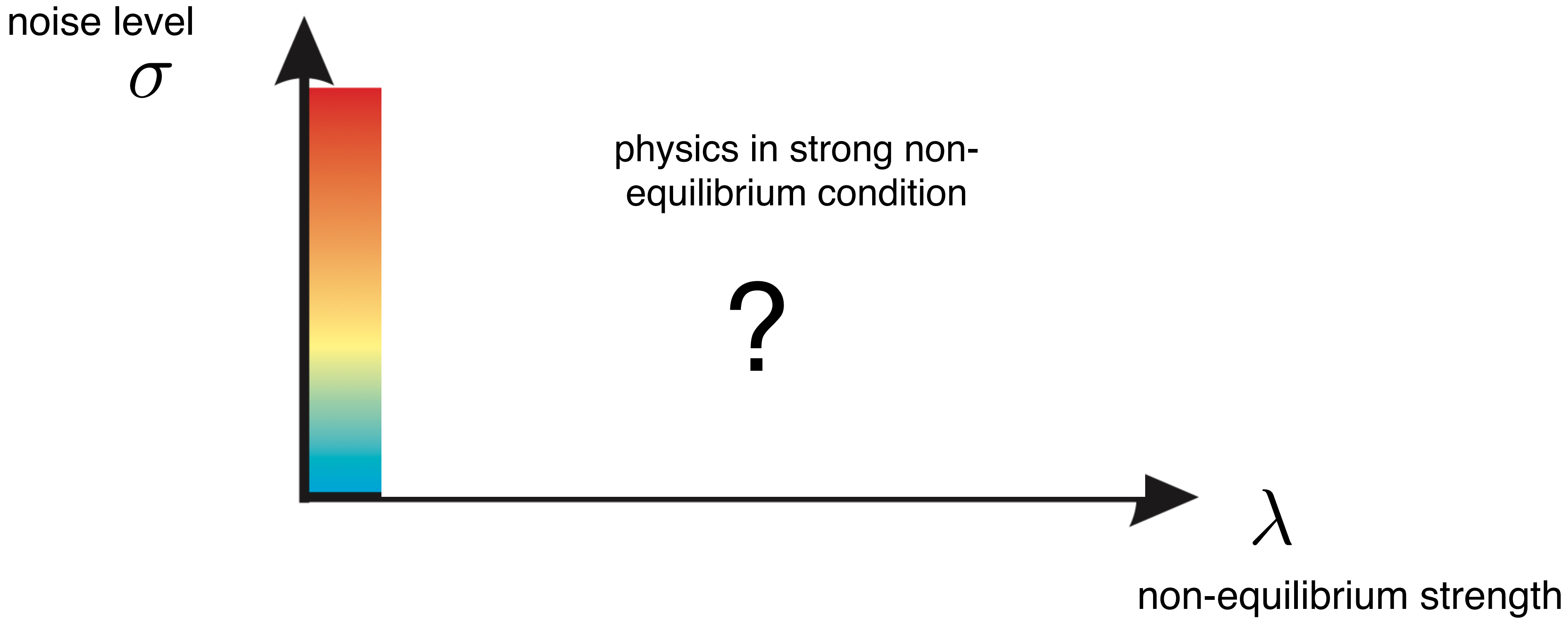
- explains qualitative features

1. temporal scaling (random uncorrelated charges) $\langle \psi^*(x,t')\psi(x,t)\rangle \sim e^{-c|t-t'|}$
2. noise level dependence of crossover scale $T_v \sim e^{E_c/\sigma}$ (mapping to static 2D active smectic A liquid crystal)

Toner and Nelson, PRB (1984)

Strong non-equilibrium: Compact KPZ vortex turbulence

- In search of the phase diagram for XP condensates



Strong non-equilibrium: Compact KPZ vortex turbulence

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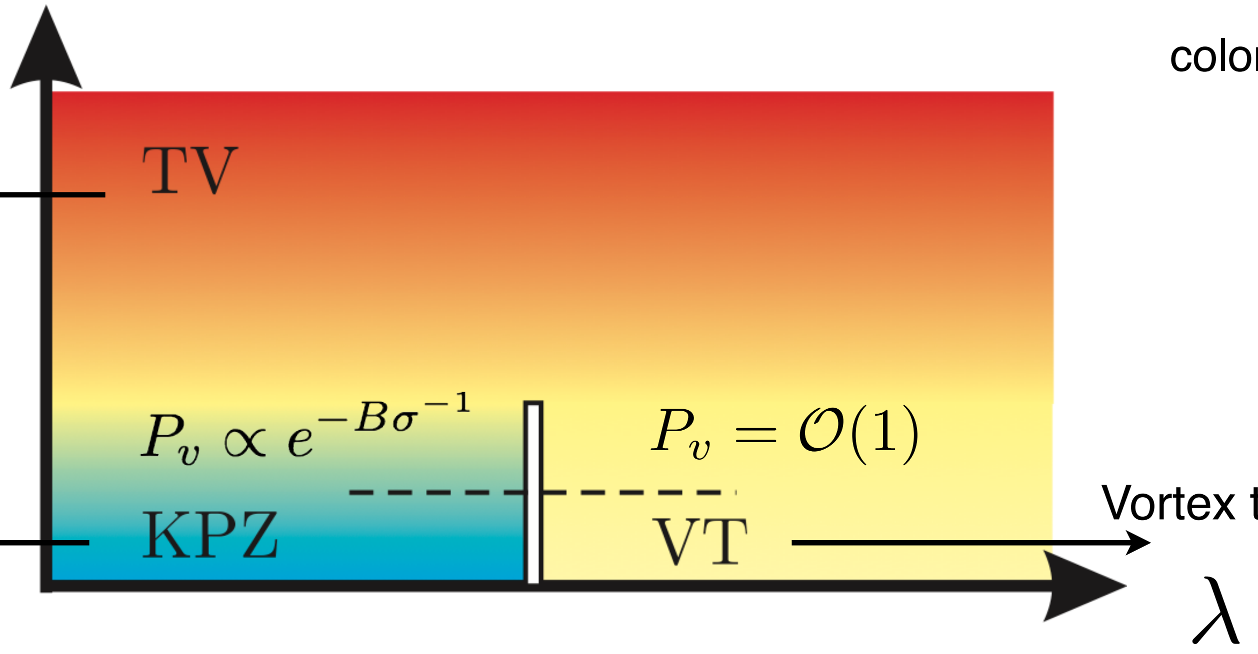

noise level

σ

Noise activated vortices (TV)

KPZ / equilibrium dominated physics

color code: vortex density on space-time plane

$$P_v$$

Vortex turbulence (VT)

much bigger parameter space & new phases: Veresci, PRR (2023)

first order non-equilibrium phase transition

non-equilibrium strength

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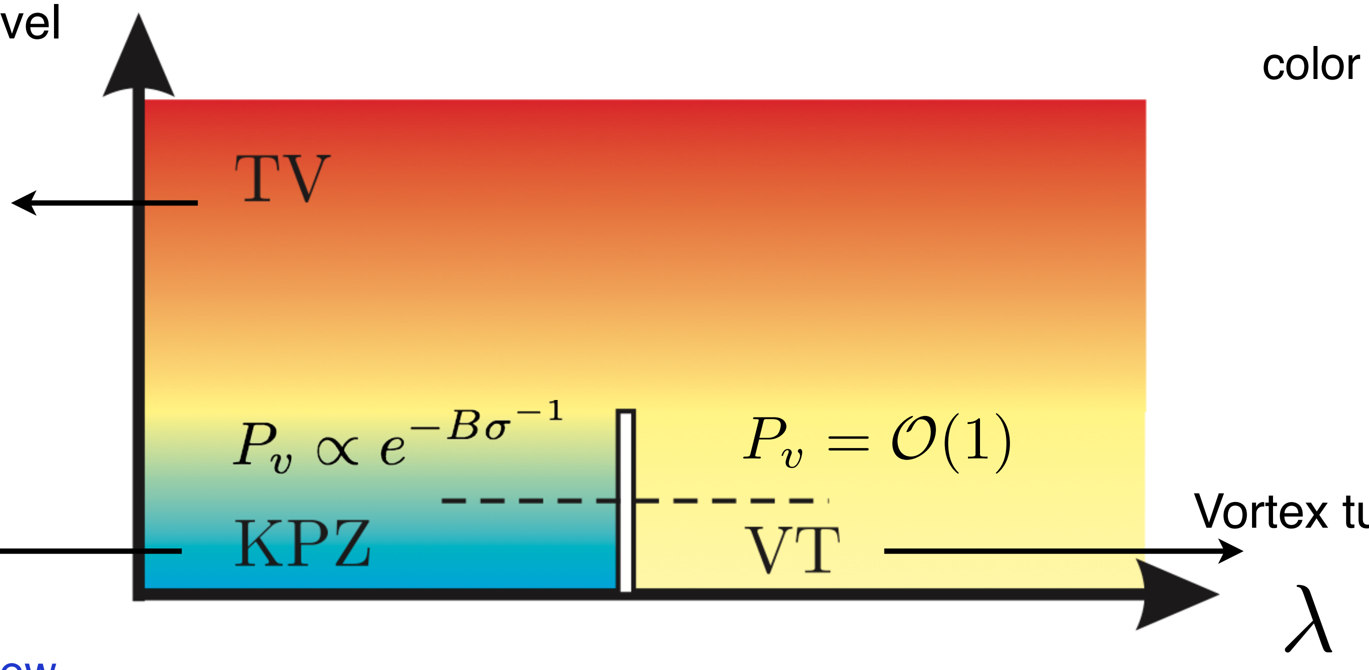

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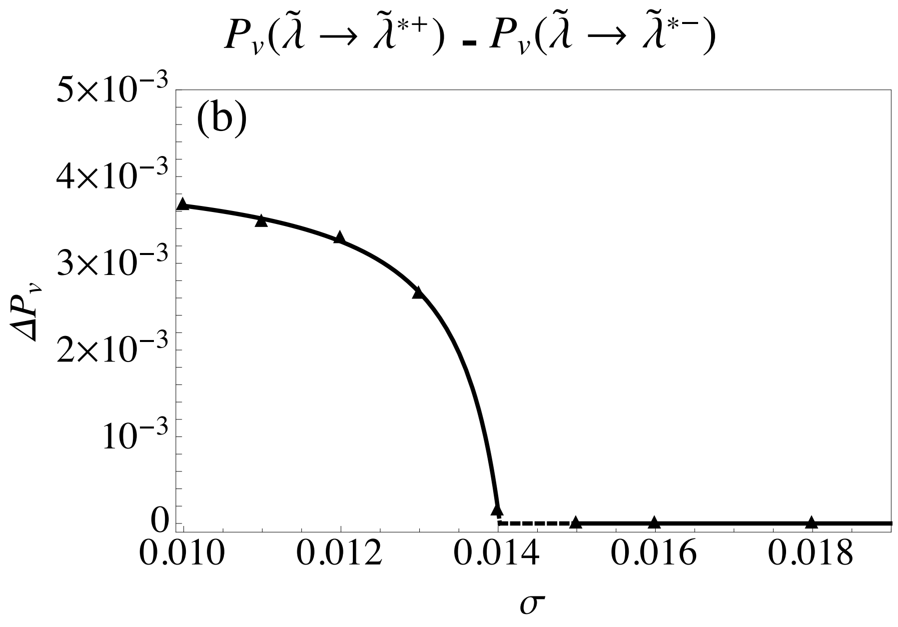

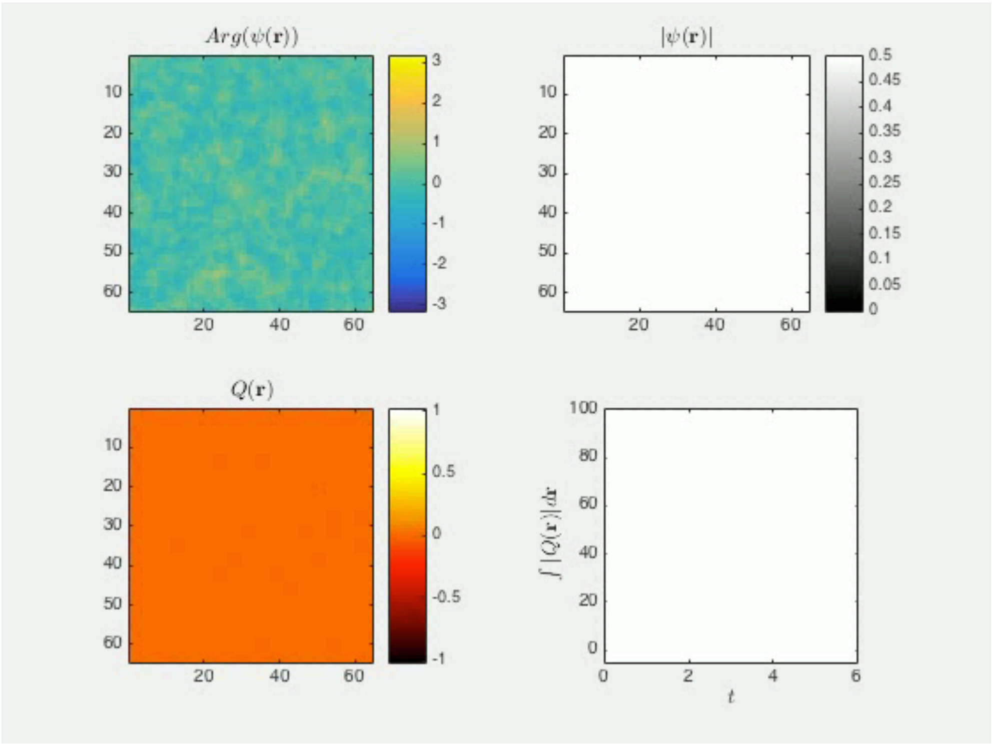

➡ deterministic limit: how does the system generate its own noise?

Phase transition driven by non-equilibrium strength

- qualitative reason: competition in lattice regularized model (compact KPZ)

$$\partial_t \vec{\chi} = \underbrace{S\vec{V}_E[\vec{\chi}]}_{\text{equilibrium}} + \underbrace{A\vec{V}_N[\vec{\chi}]}_{\text{non-equilibrium}} + \vec{\xi}$$

lattice site

$$\chi_i = \nabla\theta_i = \theta_{i+1} - \theta_i$$

$$V_{E/N,i} = \frac{\delta}{\delta\chi_i}\mathcal{V}_{E/N}$$

$$\mathcal{V}_E = K\sum_j \cos\chi_j \qquad \mathcal{V}_N = \lambda\sum_j \sin\chi_j$$

$$D_{ij} = \delta_{ij+1} - \delta_{ij}$$

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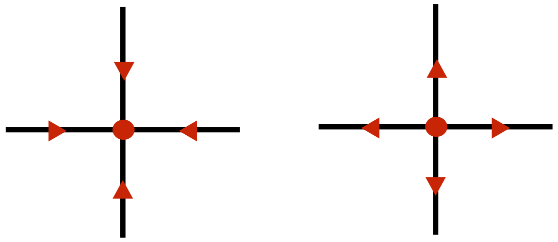
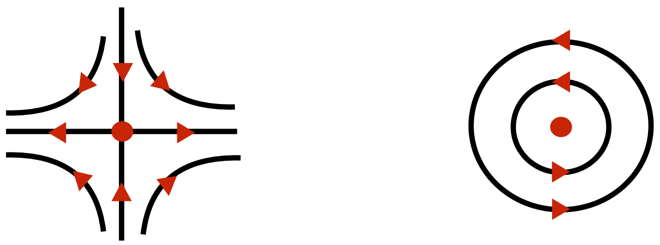
$$S = D + D^T = S^T$$

fixed point structures



attractive repulsive

$$A = D - D^T = -A^T$$



hyperbolic elliptic

=> emergent Liouville thm.

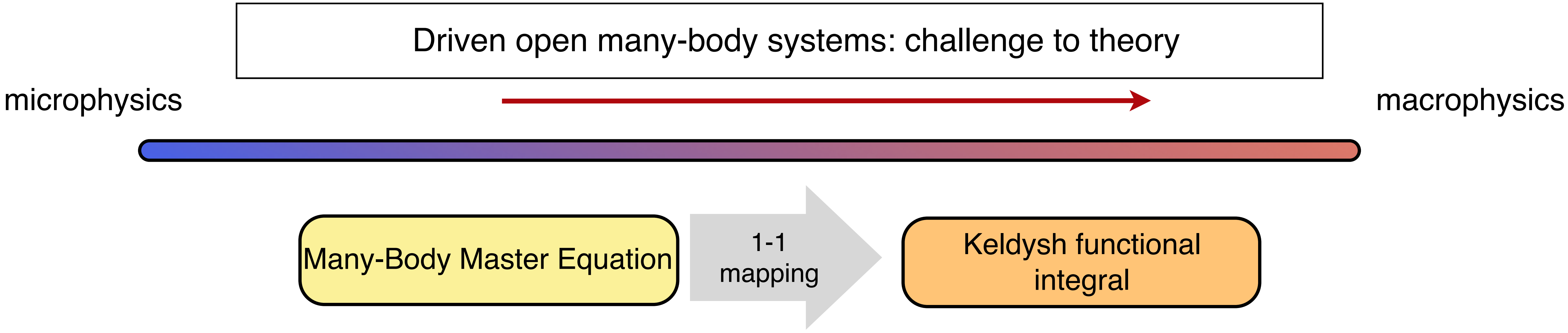
$$\mathrm{div}[A\vec{V}_N] = 0$$

➡ high dimensional configuration space

➡ transition to chaos

Summary so far

Review: L. Sieberer, M. Buchhold, SD, *Keldysh Field Theory for Driven Open Quantum Systems*, Reports on Progress in Physics (2016)



Driven open many-body systems: challenge to theory

microphysics

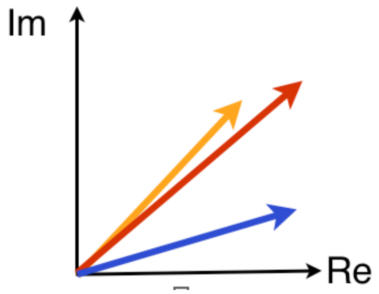
macrophysics

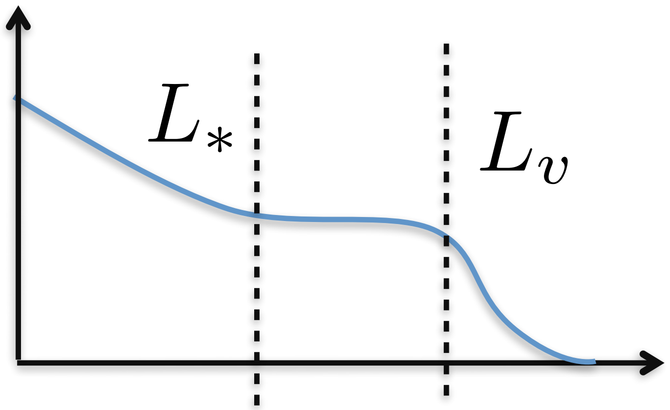
Many-Body Master Equation

1-1 mapping

Keldysh functional integral

- mapping opens up QFT toolbox:
  - symmetries: eq. vs. non-eq.
  - control of IR fluctuations: understanding low dimensional gapless phases out of equilibrium
  - flexible choice of degrees of freedom: KPZ vs. vortices





2D: $L_v \ll L_*$ 1D: $L_v \gg L_*$

(time scales)

4. Principles for universality in driven open quantum matter

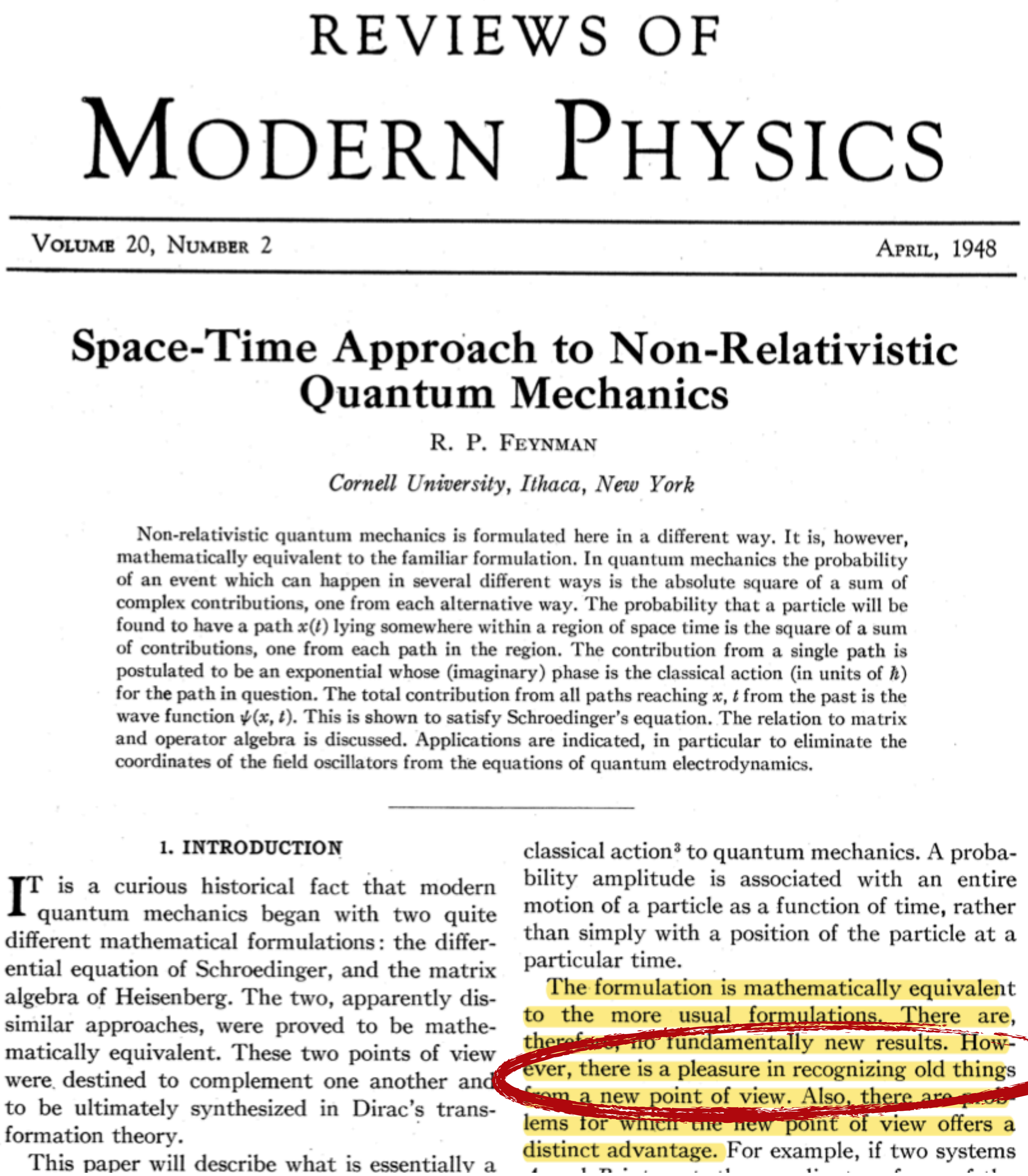
microphysics → macrophysics

- Field theory: a tool to distill universal and emergent macroscopic phenomena from a given microscopic physics
- Keldysh field theory: enabling this transition in out-of-equilibrium systems

- Guiding principles:

- equilibrium vs. non-equilibrium stationary states
  - ➡ equilibrium conditions encoded in thermal symmetry, violated explicitly in Lindblad-Keldysh field theory
- mixed vs. pure states
  - ➡ classical and quantum scaling solutions can exist in and out-of-equilibrium
- weak vs. strong symmetries: fine structure of symmetries in Keldysh framework
  - ➡ weak symmetry: Goldstone modes if spontaneously broken
  - ➡ strong symmetry: conservation laws and hydrodynamic modes (Noether theorem)

Weak vs. strong symmetries in the Keldysh formalism



REVIEWS OF
MODERN PHYSICS

VOLUME 20, NUMBER 2 APRIL, 1948

Space-Time Approach to Non-Relativistic Quantum Mechanics

R. P. FEYNMAN

Cornell University, Ithaca, New York

Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be found to have a path $x(t)$ lying somewhere within a region of space time is the square of a sum of contributions, one from each path in the region. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of $\hbar$) for the path in question. The total contribution from all paths reaching x, t from the past is the wave function $\psi(x, t)$. This is shown to satisfy Schroedinger's equation. The relation to matrix and operator algebra is discussed. Applications are indicated, in particular to eliminate the coordinates of the field oscillators from the equations of quantum electrodynamics.

1. INTRODUCTION

IT is a curious historical fact that modern quantum mechanics began with two quite different mathematical formulations: the differential equation of Schroedinger, and the matrix algebra of Heisenberg. The two, apparently dissimilar approaches, were proved to be mathematically equivalent. These two points of view were destined to complement one another and to be ultimately synthesized in Dirac's transformation theory.

This paper will describe what is essentially a

classical action[3] to quantum mechanics. A probability amplitude is associated with an entire motion of a particle as a function of time, rather than simply with a position of the particle at a particular time.

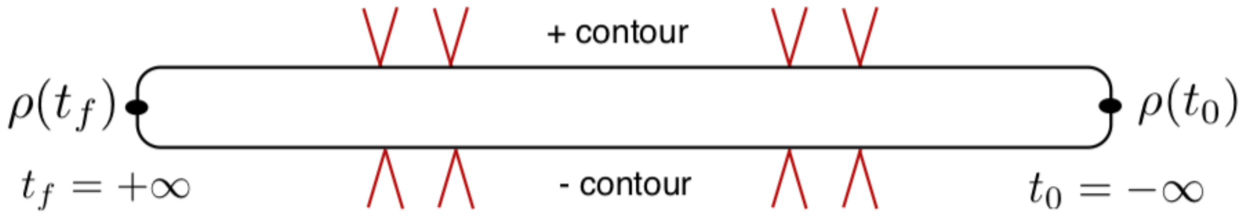
The formulation is mathematically equivalent to the more usual formulations. There are, therefore, no fundamentally new results. However, there is a pleasure in recognizing old things from a new point of view. Also, there are problems for which the new point of view offers a distinct advantage. For example, if two systems

more details: Sieberer, Buchhold, SD, ROPP (2016);
Sieberer, Buchhold, Marino, SD, arxiv (2023)

Concept: Symmetries in the Keldysh formalism

- closed time path

$$Z = \mathrm{tr}\hat{\rho}(t) =$$



$$= \int \mathcal{D}\psi_\pm e^{iS[\psi_\pm]}$$

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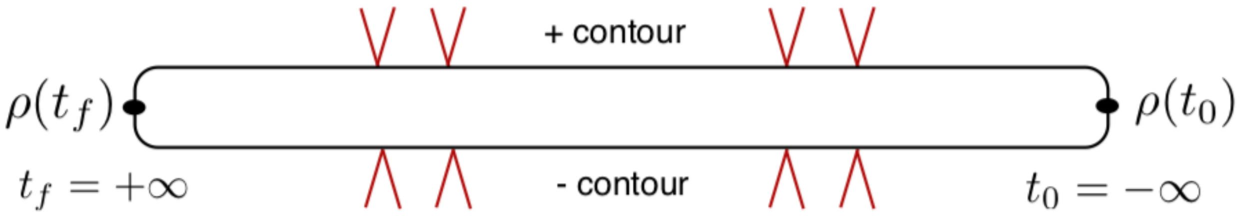

$$= \int \mathcal{D}\psi_\pm e^{iS[\psi_\pm]}$$

- Lindblad-Keldysh action $H_\pm = H[\psi_\pm], L_\pm = L[\psi_\pm]$

left action right action

$$S[\psi_\pm] = \int_{t,\mathbf{x}} \left[\psi_+^\dagger i\partial_t\psi_+ - H_+ - (+\to -) - i\sum_\alpha \gamma_\alpha \left(2L_{\alpha,-}^\dagger L_{\alpha,+} - L_{\alpha,+}^\dagger L_{\alpha,+} - L_{\alpha,-}^\dagger L_{\alpha,-}\right)\right]$$

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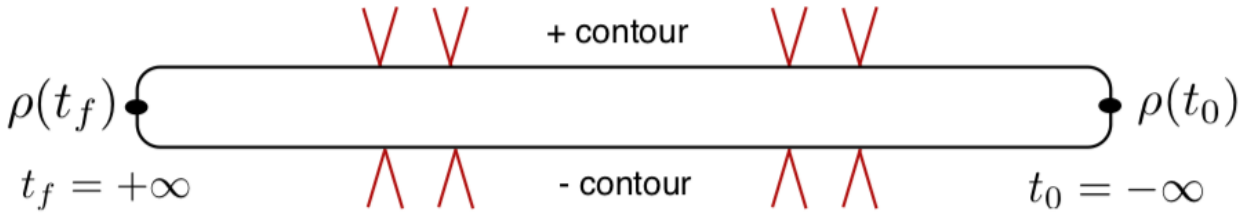

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- basis separating responses and correlations: $\psi_c = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-), \quad \psi_q = \frac{1}{\sqrt{2}}(\psi_+ - \psi_-)$

`classical field’ `quantum field’

more details: Sieberer, Buchhold, SD, ROPP (2016);
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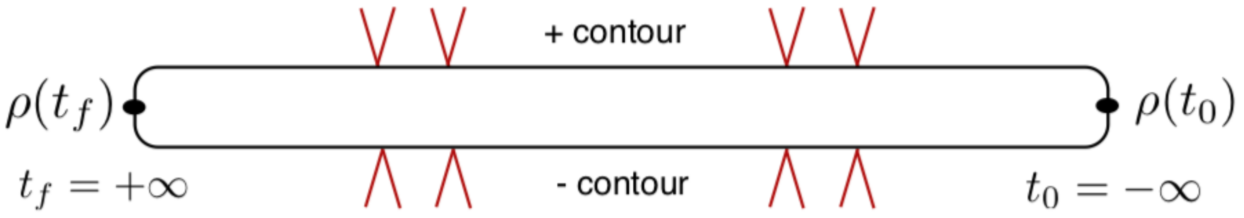

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- basis separating responses and correlations: $\psi_c = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-), \quad \psi_q = \frac{1}{\sqrt{2}}(\psi_+ - \psi_-)$

`classical field’ `quantum field’

- symmetries have contour structure, e.g. global U(1):

$$\psi_\pm(t,\mathbf{x}) \to e^{i\chi_\pm}\psi_\pm(t,\mathbf{x})$$

- symmetry generators $\chi_+, \quad \chi_- \quad \leftrightarrow \quad \chi_c = (\chi_+ + \chi_-)/2, \quad \chi_q = (\chi_+ - \chi_-)/2$

$U_+(1) \quad U_-(1) \qquad U_c(1) \qquad U_q(1)$

in-phase out-of-phase

more details: Sieberer, Buchhold, SD, ROPP (2016);
Sieberer, Buchhold, Marino, SD, arxiv (2023)

Concept: Classification of symmetries

Buca, Prosen, NJP (2013)
Lieu et al., PRL (2020)

- overview: strong and weak symmetries
- focus: continuous global symmetry U(1)

- strong/quantum symmetries: independent transformations on both contours

$$U_c(1) \times U_q(1)$$

➡ conservation laws, gapless hydrodynamic modes (Keldysh Noether theorem)

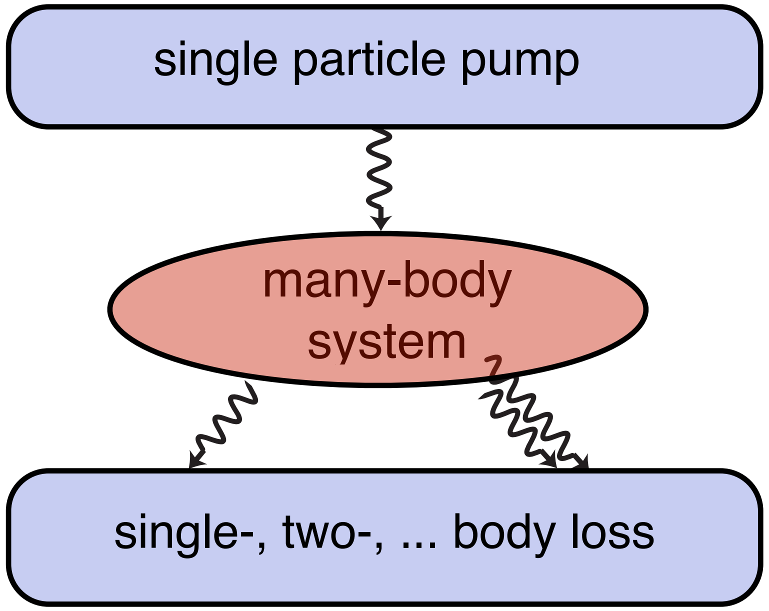
- weak/classical symmetries: both contours ‘in phase’, $U_q(1)$explicitly broken, $\chi_q = 0$

$$U_c(1)$$

➡ spontaneous breaking of ‘classical’ symmetry: gapless Goldstone modes (Keldysh Goldstone theorem)

Concept: Classical/weak symmetry

- recall generic Lindblad ϕ^4 model



$$\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho]$$

$$H = \int_{\mathbf{x}} \hat{\phi}^{\dagger}_{\mathbf{x}} \left(\frac{\triangle}{2M} - \mu\right) \hat{\phi}_{\mathbf{x}} + \frac{\lambda}{2}(\hat{\phi}^{\dagger}_{\mathbf{x}} \hat{\phi}_{\mathbf{x}})^2$$

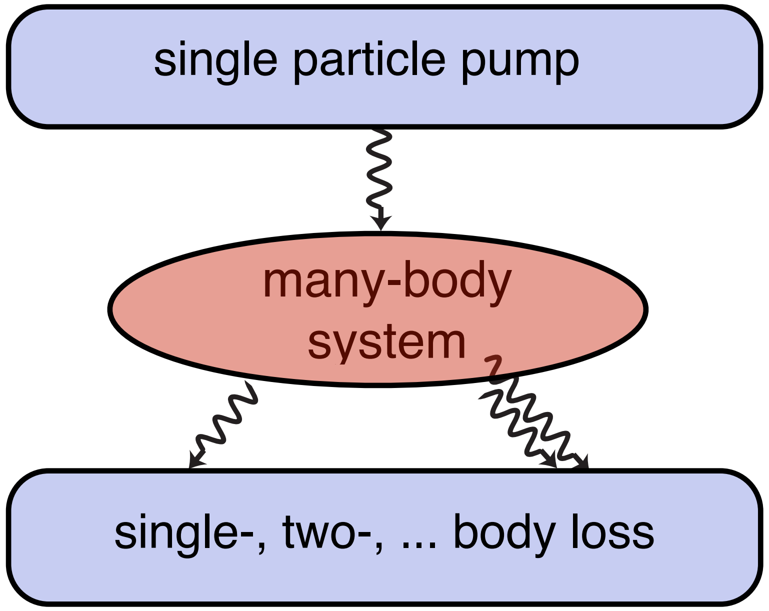
$$\mathcal{D}[\rho] = \underbrace{\gamma_p \int_{\mathbf{x}} [\hat{\phi}^{\dagger}_{\mathbf{x}}\, \rho\, \hat{\phi}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}_{\mathbf{x}} \hat{\phi}^{\dagger}_{\mathbf{x}}, \rho\}]}_{\text{single particle pump}} \quad + \quad \underbrace{\gamma_l \int_{\mathbf{x}} [\hat{\phi}_{\mathbf{x}}\, \rho\, \hat{\phi}^{\dagger}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^{\dagger}_{\mathbf{x}} \hat{\phi}_{\mathbf{x}}, \rho\}]}_{\text{single particle loss}} \quad +$$

$$\underbrace{\kappa \int_{\mathbf{x}} [\hat{\phi}^{2}_{\mathbf{x}}\, \rho\, \hat{\phi}^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^{\dagger\,2}_{\mathbf{x}} \hat{\phi}^{2}_{\mathbf{x}}, \rho\}]}_{\text{two particle loss}}$$

$U_c(1)$

Concept: Classical/weak symmetry

- recall generic Lindblad ϕ^4 model



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$$\underbrace{\kappa \int_{\mathbf{x}} [\hat{\phi}^2_{\mathbf{x}}\, \rho\, \hat{\phi}^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^{\dagger\,2}_{\mathbf{x}}\hat{\phi}^2_{\mathbf{x}}, \rho\}]}_{\text{two particle loss}}$$

- symmetries (and recall all examples above!):
  - $U_c(1)$ invariance: both contours transform 'in phase'

$$\hat{\phi} \to e^{i\chi}\hat{\phi}$$

  - but no $U_q(1)$ invariance
  - interesting physics is associated to the spontaneous breaking of $U_c(1)$

Concept: Classical/weak symmetry, spontaneous breaking

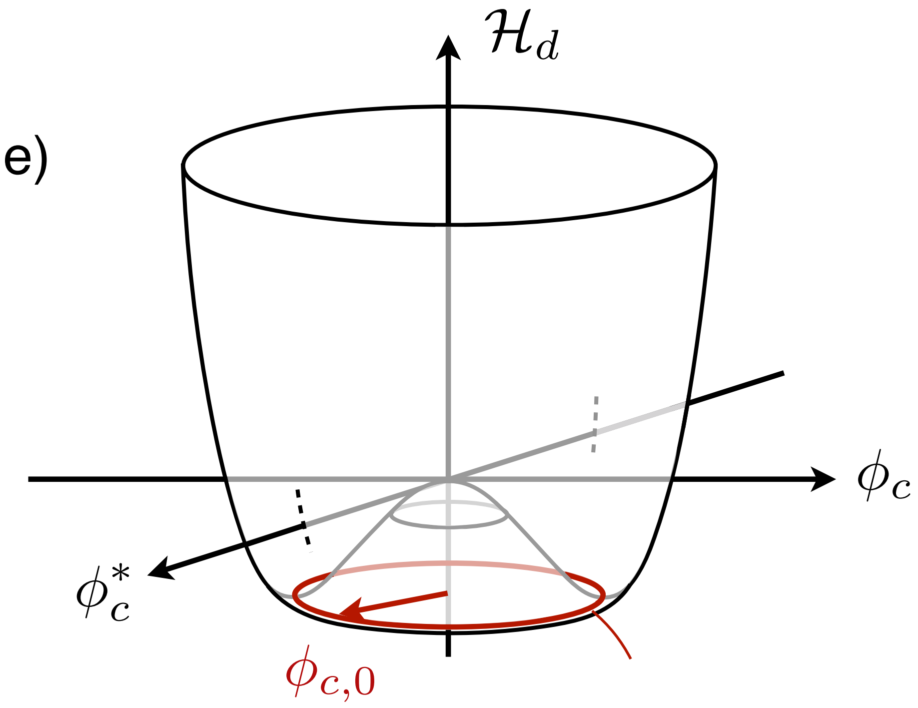
- associated Keldysh action (semiclassical limit)

Landau-type ϕ^4 functionals

$$S = \int_{t,\mathbf{x}} \left\{ \phi_q^* \left[i\partial_t \phi_c - \frac{\delta \mathcal{H}_c}{\delta \phi_c^*} + i \frac{\delta \mathcal{H}_d}{\delta \phi_c^*} \right] + 2i\gamma \phi_q^* \phi_q \right\} \qquad \mathcal{H}_\alpha = \int d^d x [r_\alpha |\phi_c|^2 + K_\alpha |\nabla \phi_c|^2 + g_\alpha |\phi_c|^4]$$

- $U_c(1)$ invariance: $\phi_c \rightarrow e^{i\chi_c} \phi_c, \quad \phi_q \rightarrow e^{i\chi_c} \phi_q$

- condensation for $r_d = \gamma_l - \gamma_p < 0$ (stat. state determined by $\mathcal{H}_d$ alone)



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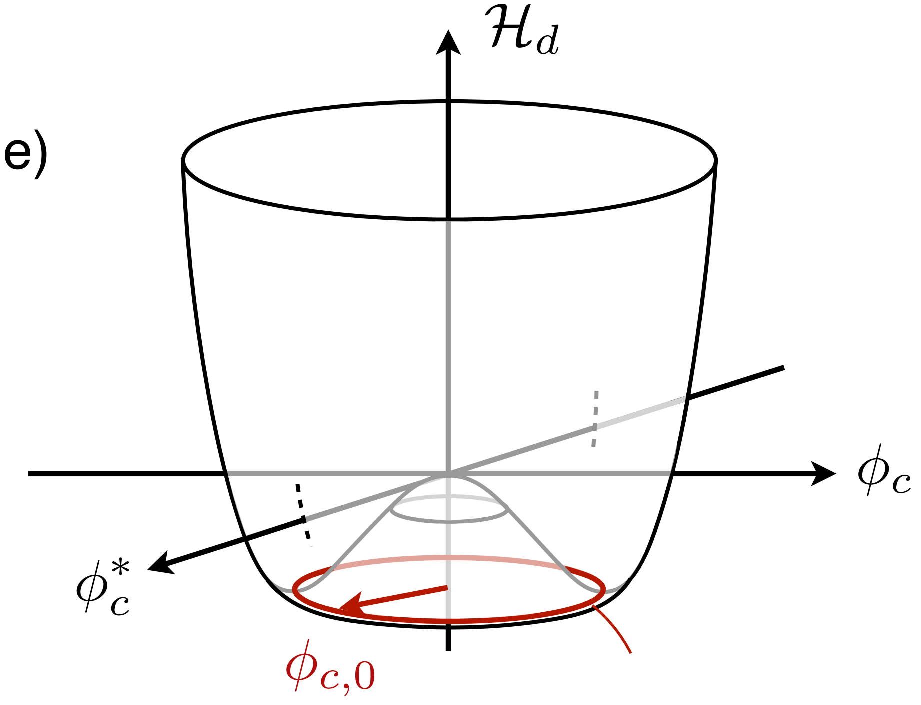
- visually clear:
  - ➡ sombrero potential with degenerate manifold of minima
  - ➡ system chooses one of the minima spontaneously
  - ➡ excitation along this manifold costs no action



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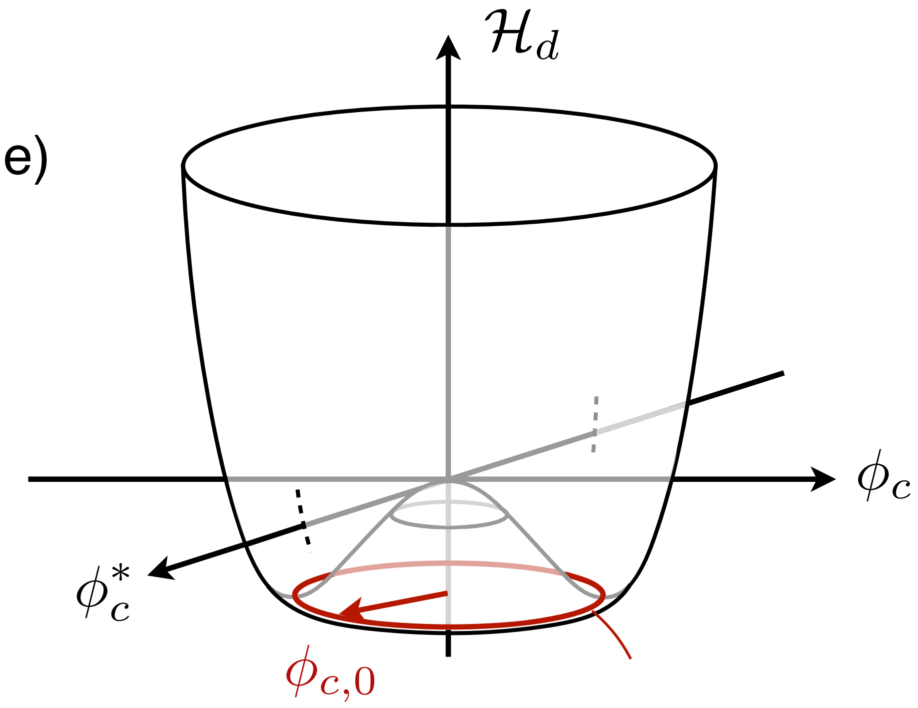

- more generally valid (incl. beyond semiclassical limit): Keldysh Goldstone theorem

more details: reviews

> If a global continuous weak symmetry is broken, there is an exact zero mode (R/A sector).

Application: KPZ in driven open Bose gas

- Goldstone mode construction:
  - promote classical symmetry generator to slow dynamic field, $\theta_c \to \theta_c(t, \mathbf{x})$
  - derive its action, $S_G[\partial_\mu \theta_c, \dots]$: depends only on derivatives of $\theta(t, \mathbf{x})$ by symmetry

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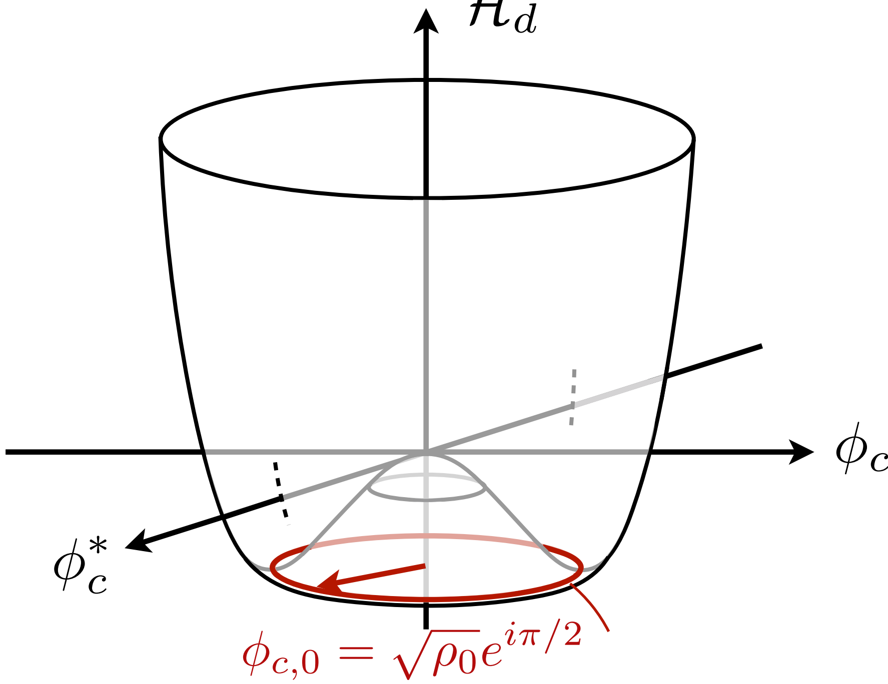

- field parametrisation (drop index c):
    - classical field: $\phi_c(t, \mathbf{x}) = \sqrt{\rho(t, \mathbf{x})} e^{i\theta(t, \mathbf{x})}$, $\rho, \theta \in \mathbb{R}$
      => identify phase as the Goldstone mode
    - => quantum field: $\phi_q(t, \mathbf{x}) = \zeta(t, \mathbf{x}) e^{i\theta(t, \mathbf{x})}$, $\zeta \in \mathbb{C}$

- functional integral: $Z = \int \mathcal{D}[\phi_c, \phi_c^*, \phi_q, \phi_q^*] e^{iS} = \int \mathcal{D}[\rho, \theta, \zeta, \zeta^*] e^{iS}$

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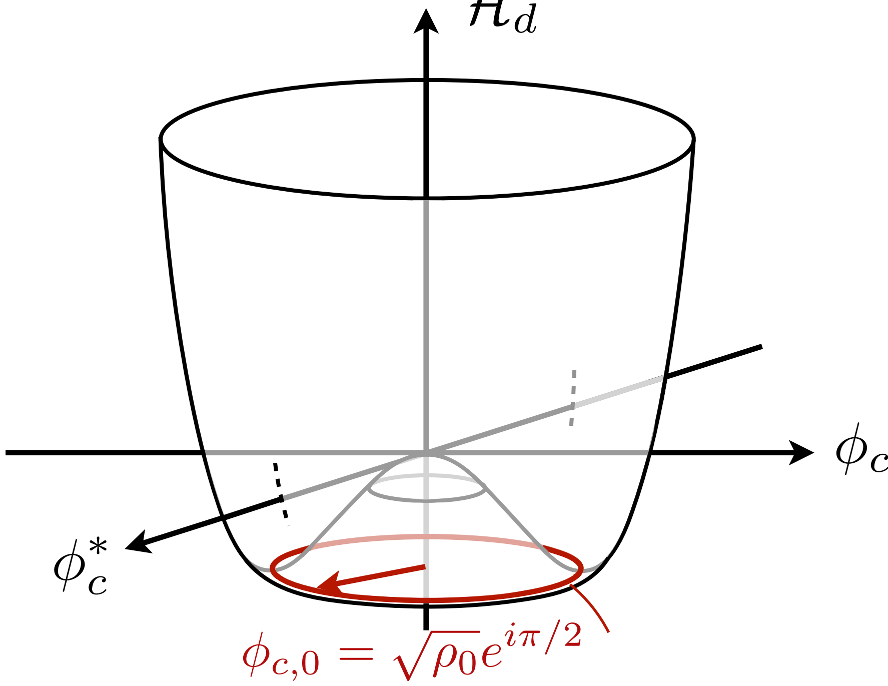

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- eliminate all gapped modes: density fluctuations $\rho(t,\mathbf{x}) = \rho_0 + \delta\rho(t,\mathbf{x})$ => Goldstone mode action

$$Z = \int \mathcal{D}\theta \mathcal{D}\tilde{\theta} e^{-S_{\text{KPZ}}}, \quad S_{\text{KPZ}} = \int_{t,\mathbf{x}} \tilde{\theta}\left[\partial_t \theta - D\nabla^2\theta - \frac{\lambda}{2}(\nabla\theta)^2 - \Delta\tilde{\theta}\right]$$

$$\tilde{\theta} = -2i\sqrt{\rho_0}\zeta_1, \quad D = K_c\left(\frac{K_d}{K_c} + \frac{u_c}{u_d}\right), \quad \lambda = 2K_d\left(\frac{K_c}{K_d} - \frac{u_c}{u_d}\right), \quad \Delta = -\frac{\gamma}{2\rho_0}\left(1 + \frac{u_c^2}{u_d^2}\right), \quad \rho_0 = -\frac{r_d}{u_d}$$

➡ role of conjugate quantum /noise field played by combination of original noise fields $\zeta = \zeta_1 + i\zeta_2$

➡ equivalence MSRJD <—> Langevin: KPZ equation ($\lambda = 0$ at equilibrium)

Concept: Conservation laws and quantum/strong symmetry

more details: Sieberer, Buchhold, SD, ROPP (2016);
Sieberer, Buchhold, Marino, SD, arxiv (2023)

- example 1: dephasing Lindbladian $\hat{L}_\alpha = \hat{n}_i$

➡ particle number conserved, $[\hat{n}_i, \hat{N}] = 0 \, \forall \, i$

➡ independent phase rotation symmetries on both contours $U_+(1) \times U_-(1) = U_c(1) \times U_q(1)$

➡ simple intuition: no number exchange with the bath

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- example 2: for any Hamiltonian, strong = weak symmetry

$$S[\psi_\pm] = \int_{t,\mathbf{x}} \Big[\underbrace{\psi_+^\dagger i\partial_t \psi_+ - H_+ - (+ \to -)}_{\text{contour diagonal}} - i \sum_\alpha \gamma_\alpha \Big(\underbrace{2L_{\alpha,-}^\dagger L_{\alpha,+}}_{\text{contour coupling}} \underbrace{- L_{\alpha,+}^\dagger L_{\alpha,+} - L_{\alpha,-}^\dagger L_{\alpha,-}}_{\text{contour diagonal}} \Big) \Big]$$

➡ the exchange of physical quantities (‘charge’) is via the fluctuation term

Concept: Conservation laws and quantum/strong symmetry

- Keldysh Noether construction:

more details: Sieberer, Buchhold, SD, ROPP (2016);
Sieberer, Buchhold, Marino, SD, arxiv (2023)

  - promote $\theta_{c,q} \rightarrow \theta_{c,q}(t, \vec{x})$
  - change of action:
  - vanishing variation: $\frac{\delta S}{\delta \theta_{q,c}} \overset{!}{=} 0 \implies \partial_\mu J^\mu_{c,q} = 0$ or $\partial_t J^0_{c,q} = -\nabla \vec{J}_{c,q}$ continuity equation

  where universally for time-local non-relativistic dynamics $J^0_c = \psi^*_c \psi_c + \psi^*_q \psi_q = \psi^*_+ \psi_+ + \psi^*_- \psi_-$

  and $\vec{J}_{c,q}$ is model specific $J^0_q = \psi^*_c \psi_q + \psi^*_q \psi_c = \psi^*_+ \psi_+ - \psi^*_- \psi_-$

- Noether charges:

$$U_c(1): \quad N_q = \int d^2x \langle J^0_q \rangle = 0$$

redundancy

$$\boxed{U_q(1): \quad N_c = \int d^2x \langle J^0_c \rangle = N}$$

➡ non-trivial classical charge (with finite expectation value) only for quantum symmetry

- constructing hydrodynamic action: $S_H = S_H[\partial_\mu \theta_q, ...]$

➡ Hamiltonian system, quantum symmetries are broken infinitesimally by infinitesimal regularization

➡ quantum symmetries are not preserved under RG transformations (spontaneously broken): emergence of dissipation at large distance

Application: KPZ in equilibrium closed Bose gas

Keldysh analog to non-linear fluctuation hydrodynamics: C. Mendl, H. Spohn, PRL (2012); H. Spohn, J. Stat. Phys. (2014)

- 1d Hamiltonian:

$$H = \int dx \, [Z|\partial_x \psi|^2 + \lambda |\psi|^4]$$

- Keldysh action (without coupling to bath)

$$S = \int dt \, dx \sum_{\sigma=\pm} \sigma(\psi^*_\sigma i\partial_t \psi_\sigma - H[\psi^*_\sigma, \psi_\sigma])$$

- two relevant strong/quantum symmetries:
  - $U(1)_q \rightarrow$ particle number conservation
  - translations $\rightarrow$ momentum conservation

➡ expected hydrodynamic equations:

- ➡ Continuity
- ➡ Euler

1D Bose Gas: Kulkarni, Lamacraft, PRA (2013);
Spin chains: De Nardis, Gopalakrishnan, Vasseur, PRL (2023)

Closed equilibrium Bose gas: Symmetry based derivation of KPZ

Keldysh analog to non-linear fluctuating hydrodynamics: C. Mendl, H. Spohn, PRL (2012); H. Spohn, J. Stat. Phys. (2014)

- derive hydrodynamics from Keldysh via Noether reasoning → naturally generates conserved noises
- expand in symmetry generator fluctuations:

$$U_q(1):\ \psi_\sigma(x,t) \to e^{i\sigma\theta_q(x,t)}\psi(x,t) \qquad T_q^x:\ \psi_\sigma(x,t) \to \psi_\sigma(x+\sigma\chi_q(x,t))$$

- global invariances → quantum fields θ_q, χ_q appear only with derivatives (omit index 'c')

$$\Delta S = \int_{t,x} \theta_q(\partial_t\rho - \partial_x(v\rho)) + \chi_q(\partial_t(\rho v) - \lambda\partial_x\rho^2 + Z\partial_x(\rho v^2 + \cancel{(\partial_x\sqrt{\rho})^2}))$$

neglect (quantum pressure terms)

- with density ρ and superfluid velocity $v\rho = \frac{1}{2i}(\psi^*\partial_x\psi - \psi\partial_x\psi^*) = \rho\partial_x\theta$
- equation of motion for θ_q, χ_q: continuity, Euler equation

Closed equilibrium Bose gas: Symmetry based derivation of KPZ

- expand $\rho = \rho_0 + \delta\rho$

$$\partial_t \delta\rho = \rho_0 \partial_x v \qquad + v \partial_x \delta\rho$$

$$\rho_0 \partial_t v = 2\lambda \rho_0 \partial_x \delta\rho + \lambda \partial_x \delta\rho^2 + Z \rho_0 v \partial_x v^2 + Z \partial_x (\delta\rho v^2)$$

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- linearized time evolution → propagating sound waves (special to 1D)
  - diagonalize via $v_\pm = v \pm c\rho, \quad c = \sqrt{2\lambda\rho_0}$ emergent KPZ from sound modes: H. van Beijeren, PRL (2012)
  - consider fluctuations around comoving frames $v_\pm(x,t) = f_\pm(x \mp ct, t)$ removes all linear terms

$$\boxed{\partial_t f_\pm = \lambda_1 \partial_x f_\pm^2 + \lambda_2 \partial_x f_\mp^2 + g_\pm \partial_x f_\mp f_\pm}$$

parameters: functions of the
ones above

Closed equilibrium Bose gas: Symmetry based derivation of KPZ

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- add back conserved noise + dissipation on the path integral → 2 coupled Burgers equations
- concrete parameter set: decouple under RG Ertas, Kardar, PRL (1992)
- KPZ scaling for dispersion of sound wave packages in comoving frame
- Keldysh → Can systematically take the $T \to 0$ limit (all non-linearities irrelevant, z = 2)

Pereira et al., PRL (2006)

Summary: two symmetry based routes to KPZ in 1D Bose gases

microphysics → macrophysics

driven open Bose gas

closed equilibrium Bose gas

- weak vs. strong symmetry

spontaneous breaking of weak symmetry

—> soft Goldstone modes in open system
(adds possibility for vortex defects)

$$S = S[\partial_\mu \theta_c^a, \dots]$$

strong symmetry

—> soft hydrodynamic modes in closed system
analogous C. Mendl, H. Spohn, PRL (2012)

$$S = S[\partial_\mu \theta_q^a, \dots]$$

- pure vs. mixed states

$P^K \sim k^0$: KPZ

($P^K \sim k^2$ dark state, fine tuning needed)

T > 0: non-linearities relevant, KPZ $z = 3/2$

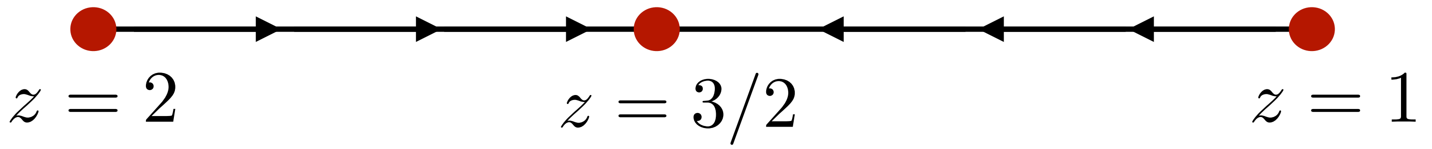
T=0: non-linearities irrelevant, suggests $z = 2$

- equilibrium vs. non-equilibrium

no (infinitesimally small breaking enough)

yes (very peculiar to 1D)

C. Fontaine, PRL (2023)



Outline

Keldysh theory general: A. Kamenev, *Field theory or non-equilibrium systems*, Cambridge University Press

Review: L. Sieberer, M. Buchhold, SD, *Keldysh Field Theory for Driven Open Quantum Systems*, Reports on Progress in Physics (2016);

L. Sieberer, M. Buchhold, J. Marino, SD, *Universality in Driven Open Quantum Matter*, arxiv (2023)

1. From the Lindblad equation to the Lindblad-Keldysh functional integral

- Lindblad equation for driven open quantum matter
- construction of Lindblad-Keldysh functional integral

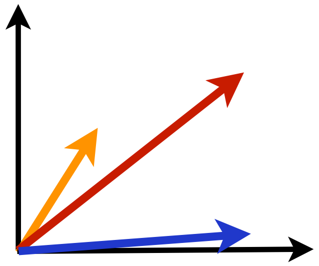
$$\partial_t \rho = -i[H, \rho] + \mathcal{L}[\rho]$$

$$\Downarrow$$

$$\mathrm{e}^{\mathrm{i}\Gamma[\Phi]} = \int \mathcal{D}\delta\Phi \mathrm{e}^{\mathrm{i}S_M[\Phi+\delta\Phi]}$$

2. KPZ equation in exciton-polariton condensates

- background: semiclassical limit, classifying eq. vs. non-eq. states
- from XP to KPZ: absence of algebraic order out of equilibrium
- compact KPZ and non-equilibrium phase transition

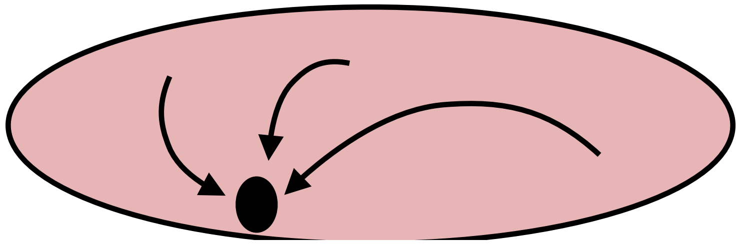


3. Principles of universality in driven open quantum matter

- the principles: eq. vs. non-eq.; pure vs. mixed states; weak vs. strong symmetries
- application: 1D KPZ in open vs. closed systems

4. Macroscopic non-equilibrium phenomena from weak non-equilibrium drive

- non-equilibrium O(N) models: phase structure, limit cycles
- novel non-equilibrium criticality at onset of a limit cycle
- route towards KPZ via breaking of time translation symmetry



5. Quantum aspects: topology in driven open quantum matter

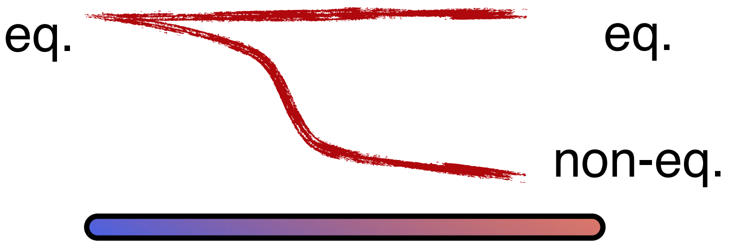
- fermion topological dark states in Lindblad evolution
- universality of topological response: pure states, mixed states

$$\frac{\theta}{4\pi} \int A_+ \, dA_+ - A_- \, dA_-$$

4. Macroscopic non-equilibrium phenomena from weak non-equilibrium drive

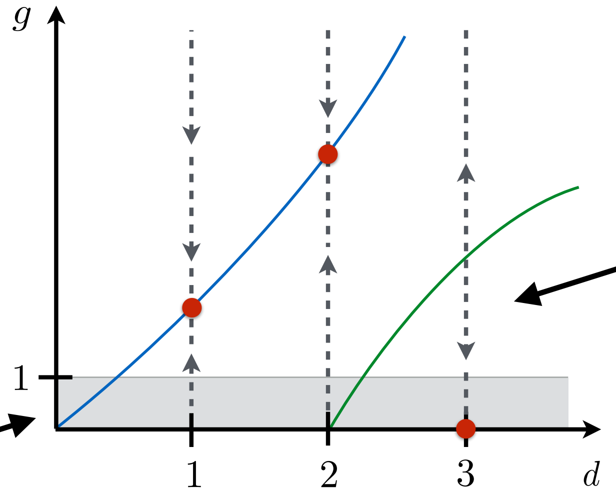
- archetypical example: KPZ in low dimension

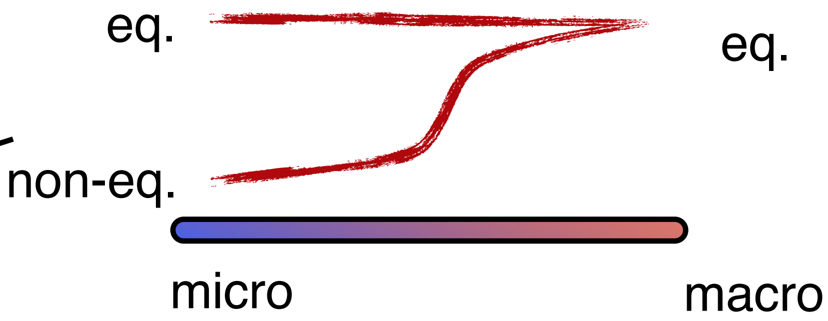
physics of gapless modes in a stable phase of matter


eq.

eq.

non-eq.




eq.

eq.

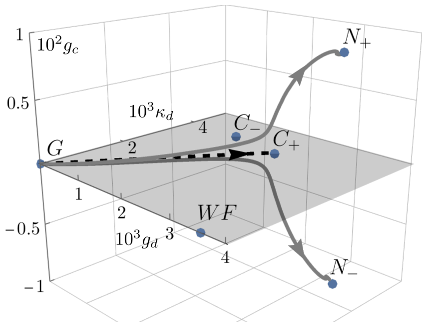
non-eq.

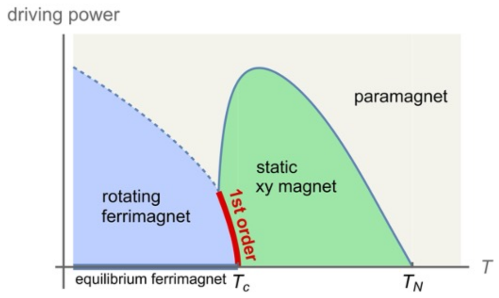
micro

macro

- non-equilibrium O(N) models: phase structure, limit cycles
- novel non-equilibrium criticality at onset of a limit cycle

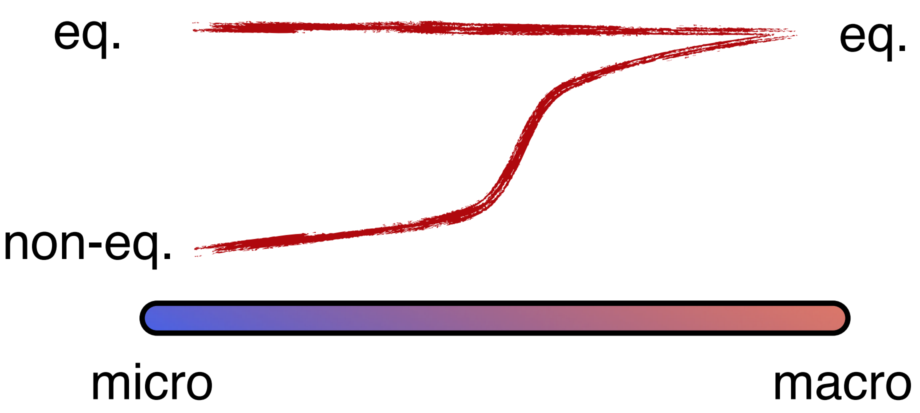
physics of gapless critical modes at a second order phase transition



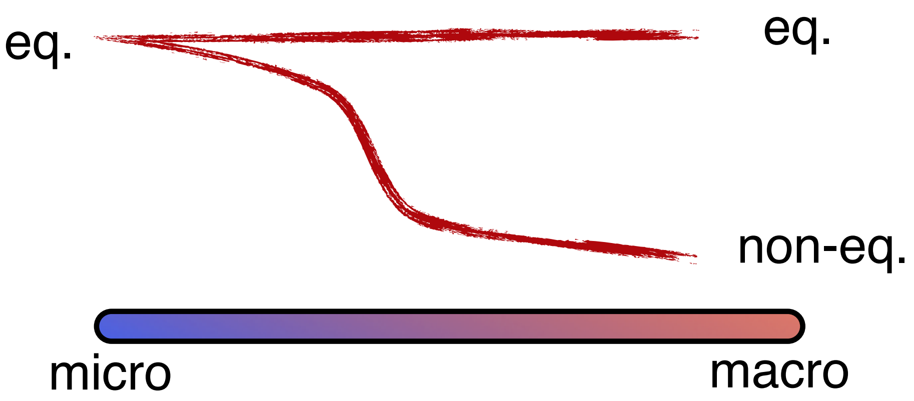


- spinoff, in stable phase: route towards KPZ via breaking of time translation symmetry

Emergent non-equilibrium: Dynamical limit cycles as candidates



vs.



- dynamical limit cycle phases / time crystalline order: non-equilibrium collective effects, in classical and quantum systems, ruled out at equilibrium

P. Bruno, PRL (2013); Volovik, JETP Letters (2013); Watanabe, Oshikawa, PRL (2015)

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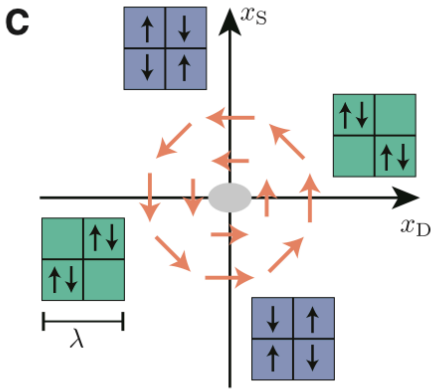
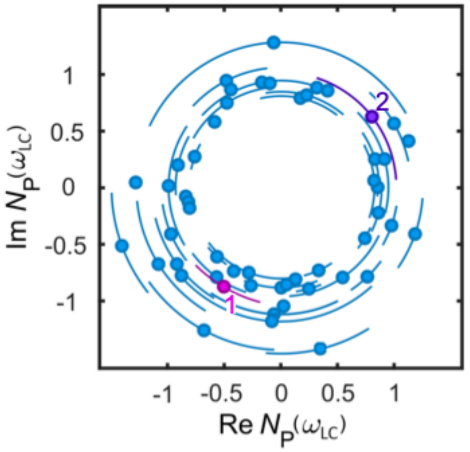
- experimental realizations in cold atomic systems



Dogra et al., Science (2019)



Kongkhambut et al., Science (2022)

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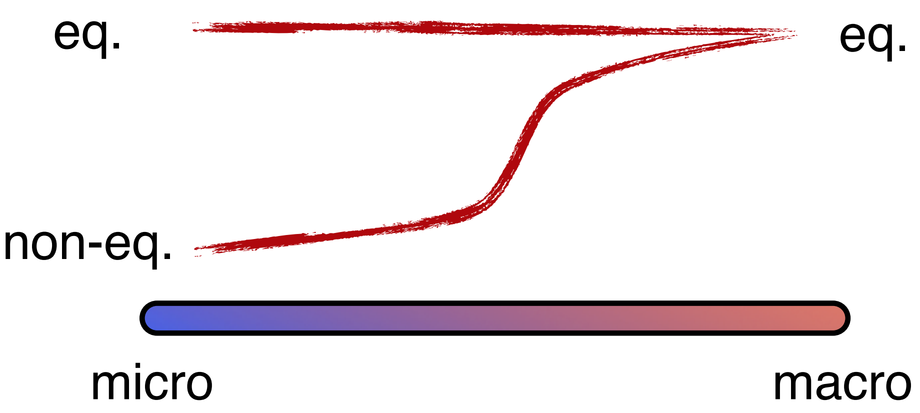

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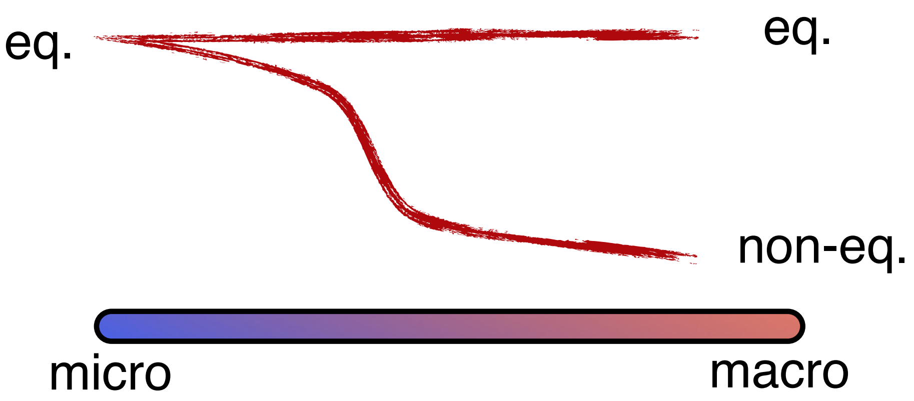

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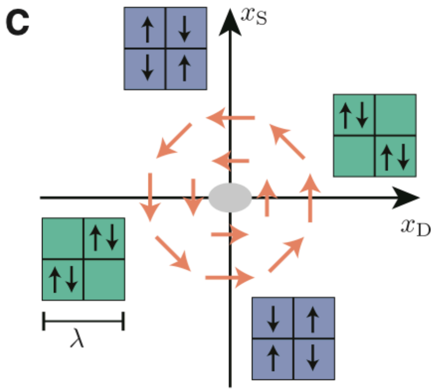

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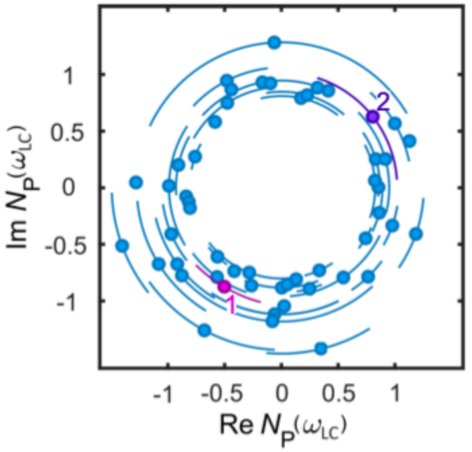
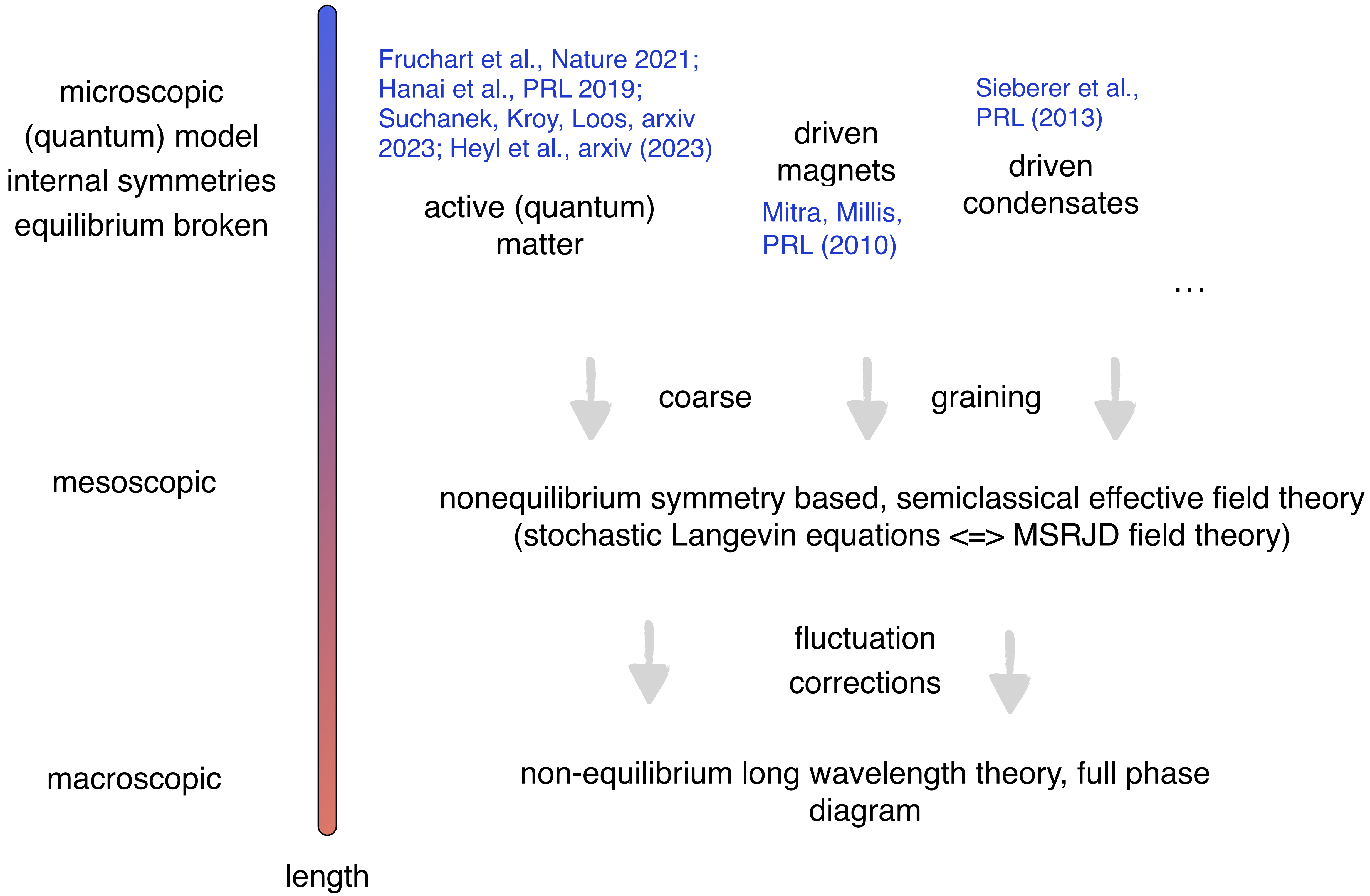


Kongkhambut et al., Science (2022)

Approach: Effective field theory



microscopic
(quantum) model
internal symmetries
equilibrium broken

mesoscopic

macroscopic

length

Fruchart et al., Nature 2021; Hanai et al., PRL 2019; Suchanek, Kroy, Loos, arxiv 2023; Heyl et al., arxiv (2023)

active (quantum)
matter

driven
magnets

Mitra, Millis, PRL (2010)

Sieberer et al., PRL (2013)

driven
condensates

…

coarse graining

nonequilibrium symmetry based, semiclassical effective field theory
(stochastic Langevin equations <=> MSRJD field theory)

fluctuation
corrections

non-equilibrium long wavelength theory, full phase diagram

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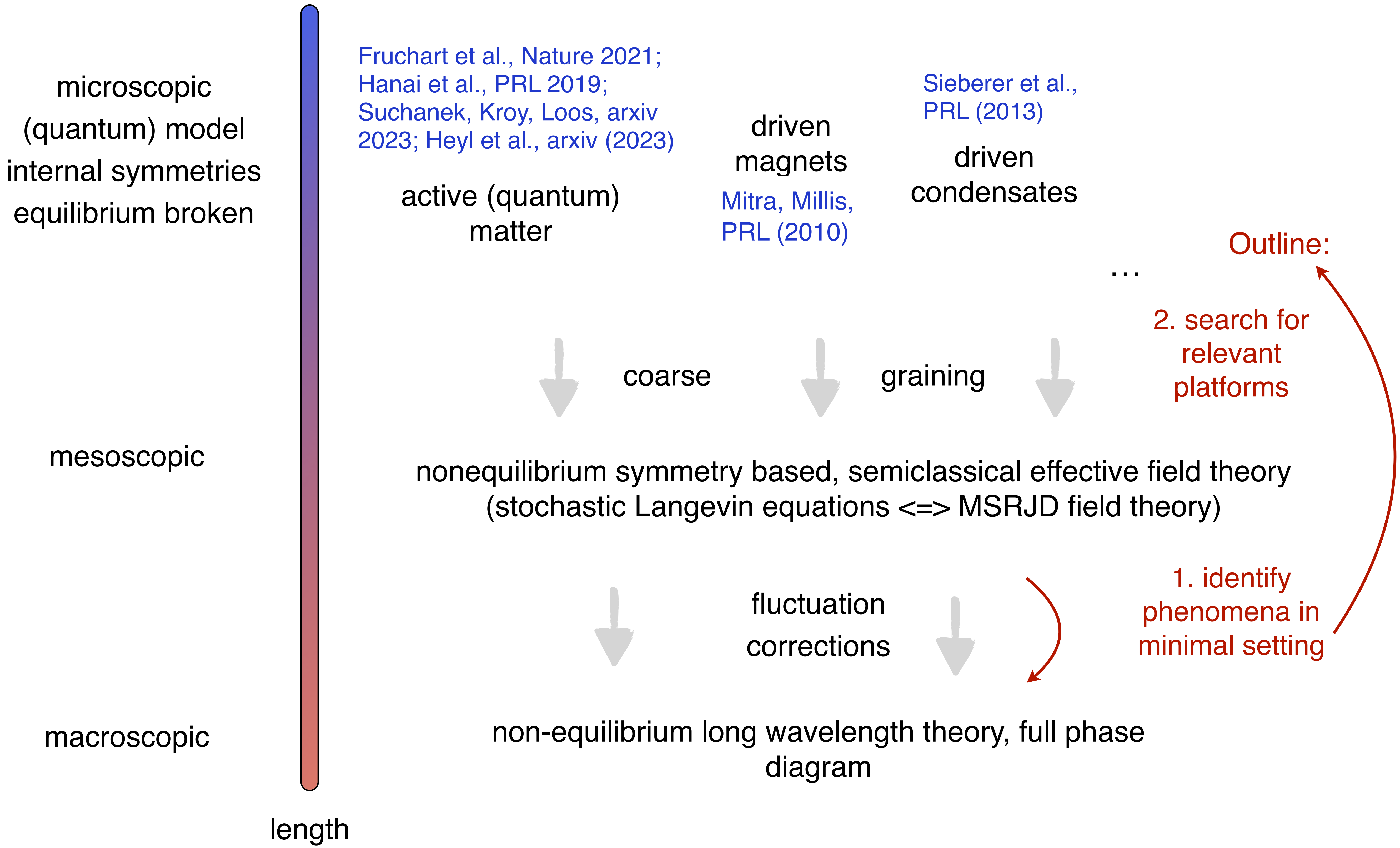

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…

Outline:

2. search for relevant platforms

coarse graining

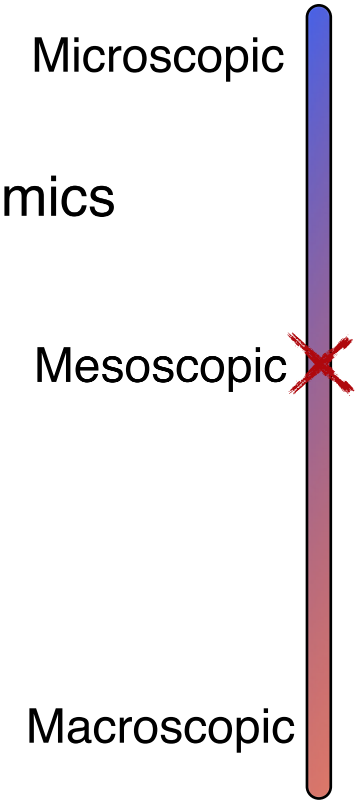
nonequilibrium symmetry based, semiclassical effective field theory
(stochastic Langevin equations <=> MSRJD field theory)

fluctuation corrections

1. identify phenomena in minimal setting

non-equilibrium long wavelength theory, full phase diagram

Effective field theory: The non-equilibrium $O(N)$model



- equilibrium: (mesoscopic) $O(N)$ model as workhorse of stat. mech. & near-equilibrium dynamics
  - purely symmetry based

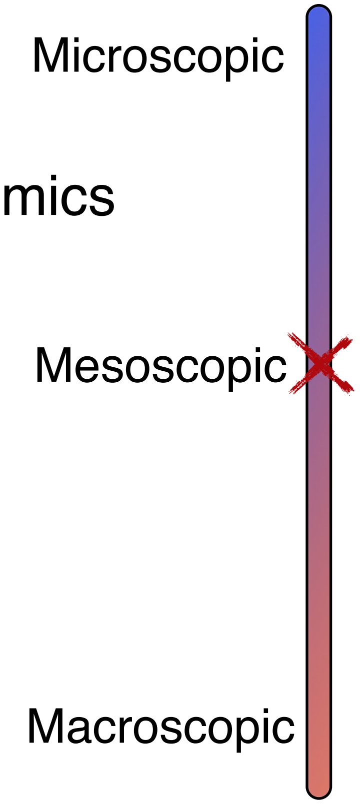
relativistic $O(N)$ model

ϕ^4 potential

$$\partial_t^2 \overrightarrow{\phi} \qquad + (r - Z'\nabla^2 + \lambda\rho)\overrightarrow{\phi} \qquad = 0$$

$$\rho = \overrightarrow{\phi} \cdot \overrightarrow{\phi}$$

Effective field theory: The non-equilibrium $O(N)$model



- equilibrium: (mesoscopic) $O(N)$ model as workhorse of stat. mech. & near-equilibrium dynamics
    - purely symmetry based
    - slow dynamics of order parameter ϕ & possibly conservation laws

Hohenberg-Halperin model A

Hohenberg, Halperin, RMP (1977)

noise (exploring configurations beyond deterministic)

$$\partial_t^2 \overrightarrow{\phi} + (2\gamma - Z\nabla^2 \qquad)\partial_t \overrightarrow{\phi} \qquad\qquad + (r - Z'\nabla^2 + \lambda\rho)\overrightarrow{\phi} + \vec{\xi} = 0$$

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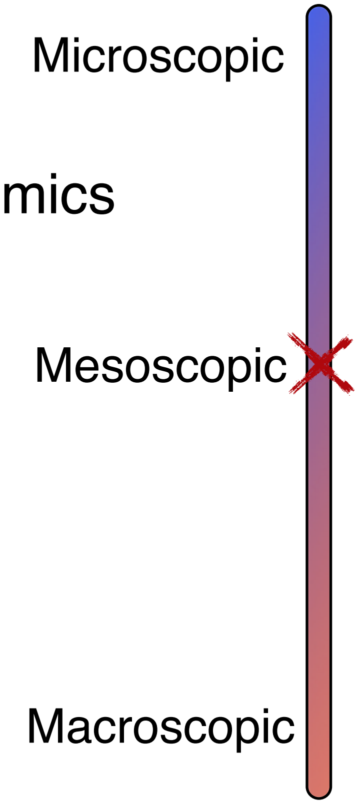
Zelle, Daviet, Rosch, SD, arxiv:2304.09207

Effective field theory: The non-equilibrium $O(N)$model


Microscopic

Mesoscopic

Macroscopic

- equilibrium: (mesoscopic) $O(N)$ model as workhorse of stat. mech. & near-equilibrium dynamics
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  - slow dynamics of order parameter ϕ & possibly conservation laws
- non-equilibrium setup:
  - non-thermal pumps & losses (e.g. laser) → nonconservative interactions
  - no conservation laws → no hydrodynamic modes

nonconservative forces
(breaking equilibrium / thermal symmetry)

$$\partial_t^2 \overrightarrow{\phi} + (2\gamma - Z\nabla^2 + \boxed{u\rho)}\partial_t \overrightarrow{\phi} + \boxed{u'\,\partial_t\rho\,\overrightarrow{\phi}} + (r - Z'\,\nabla^2 + \lambda\rho)\overrightarrow{\phi} + \vec{\xi} = 0$$

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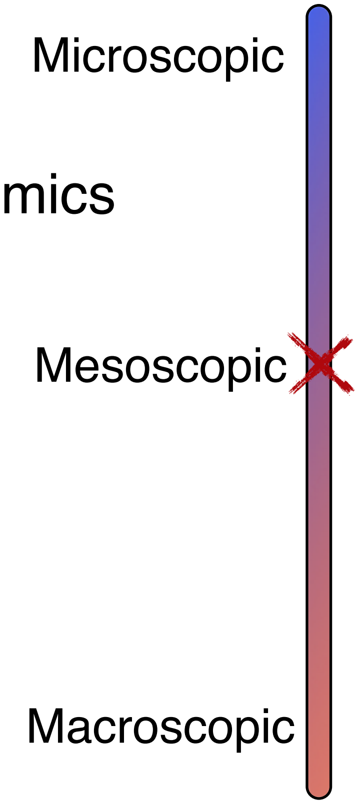

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$$\rho = \overrightarrow{\phi} \cdot \overrightarrow{\phi}$$

pump > loss ⇒ “antidamping” $\gamma < 0$:
stabilized by u, u', triggers time crystalline limit
cycle

tune $r < 0$
triggers equilibrium $O(N)$
transition

Phases of the non-equilibrium $O(N)$ model

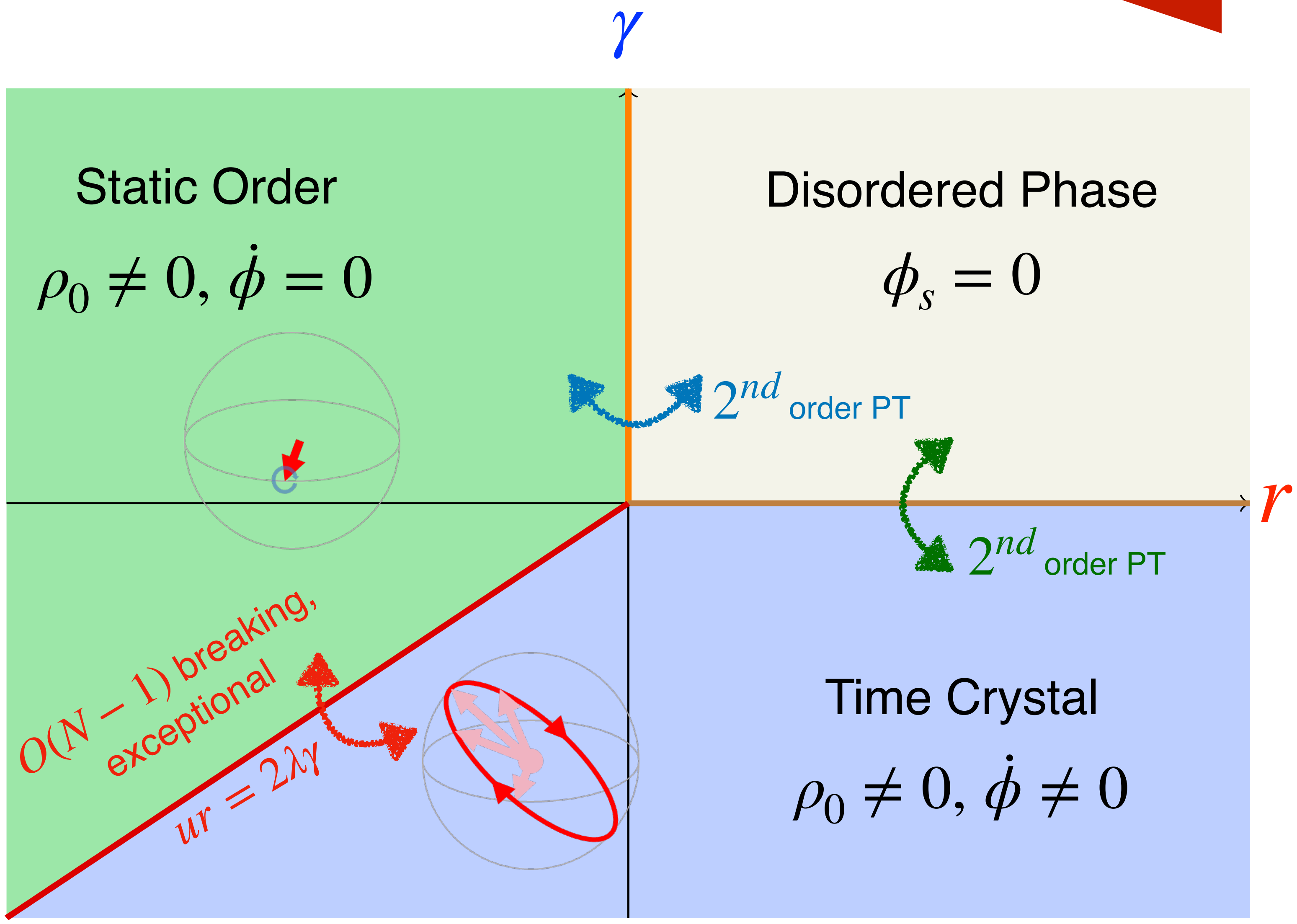
- homogeneous (0+1), non-linear mean field (deterministic) equation

$$\partial_t^2 \vec{\phi} + (2\gamma - \xcancel{Z\nabla}^2 + u\rho)\partial_t \vec{\phi} + u'\partial_t\rho\,\vec{\phi} + (r - \xcancel{Z\nabla}^2 + \lambda\rho)\vec{\phi} + \xcancel{\vec{\xi}} = 0$$

- disordered phase: $\phi_s = 0$
  No symmetries broken, $O(N)$ intact

- statically ordered phase: $\phi_s = \sqrt{\rho_0}e_1$
  usual symmetry breaking
  $O(N) \rightarrow O(N-1)$

- time crystalline order:
  $\phi_s(t)$ periodic function
  a) rotation on a circle
  b) amplitude oscillations along fixed axis



Aspects of phase diagram I: emergent equilibrium

- full problem: (d+1) dimensions, stochastic equation (or functional integral)

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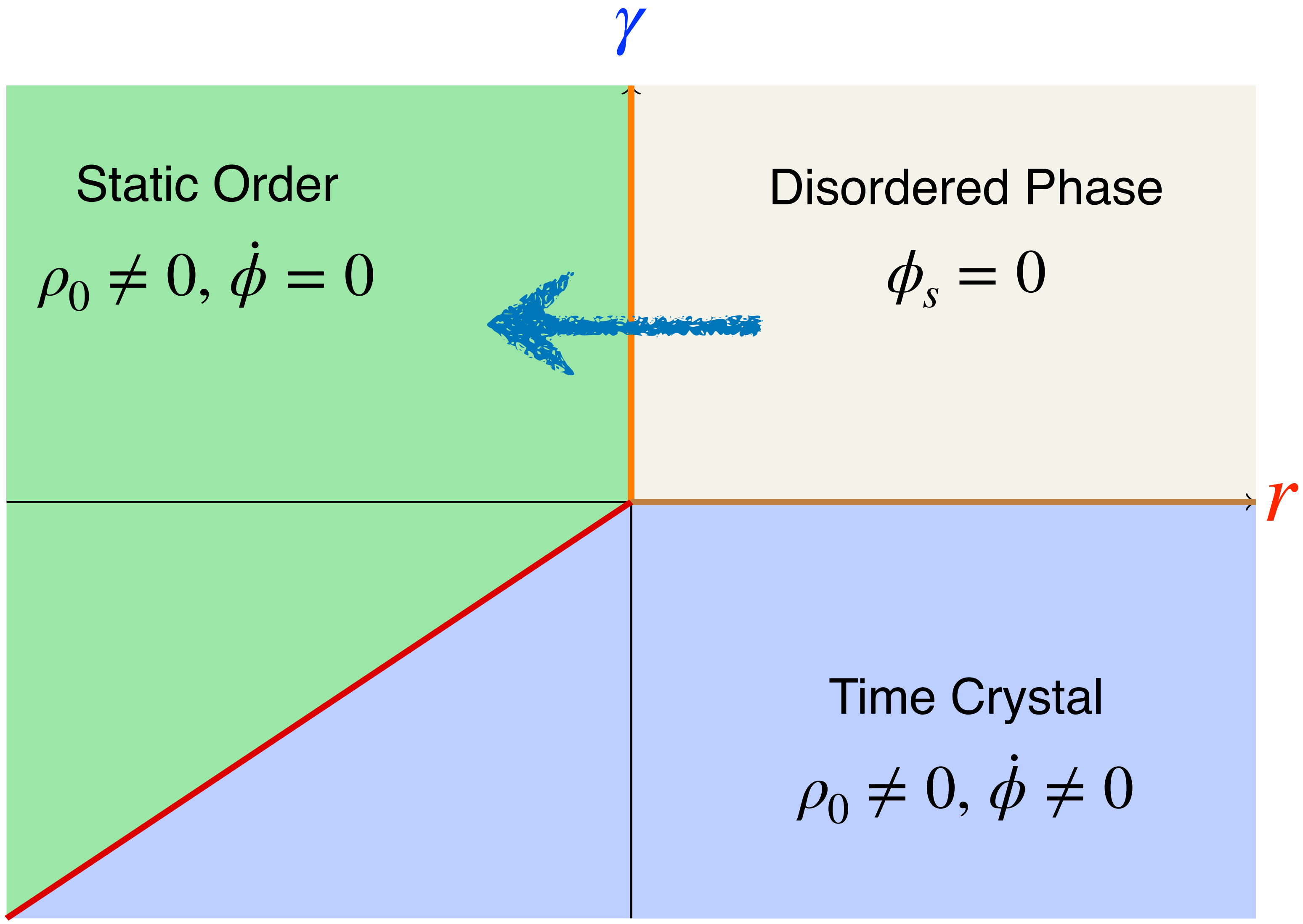
- emergent equilibrium behavior (non-equilibrium perturbation irrelevant)

- phase transition in Model A universality class of Hohenberg-Halperin

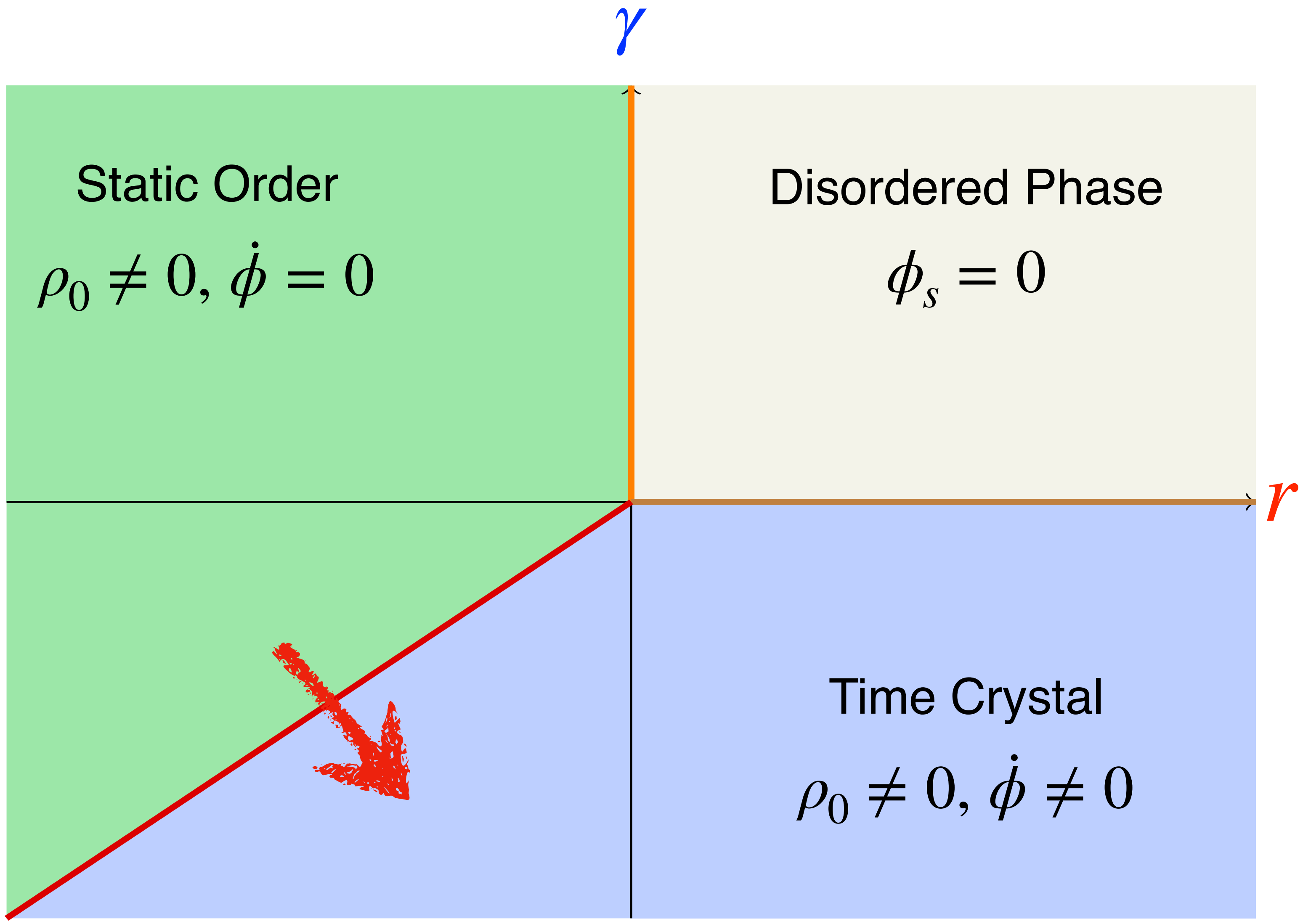


Aspects of phase diagram II: the CEP transition

- full problem: (d+1) dimensions, stochastic equation (or functional integral)

$$\partial_t^2 \overrightarrow{\phi} + (2\gamma - Z\nabla^2 + u\rho)\partial_t \overrightarrow{\phi} + u'\partial_t\rho\,\overrightarrow{\phi} + (r - Z'\,\nabla^2 + \lambda\rho)\,\overrightarrow{\phi} + \vec{\xi} = 0$$

- order-to-order transition: symmetry protected Critical Exceptional Point (CEP)
- giant non-equilibrium fluctuations, super-thermal mode occupation: 2 possibilites
  - symmetry restoration (weak interaction)
  - fluctuation induced first order transition controlled, full resummation of diagrams possible (strong int.)



Zelle, Daviet, Rosch, SD, arxiv:2304.09207, to appear in PRX

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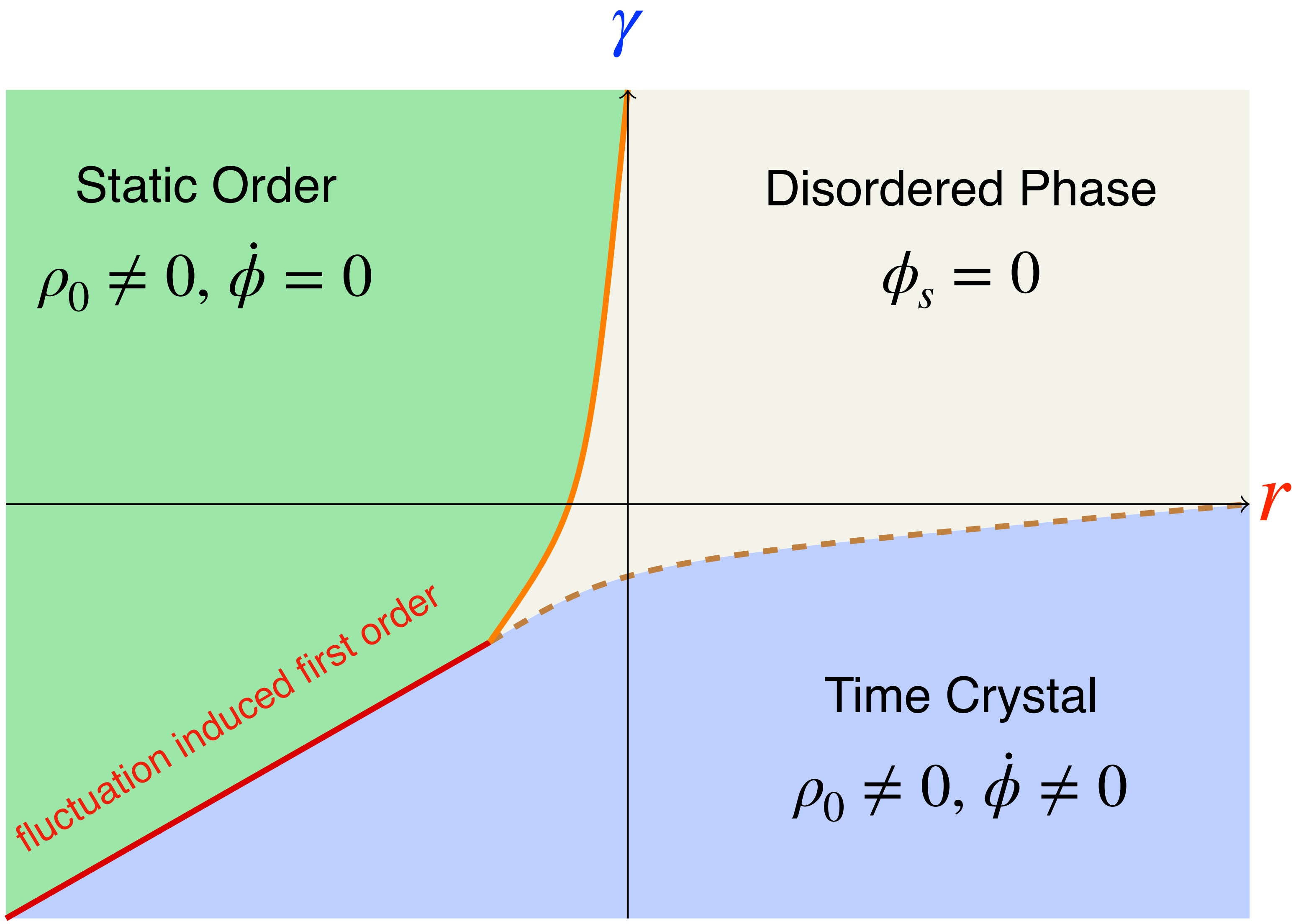

Daviet, Zelle, Rosch, SD, to appear in PRL

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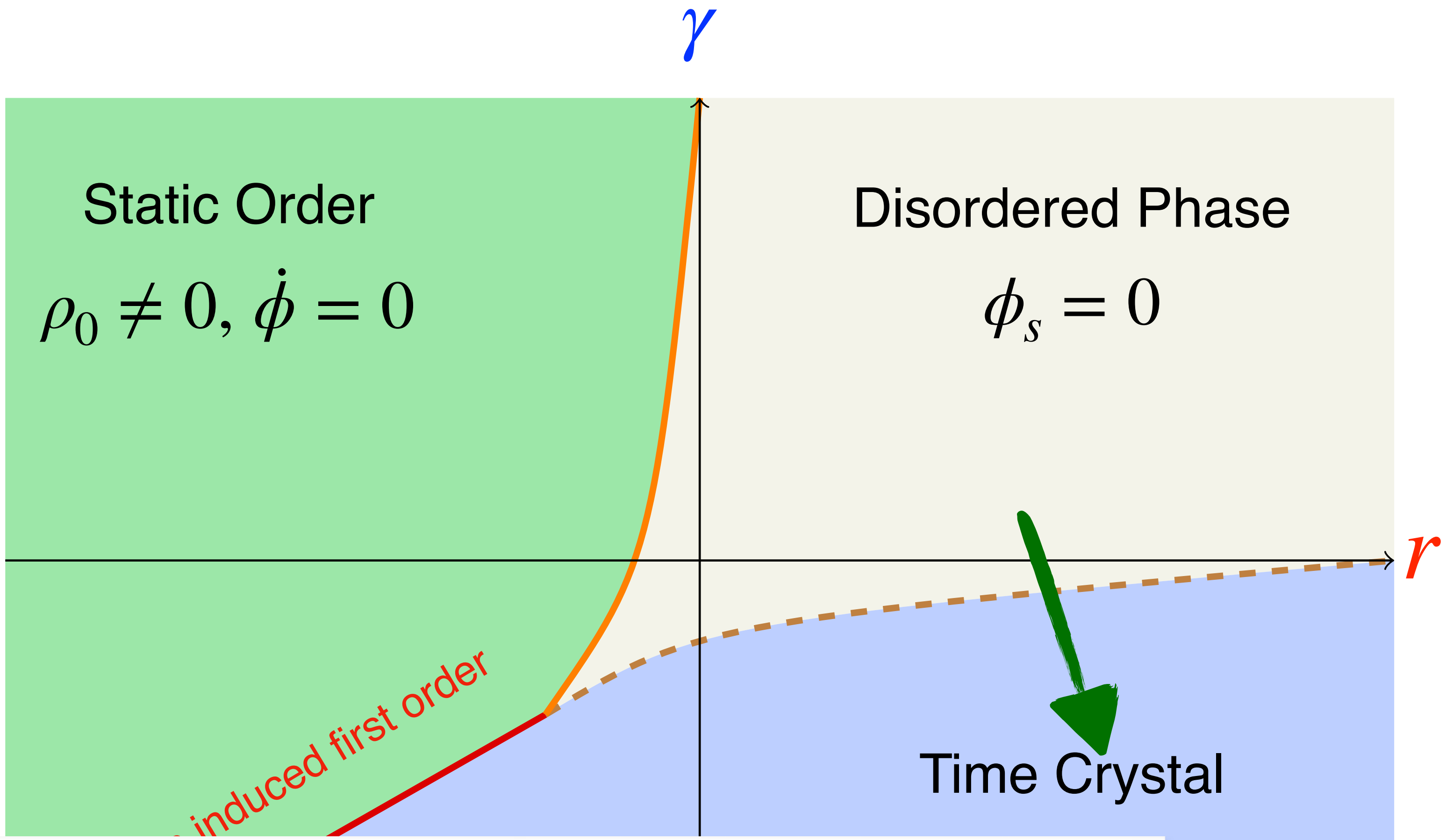


direct transition → novel universality, focus here

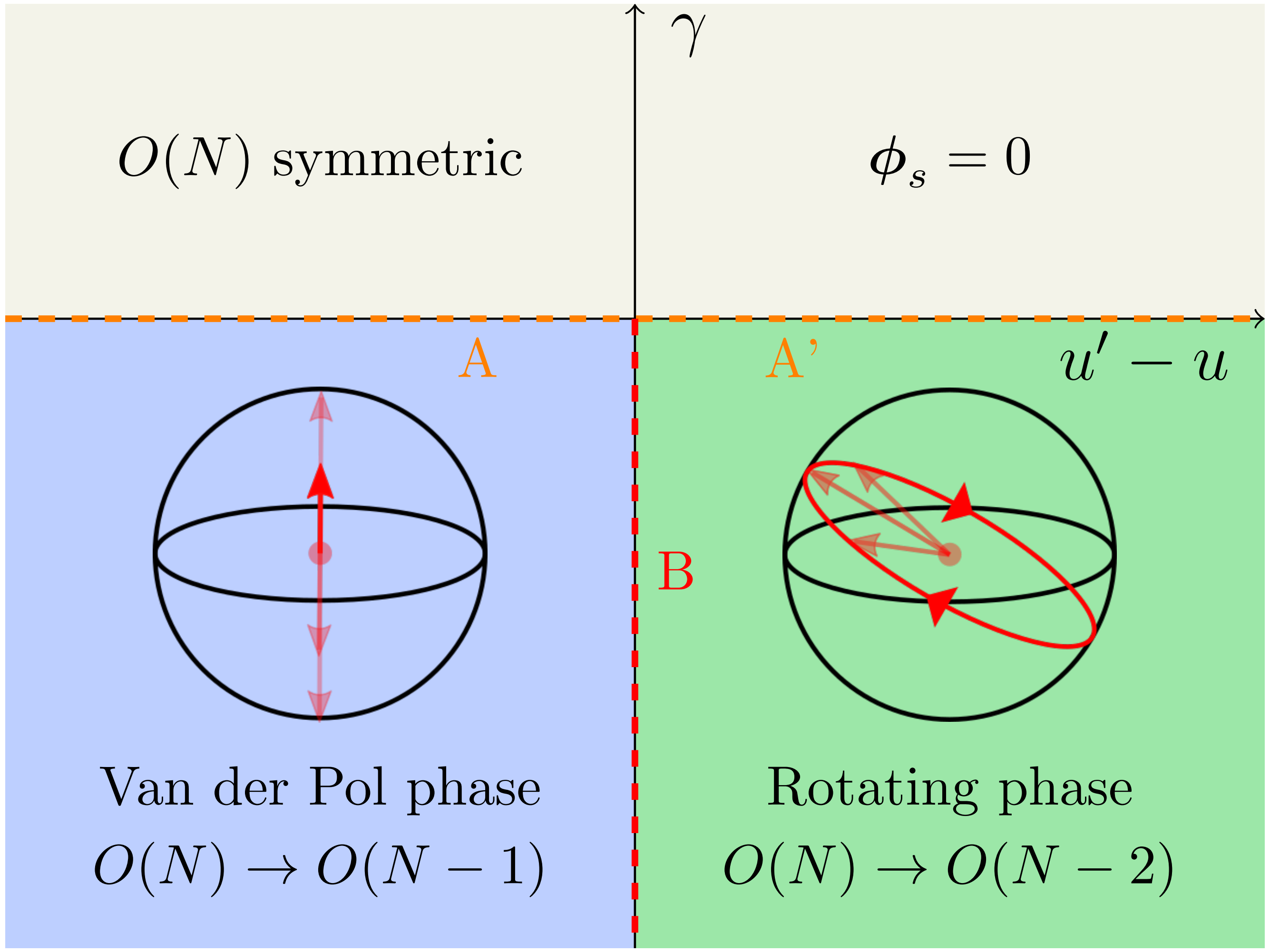
Daviet, Zelle, Rosch, SD, to appear in PRL

Direct transition: oscillatory vs. rotating phases

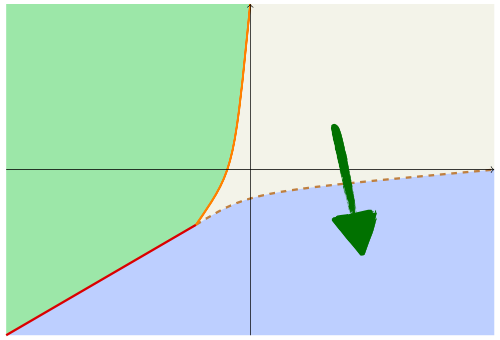
- competing non-equilibrium couplings

$$\partial_t^2\overrightarrow{\phi} + (2\gamma - Z\nabla^2 + u\rho)\partial_t\overrightarrow{\phi} + u'\partial_t\rho\,\overrightarrow{\phi} + (r - Z'\nabla^2 + \lambda\rho)\overrightarrow{\phi} + \vec{\xi} = 0$$

- generalization of van der Pol (vdP) oscillator to
  - ➡ $O(N)$ field (vdP: $N = 1$)
  - ➡ d dimensions
  - ➡ noise
- two possible phases
  - ➡ $u' - u < 0$: vdP oscillations of amplitude along spontaneously picks axis
  - ➡ $u' - u > 0$ constant rotation, fixed amplitude, spontaneously picks orbit (only for $N > 1$)



Direct transition — emergent symmetry



- at transition $\gamma = 0$:
  - Finite frequency scale $\sqrt{r} = \omega_0$
  - Hinders straight-forward RG

- field parametrization for $\gamma \approx 0$: fast scale ω_0
  - $\overrightarrow{\phi}(t) = \vec{\chi}_1(t)\cos\omega_0 t + \vec{\chi}_2(t)\sin\omega_0 t$
  - $\chi_1, \chi_2 \in \mathbb{R}^N$: slow degrees of freedom

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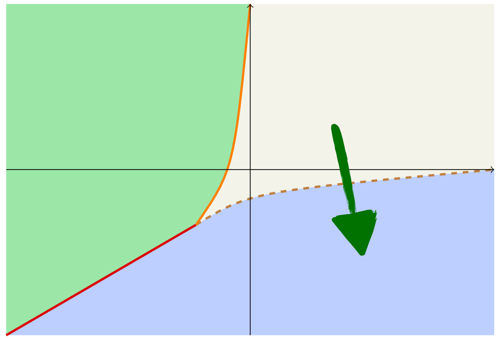

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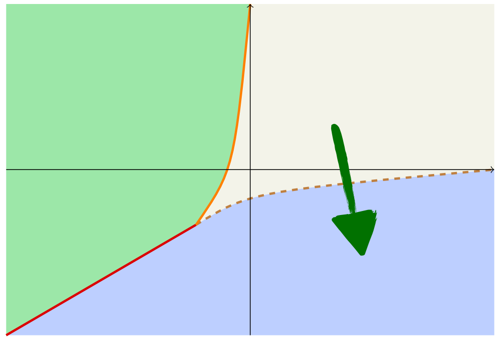

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- symmetries of slow degrees of freedom (rotating wave approximation, RWA):
  - $O(N)$: $\vec{\chi}_{1,2} \to R\vec{\chi}_{1,2}$
  - time translation acts as $SO(2)$:

$$\begin{pmatrix} \vec{\chi}_1 \\ \vec{\chi}_2 \end{pmatrix} \to \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \vec{\chi}_1 \\ \vec{\chi}_2 \end{pmatrix}$$

  - reason:
    - arbitrary shift $\omega_0 t \to \omega_0 t + \alpha$ in parametriz. of $\overrightarrow{\phi}$

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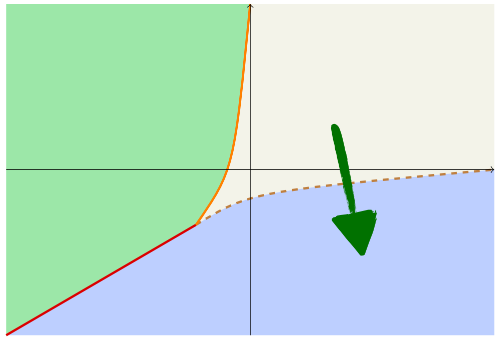

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➡ direct transition: $O(N) \times SO(2)$ breaking

The $O(N) \times SO(2)$ model

- effective dynamics at transition (RWA, or symmetry based):

$$\partial_t \boldsymbol{\chi}_a + \frac{\delta H_d}{\delta \boldsymbol{\chi}_a} + \epsilon_{ab} \frac{\delta H_c}{\delta \boldsymbol{\chi}_b} + \boldsymbol{\xi}_a = 0, \quad (a,b) \in \{1,2\}$$

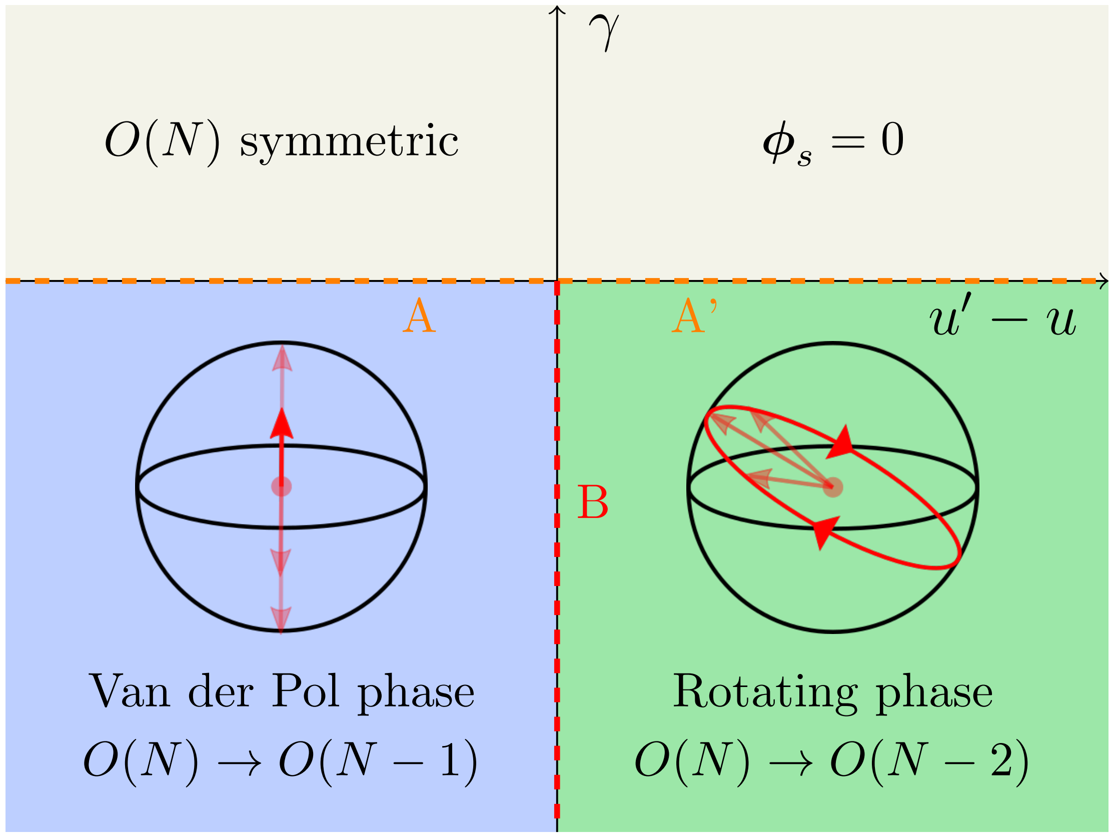
$$H_l = \int d^d \boldsymbol{x} \frac{Z_l}{2} \left[(\boldsymbol{\nabla} \boldsymbol{\chi}_1)^2 + (\boldsymbol{\nabla} \boldsymbol{\chi}_2)^2 \right] + \frac{\gamma_l}{2} \rho + \frac{g_l}{8} \rho^2 + \frac{\kappa_l}{2} \tau$$

$O(N) \times SO(2)$ invariants

$$\begin{cases} \rho = \left(\boldsymbol{\chi}_1^2 + \boldsymbol{\chi}_2^2 \right) \\ \tau = \frac{1}{4} \left(\boldsymbol{\chi}_1^2 - \boldsymbol{\chi}_2^2 \right)^2 + \left(\boldsymbol{\chi}_1 \cdot \boldsymbol{\chi}_2 \right)^2 \end{cases}$$

$\kappa_d > 0$: ρ condenses ⇒ rotation

$\kappa_d < 0$: τ condenses ⇒ vdP oscillation



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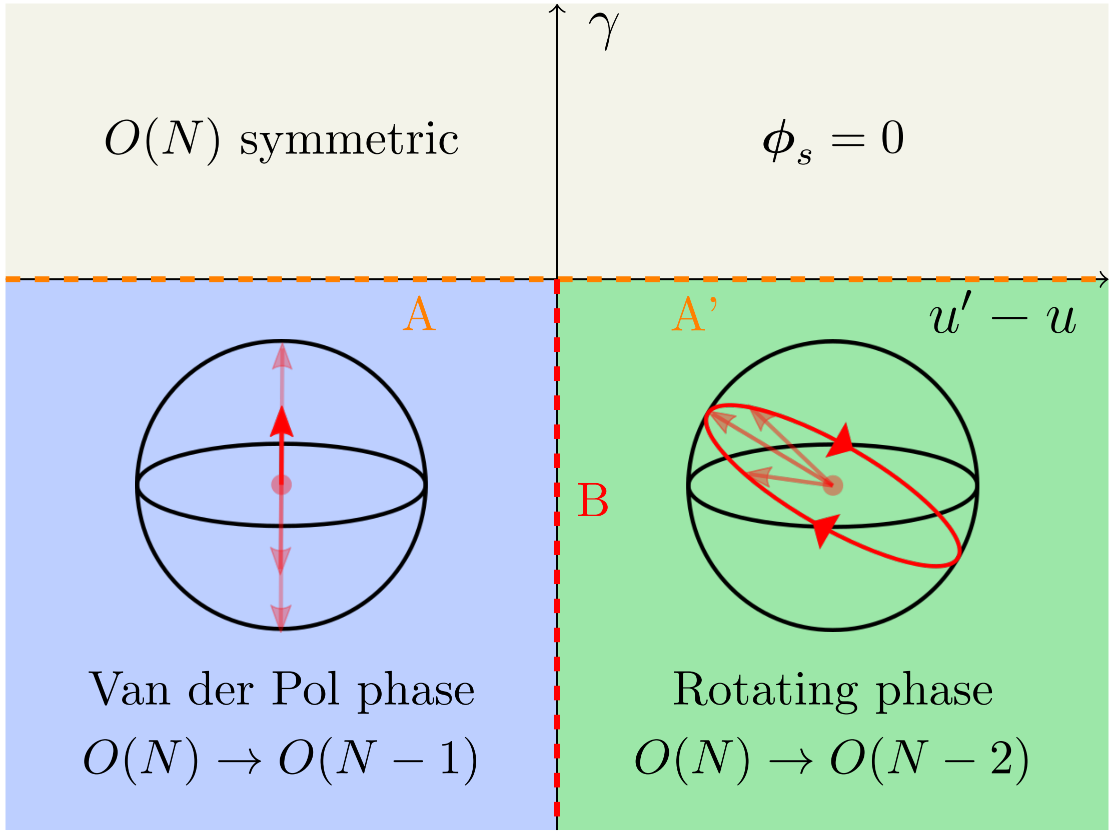
- equivalent: generalized noisy Gross-Pitaevski

$$\boldsymbol{\psi} = \boldsymbol{\chi}_1 + i \boldsymbol{\chi}_2 \in \mathbb{C}^N$$

$$(i\partial_t - Z\nabla^2 + i\gamma)\boldsymbol{\psi} + \frac{g}{2}(\boldsymbol{\psi} \cdot \boldsymbol{\psi}^*)\boldsymbol{\psi} + \frac{\kappa}{2}(\boldsymbol{\psi} \cdot \boldsymbol{\psi})\boldsymbol{\psi}^* + \boldsymbol{\xi} = 0$$

$$Z = Z_c + iZ_d, \; \gamma = \gamma_d, \; g = g_c + ig_d, \; \kappa = \kappa_c + i\kappa_d$$

➡ $N = 1$ van der Pol ⇔ driven-dissipative Bose condensation



driven Ising: Avni et al., arxiv (2023)

The $O(N) \times SO(2)$ model: Equilibrium vs. non-equilibrium

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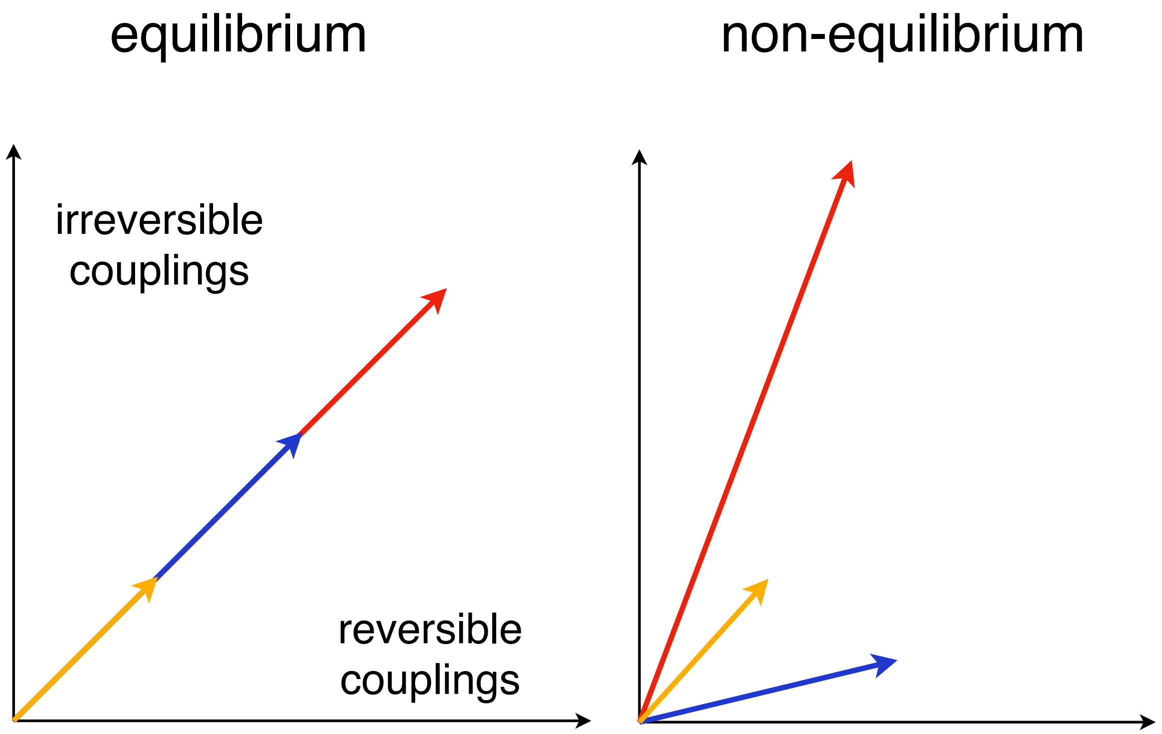
- equilibrium limit: thermal symmetry
- satisfied iff

$$H_c = JH_d,\ J = \frac{Z_c}{Z_d} = \frac{g_c}{g_d} = \ldots$$



The $O(N) \times SO(2)$ model: Equilibrium vs. non-equilibrium

- effective dynamics at transition (RWA, or symmetry based):

$$\partial_t \boldsymbol{\chi}_a + \frac{\delta H_d}{\delta \boldsymbol{\chi}_a} + \epsilon_{ab} \frac{\delta H_c}{\delta \boldsymbol{\chi}_b} + \boldsymbol{\xi}_a = 0, \quad (a, b) \in \{1, 2\}$$

$$H_l = \int d^d \boldsymbol{x} \frac{Z_l}{2} \left[(\boldsymbol{\nabla} \boldsymbol{\chi}_1)^2 + (\boldsymbol{\nabla} \boldsymbol{\chi}_2)^2 \right] + \frac{\gamma_l}{2} \rho + \frac{g_l}{8} \rho^2 + \frac{\kappa_l}{2} \tau$$

$O(N) \times SO(2)$ invariants

$$\begin{cases} \rho = \left(\boldsymbol{\chi}_1^2 + \boldsymbol{\chi}_2^2 \right) \\ \tau = \frac{1}{4} \left(\boldsymbol{\chi}_1^2 - \boldsymbol{\chi}_2^2 \right)^2 + \left(\boldsymbol{\chi}_1 \cdot \boldsymbol{\chi}_2 \right)^2 \end{cases}$$

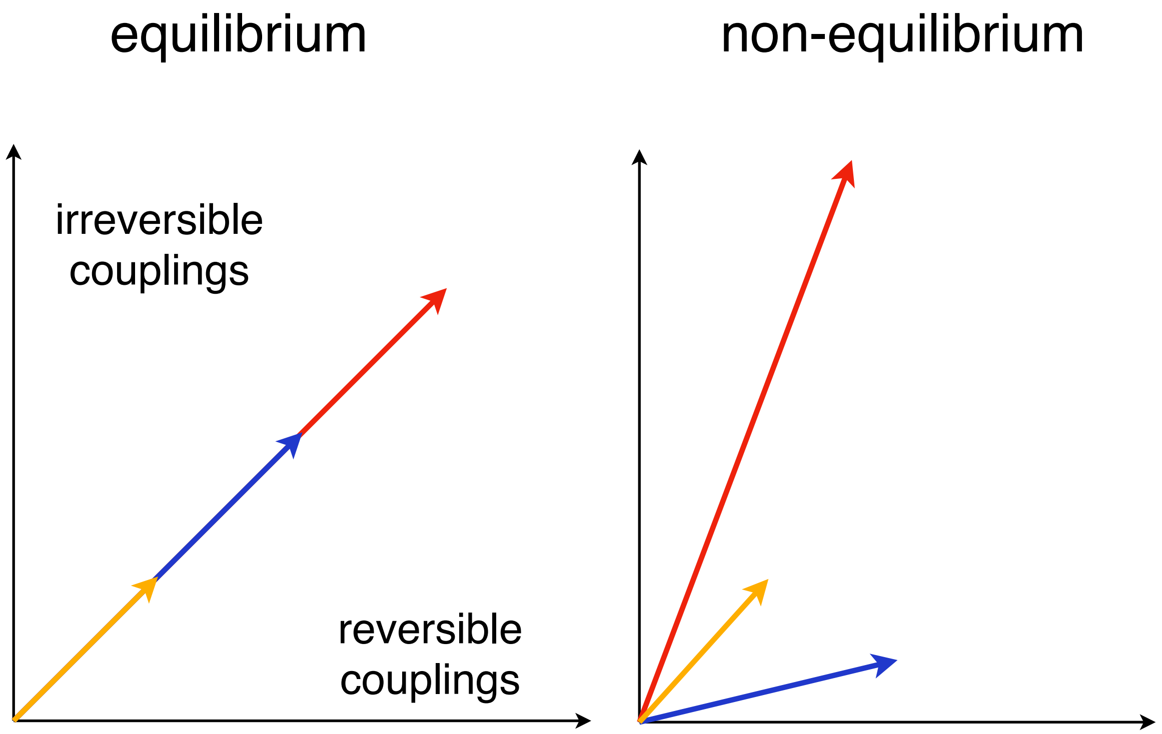
$\kappa_d > 0$: ρ condenses ⇒ rotation

$\kappa_d < 0$: τ condenses ⇒ vdP oscillation

- equilibrium limit: thermal symmetry
- satisfied iff

$$H_c = J H_d, \; J = \frac{Z_c}{Z_d} = \frac{g_c}{g_d} = \ldots$$

- $N = 1$: emergent equilibrium (with universal neq corrections) Sieberer, Huber, Altman, SD PRL (2013)



coarse graining / RG

Microscopi

Mesoscopic

Macroscopic

equilibrium broken

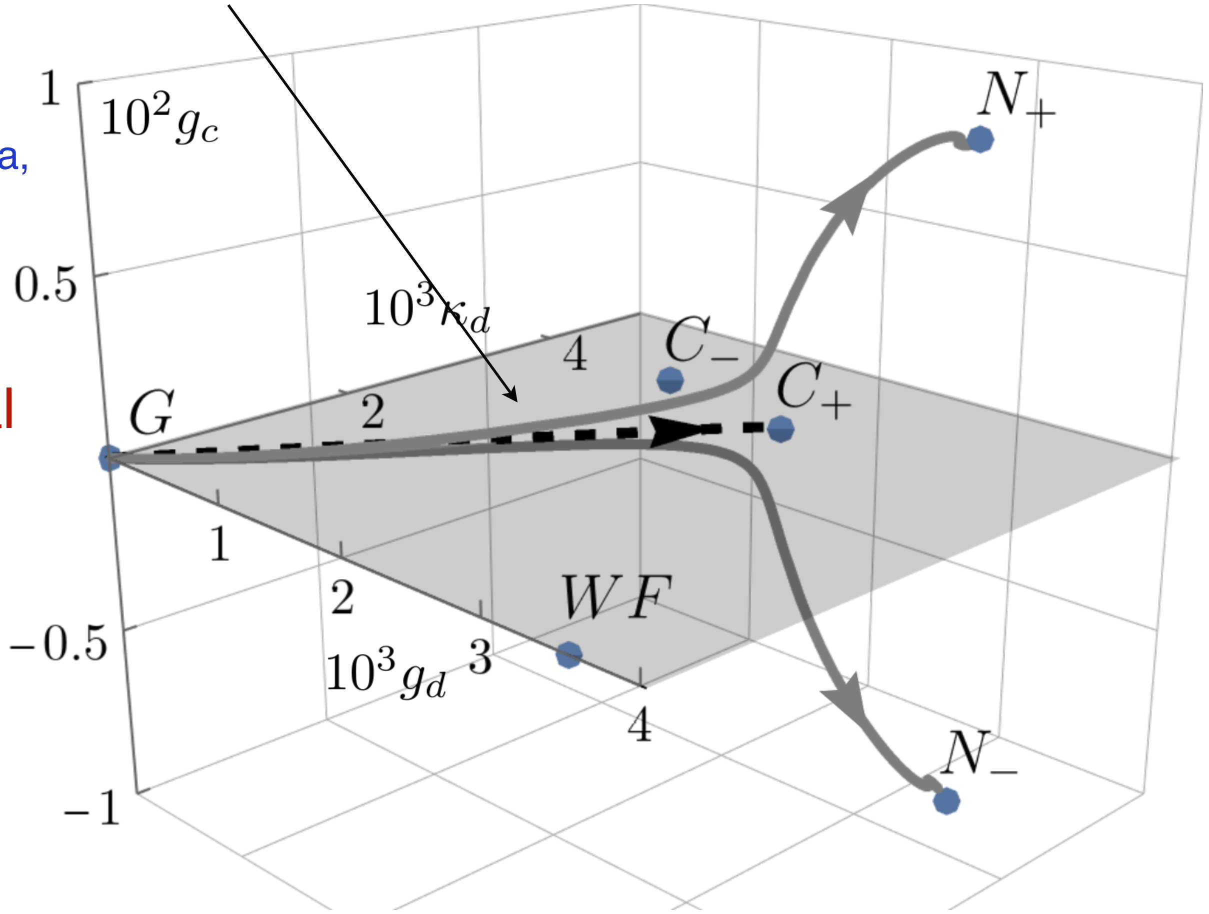
equilibrium restored

RG flow of the $O(N) \times SO(2)$ model, $N \geq 2$

- dynamic RG in $d = 4 - \epsilon$ to leading two-loop order:
  - equilibrium: $O(N) \times O(2)$, well known (frust. magnets)

    Calabrese, Parruccini, Nucl. Phys. B (2004); Delamotte, Mouhanna, Tissier PRB (2004)
  - non-equilibrium perturbations are relevant
    - ➡ equilibrium fixed points unstable to infinitesimal breaking of equilibrium conditions, opposite $N = 1$: spontaneous breaking of eq.
    - ➡ pair of novel perturbatively accessible non-equilibrium fixed points
    - ➡ new non-equilibrium universality class
  - $\kappa_d^* > 0$: transition to rotation 2^{nd} order (to oscillation: fluctuation induced 1^{st} order)

equilibrium surface



3 dimensional projection of RG flow

Nonequilibrium fixed point of $O(N) \times SO(2)$ model

critical exponents: scaling form

$$\chi^R(\boldsymbol{q},t) \sim q^{-2+\eta'+z}\tilde{\chi}^R(tq^z, iq^{\eta-\eta_c}, q\gamma^{-\nu}), \quad C(\boldsymbol{q},t) \sim q^{-2+\eta}\tilde{C}(tq^z, iq^{\eta-\eta_c}, q\gamma^{-\nu})$$

response (dynamic susceptibility)

correlations

z: dynamical exponent $\tau \sim x^z$, mean field: $z_{MF} = 2$

ν: correlation length $\xi \sim \gamma^{-\nu}$, mean field: $\nu_{MF} = 1/2$

η: anomalous dimension, mean field $\eta = 0$

η': anomalous response, $\eta \neq \eta' \Rightarrow$ asymptotic violation of FDR

η_c: competition of coherent/dissipative effects

	$N=2$	$N=3$
$Re[\eta - \eta']$	$-0.343\epsilon^2$	$-1.46\epsilon^2$
η	$-0.353\epsilon^2$	$-1.49\epsilon^2$
ν, z, η_c	See Daviet, Zelle, Rosch, SD: arXiv:2312.13372	

$$\frac{2T_{eff}(q)}{\omega}\,\mathrm{Im}\,\chi^R(q,\omega) = \mathscr{C}(q,\omega) \text{ defines } T_{eff}(q)$$

➡ $T_{eff} \sim q^{\eta-\eta'}$ diverges universally at transition into rotating phase

➡ can be measured via Stokes - Anti-Stokes

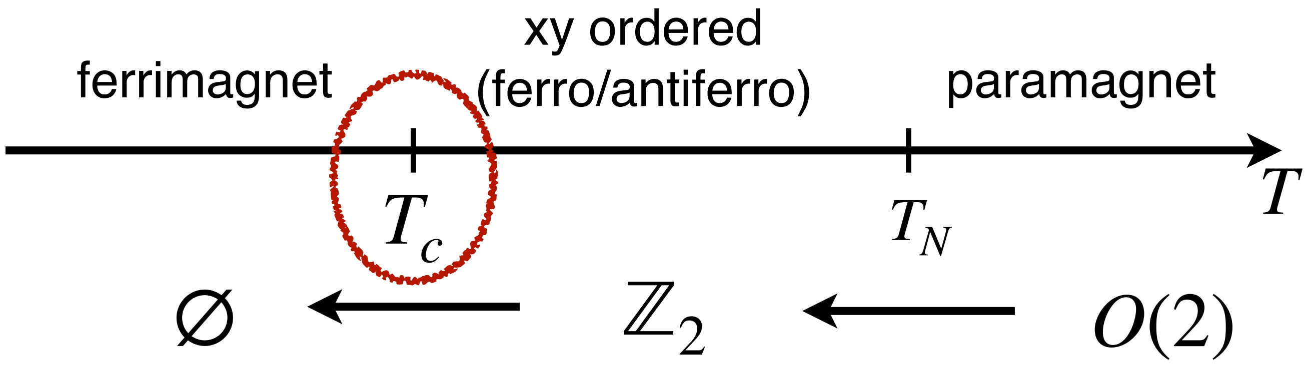
Where to look for it — I. Driven Ferrimagnet

- strategy I: driven $O(N)$ models, additional $SO(2)$ emergent

disordered

- example $N = 2$: Ferrimagnet
  - 3d magnet: $SO(2)$ in xy plane, non-commuting $\mathbb{Z}_2$ along z-axis gives $O(2) \simeq \mathbb{Z}_2 \rtimes SO(2)$
    - laser drive (couples Goldstone and Ising mode)

equilibrium phase diagram



exp: e.g. Sürgers et al. Nat. Comm. (2014); Omi et al. PRB (2021)

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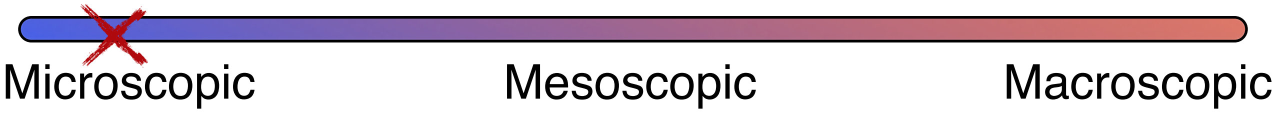

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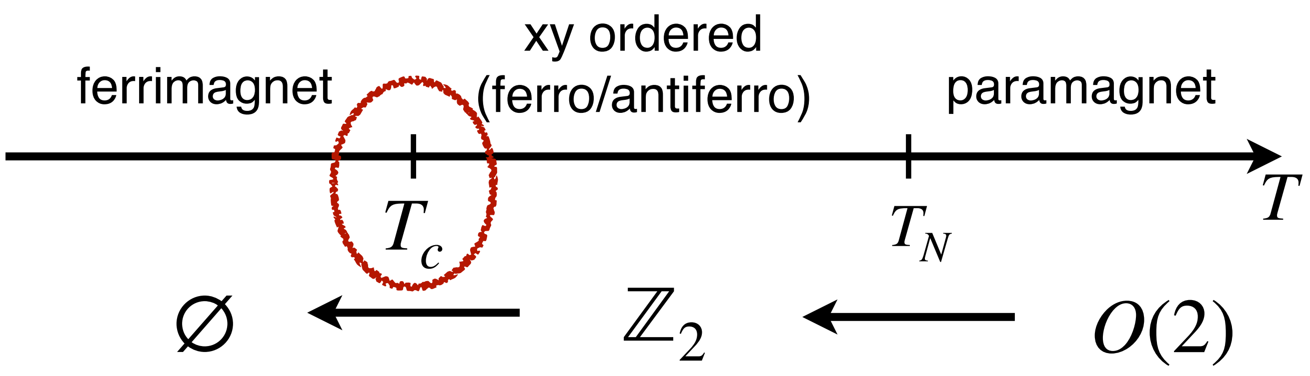
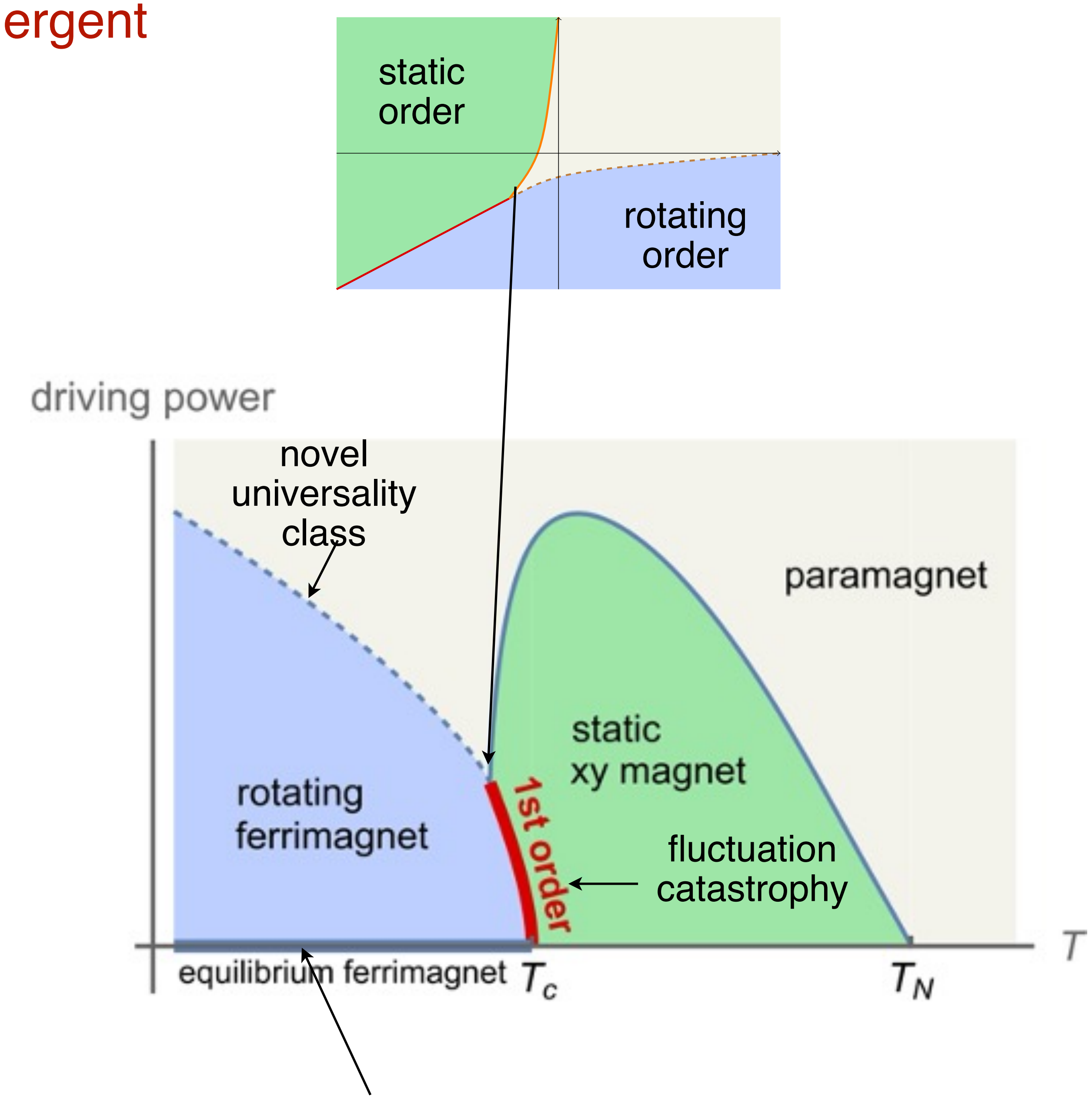


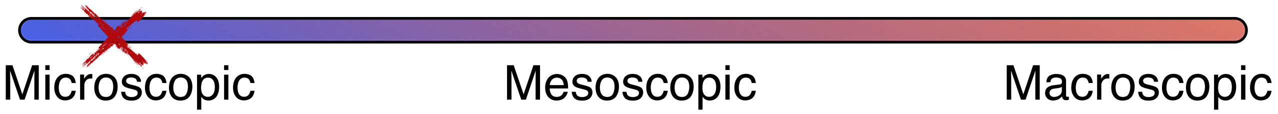


infinitesimal drive sufficient to activate limit cycle

exp: e.g. Sürgers et al. Nat. Comm. (2014); Omi et al. PRB (2021)

➡ driven magnets: small non-equilibrium perturbation gives rise to strong macroscopic effect!

Where to look for it — II. Exciton-polaritons, magnon condensates



- strategy II: realize $O(N) \times SO(2)$ directly & break equilibrium

- breaking of equilibrium conditions: lossy & incoherently pumped systems
- ingredients: two $U(1)$ symmetric d.o.f.s and one exchange symmetry

$$U(1)_+ : \psi_+ \to \exp(i\theta_+)\psi_+, \quad U(1)_- : \psi_- \to \exp(i\theta_-)\psi_-, \quad \mathbb{Z}_2 : \begin{cases} \psi_+ \to \psi_- \\ \psi_- \to \psi_+ \end{cases}$$

- or equivalently

$$U(1)_s : \psi_\pm \to \exp(i\theta_s)\psi_\pm, \quad U(1)_a : \psi_\pm \to \exp(\pm i\theta_a)\psi_\pm, \quad \mathbb{Z}_2 : \begin{cases} \psi_+ \to \psi_- \\ \psi_- \to \psi_+ \end{cases}$$

$$U(1)_s \simeq SO(2) \qquad U(1)_a \rtimes \mathbb{Z}_2 \simeq O(2)$$

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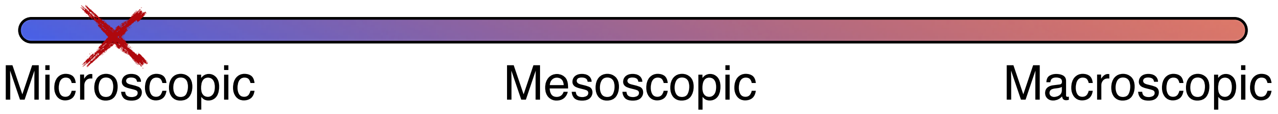

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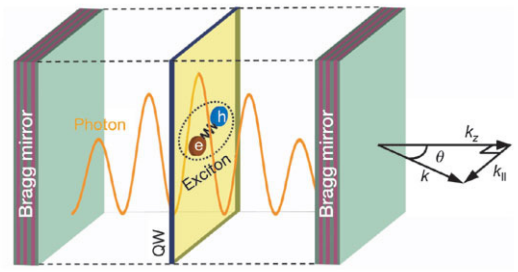
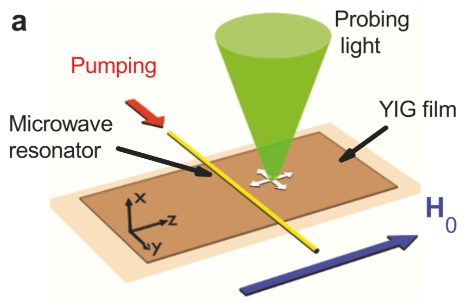
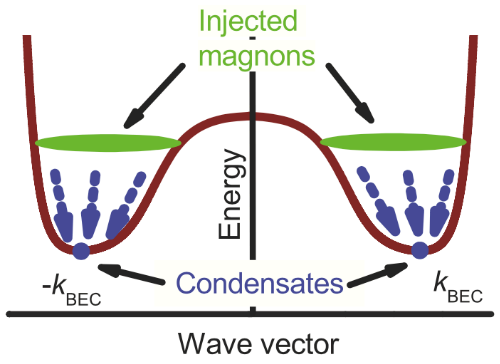
- exciton-polaritons: polarization degree of freedom
- magnon condensates in yttrium iron garnet (YIG): condensation in $\pm k_{\mathrm{BEC}}$

Exciton-polaritons



YIG





➡ realizations of relevant symmetry group in various current platforms

Limit-cycle phase: Goldstone modes

in preparation

- limit cycle as spontaneous breaking of time translation symmetry:
  - equation of motion (for slow d.o.f.) / action is time translation invariant
  - limit cycle breaks time translations spontaneously
  - Goldstone theorem: there must exist a soft mode

- example: rotating phase

symmetry breaking: $O(N) \rightarrow O(N-2)$

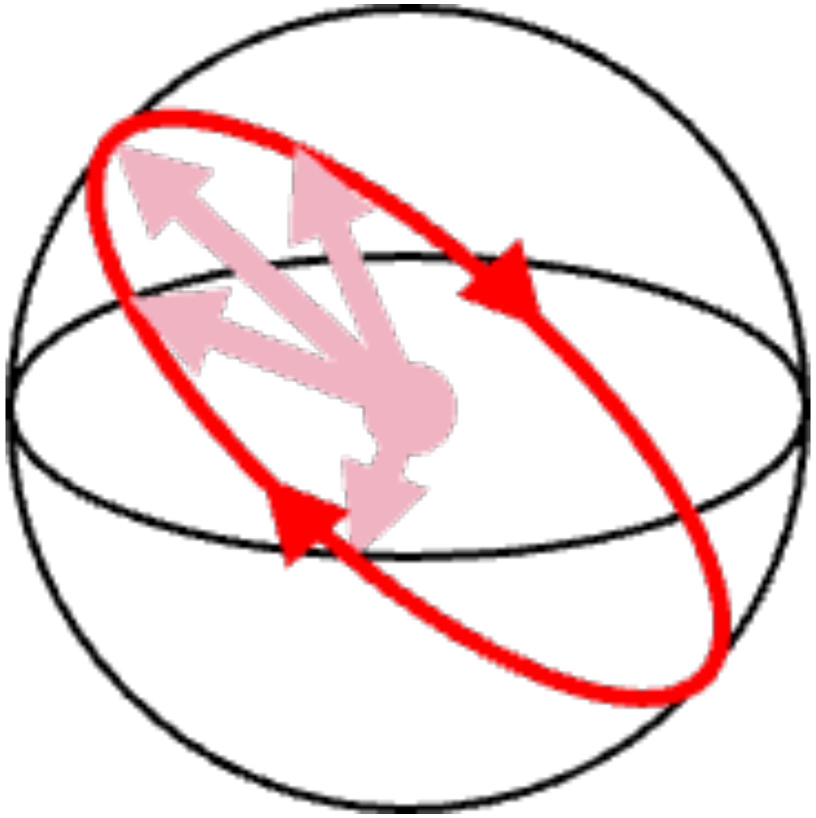
=> $N-1+N-2=2N-3$ Goldstone modes

$2N-4$ rotations of the circle

+ rotation along circle

= $2N-3$ gapless modes

Goldstone of time translation



Goldstone mode of time translation

in preparation

- limit cycle as spontaneous breaking of time translation symmetry:
  - formally: $\phi_s(t)$ limit cycle solution $\Longrightarrow$ $\phi_s(t+t_0)$ also limit cycle solution

  ➡ continuous manifold of stable states, parameterized by t_0

  - Goldstone construction: promote constant symmetry generator to slow dynamic field, $t_0 \to \theta(t, \mathbf{x})$

    derive its EoM

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    derive its EoM

- Example: $N = 1$ van der Pol oscillator

$$\partial_t^2\phi + (2\gamma + u\phi^2 - Z_1\nabla^2)\partial_t\phi + (\omega_0^2 + \lambda\phi^2 - Z_2\nabla^2)\phi + \xi = 0$$

  - discrete internal symmetry only: $\mathbb{Z}_2 : \ \phi \to -\phi$
  - time crystal for $\gamma < 0 : \ \phi_s(t) \approx A\cos\omega_D t$; resulting EoM:

$$\partial_t\theta - K\,\nabla^2\theta + g\,(\nabla\theta)^2 + \xi_\theta = 0$$

➡ limit cycles naturally host gapless excitations

➡ mechanism does not need continuous internal symmetry

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$$\partial_t \theta - K\,\nabla^2\theta + g\,(\nabla\theta)^2 + \xi_\theta = 0 \quad \longleftarrow \quad \boxed{\text{KPZ equation!}}$$

Kardar, Parisi, Zhang, PRL (1986)

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Goldstone mode of time translation: A route to KPZ matter

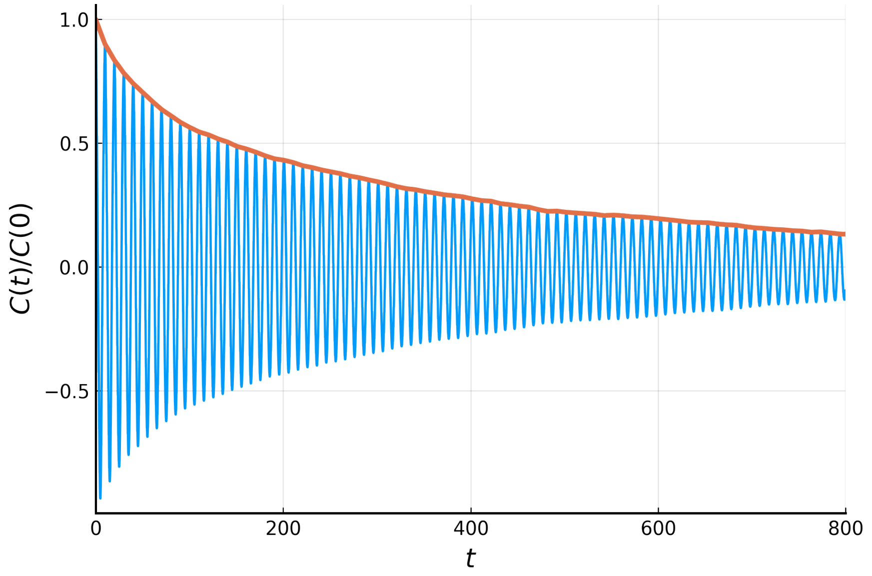
- Goldstone mode of TT breaking in van der Pol oscillator & KPZ equation

$$\partial_t\theta - K\nabla^2\theta + g(\nabla\theta)^2 + \xi = 0$$

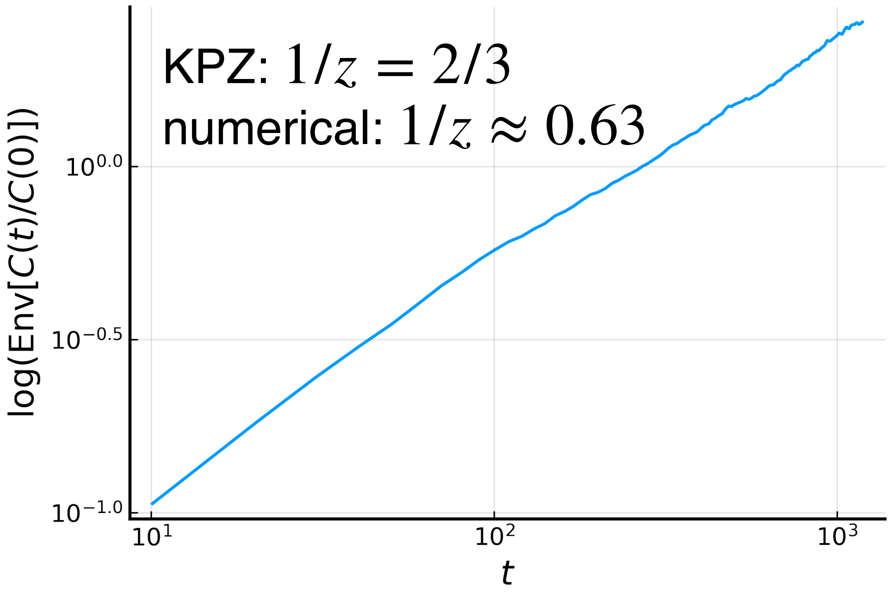
Kardar, Parisi, Zhang, PRL (1986)

- preliminary numerical evidence (N=1 van der Pol equation in 1D on 128 lattice sites)

stretched exponential decay of two-time correlation function



KPZ exponent β



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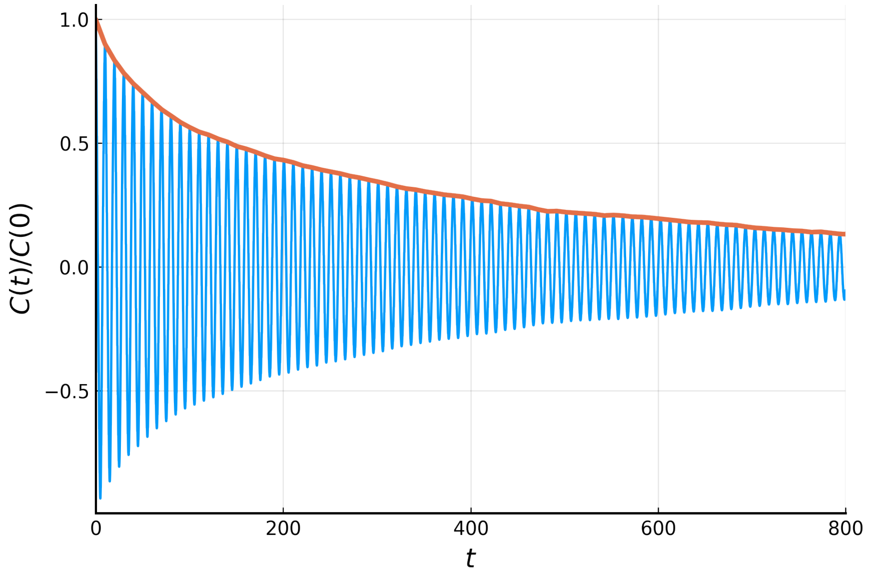

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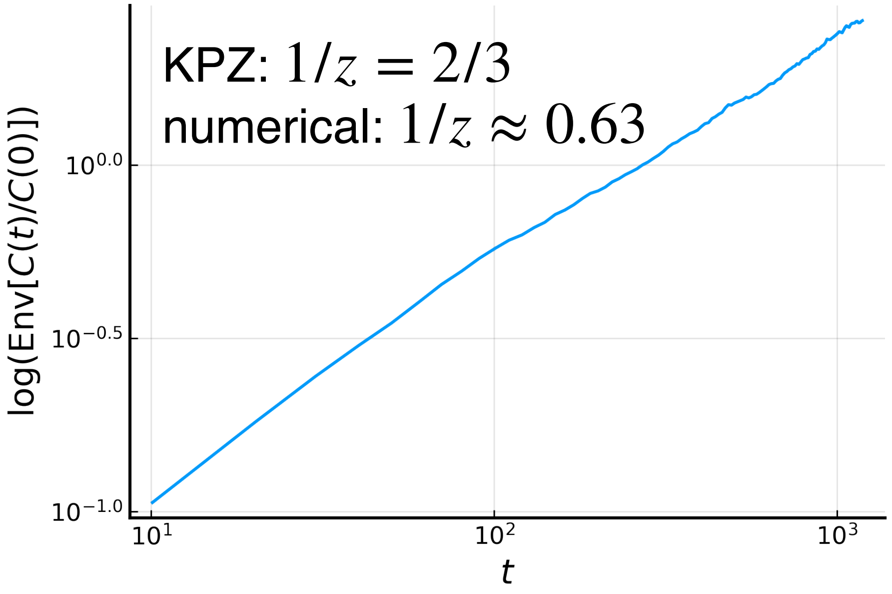


KPZ: $1/z = 2/3$

numerical: $1/z \approx 0.63$

- route to realizations of KPZ matter in $d = 2$?
  - exciton-polariton condensates
  - 2d arrays of current-driven Josephson junctions
  - magnon condensates

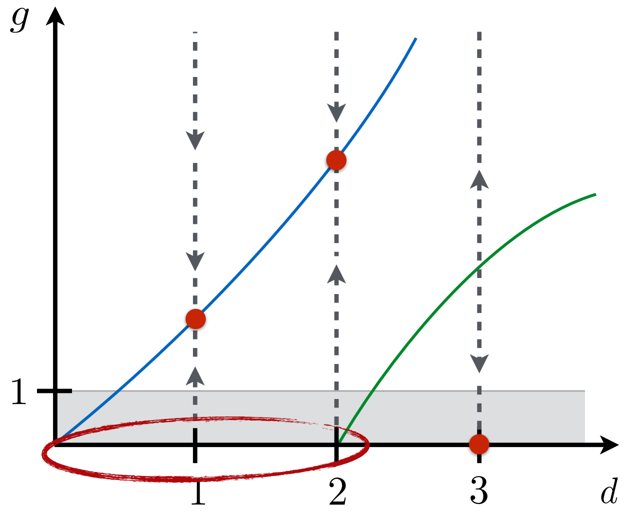
➡ robust (time translation symmetry protected) realization of KPZ dynamics with many possible incarnations

Summary and perspective: Active quantum matter

- quantum materials are usually hard to drive away from thermal equilibrium
- but soft modes can be activated: weak non-equilibrium perturbation can lead to strong non-equilibrium macroscopic effects

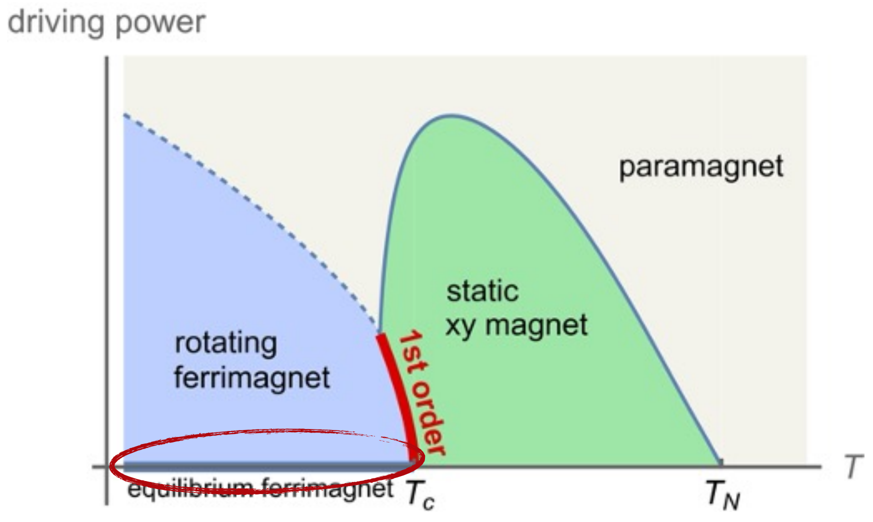
gapless phases

KPZ



$d \leq 2, N = 2$

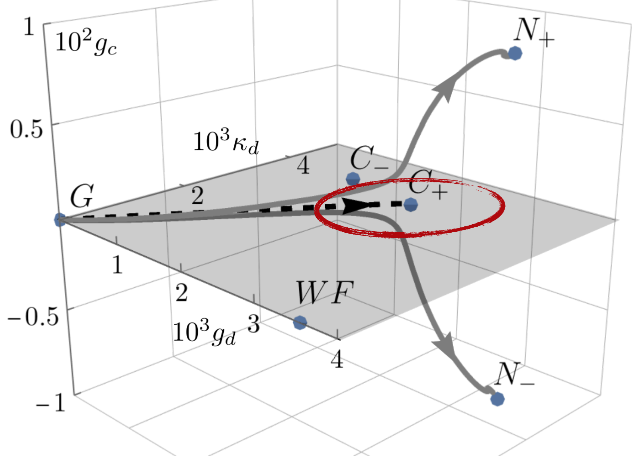
driven ferrimagnet



$d \leq 4, N \geq 2$

gapless (critical) points

onset of TT breaking



$d \leq 4, N \geq 2$

➡ non-equilibrium sensitivity of non-linear sigma models for bigger groups?

➡ role of topological defects out-of-equilibrium?

➡ general mechanism stabilizing new universality classes of nonthermal matter?

5. Quantum aspects: Topology in driven open quantum matter

- dissipation engineering: inducing topological states out of equilibrium

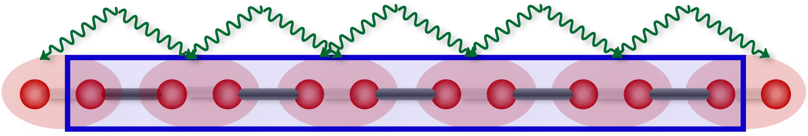
equilibrium vs. non-equilibrium

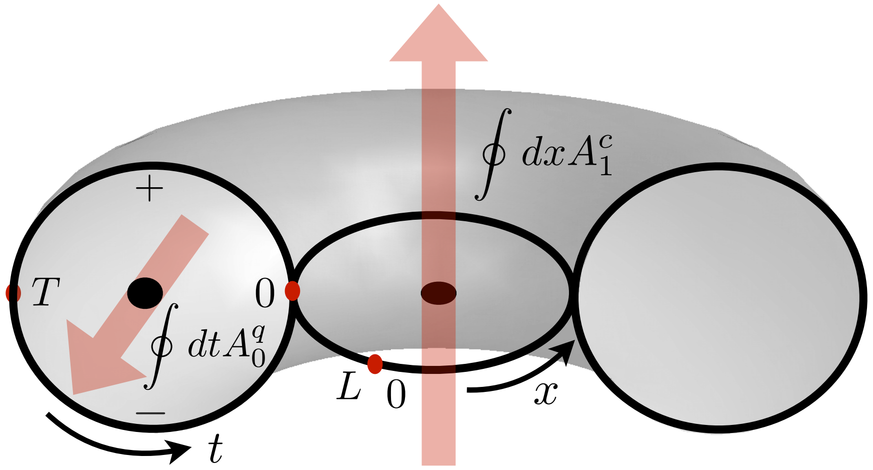
- ‘topology beats dissipation’: response of a pure non-equilibrium topological insulator

weak vs. strong symmetry

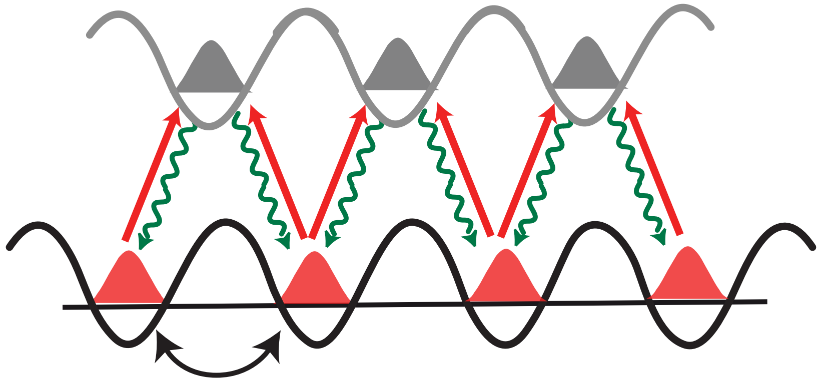
- ‘topology beats mixedness’: quantised non-linear response for mixed states

mixed vs. pure states





Pure states: Order by dissipation



SD, A. Micheli, A. Kantian, B. Kraus, H.P. Büchler, P. Zoller, Nat. Phys. (2008);
B. Kraus, H.P. Büchler, SD, A. Micheli, A. Kantian, P. Zoller, PRA (2008);
F. Verstraete, M. Wolf, J. I. Cirac, Nature Physics **5**, 633 (2009).

Microscopic
Quantum Optics

“Thermodynamic”
Many-body physics

Long wavelength
Statistical mechanics

Guiding Question

- Equilibrium: dissipation/friction enhances the statistical degree of disorder, entropy increases

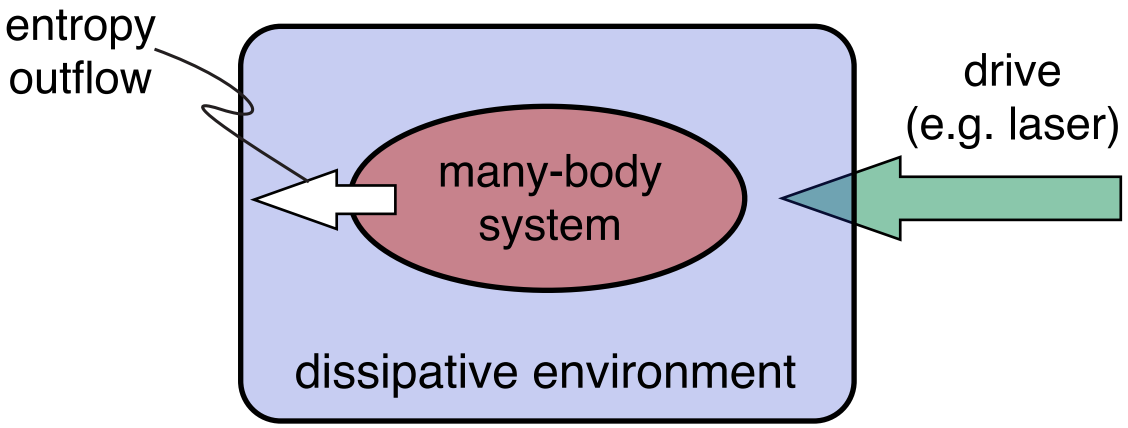
$$\partial_t S > 0$$

second law of thermodynamics

- Q1: Can this be reversed in a many-body quantum system?

$$\partial_t S < 0$$

- A: Yes, if dissipation is suitably combined with coherent drive



$$\partial_t S_{\mathrm{syst+env}} > 0 \quad \text{but} \quad \partial_t S_{\mathrm{syst}} < 0$$

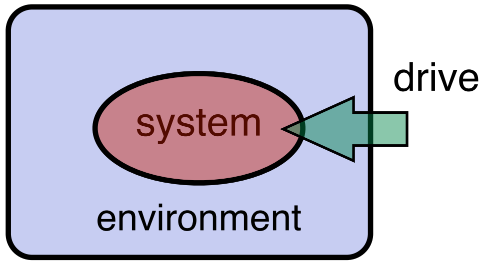
- Q2: Is it possible to create genuine many-body quantum mechanical order by dissipation?
  - Long range phase coherence
  - Entanglement
  - Topological order

Concept: Dark states in Lindblad equations

- quantum master equation

Lindblad operators

$$\partial_t \rho = -i\underbrace{\cancel{[H,\rho]}}_{\text{coherent evolution}} + \kappa \underbrace{\sum_i (L_i \rho L_i^\dagger - \tfrac{1}{2}\{L_i^\dagger L_i, \rho\})}_{\text{driven-dissipative evolution}}$$



- Key concept: Dark states

$$L_i|D\rangle = 0 \quad \forall i$$

➡ time evolution stops when $\rho = |D\rangle\langle D|$ *

---

* for e.g. $H|D\rangle = E|D\rangle$

Concept: Dark states in Lindblad equations

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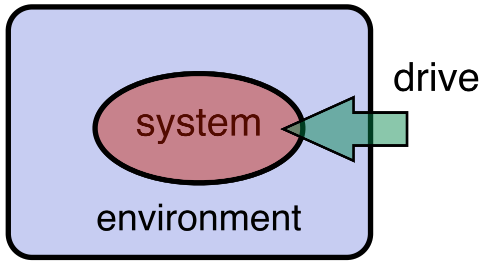
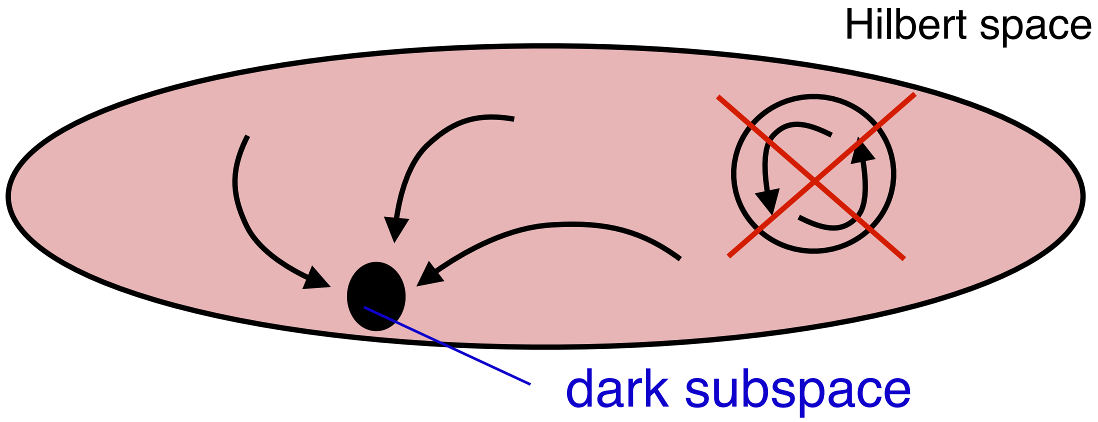


- Interesting situation: unique dark state solution



➡ you can enter, but never leave *

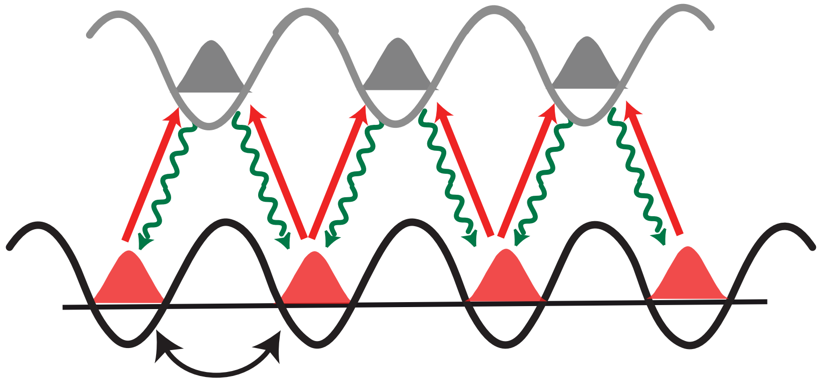
➡ directed motion in Hilbert space $\rho \overset{t\to\infty}{\longrightarrow} |D\rangle\langle D|$

➡ dissipation removes entropy, increases purity

---

* for e.g. $H|D\rangle = E|D\rangle$

Order by dissipation:
Bosons



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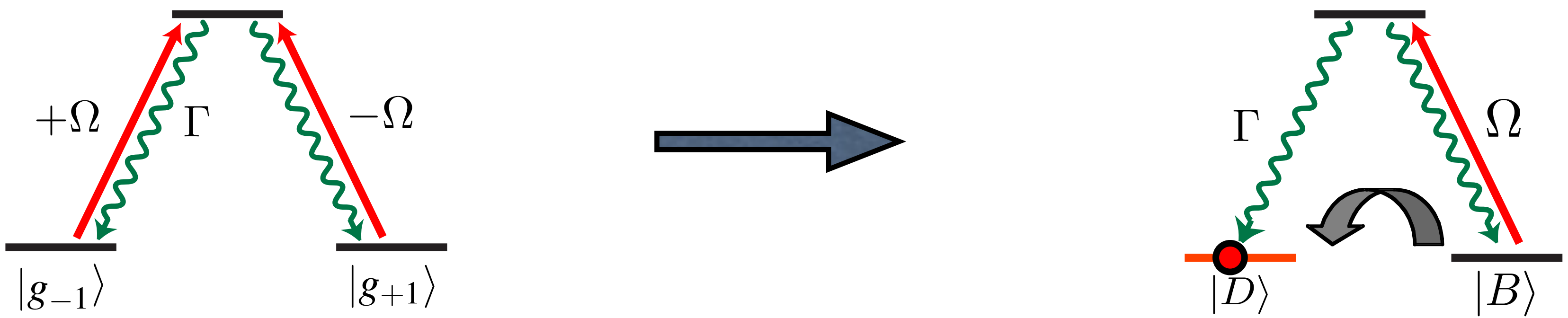
Dark states: From quantum optics to many particles

- optical pumping: three internal (electronic) levels Aspect et al., PRL (1998); Kasevich, Chu, PRL (1992)



dark state

$$|D\rangle \sim |g_{+1}\rangle + |g_{-1}\rangle$$

bright state

$$|B\rangle \sim |g_{+1}\rangle - |g_{-1}\rangle$$

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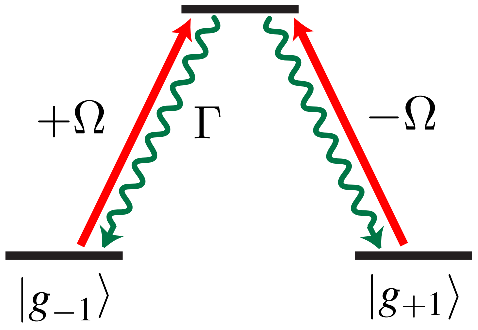
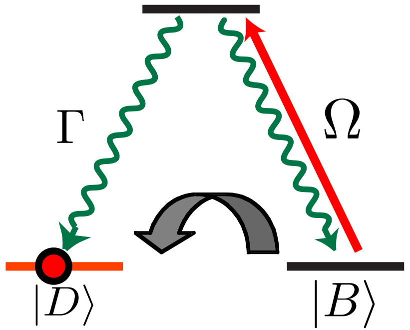
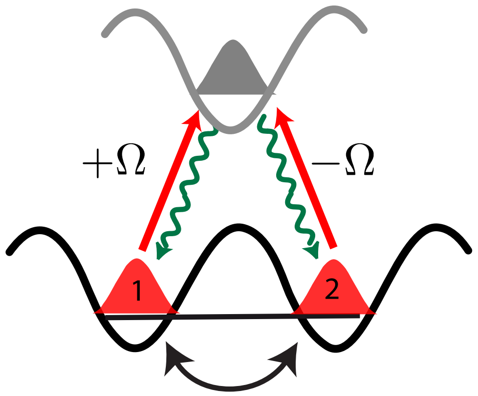


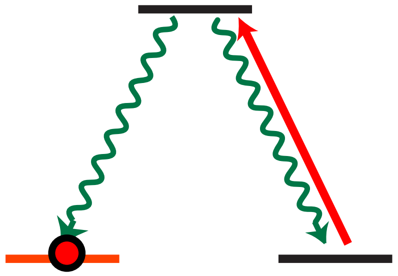


dark state

$$|D\rangle \sim |g_{+1}\rangle + |g_{-1}\rangle$$

- 1 atom on 2 sites: external (spatial) degrees of freedom (atoms on optical lattice)



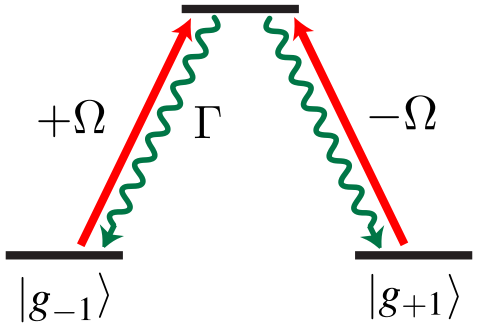


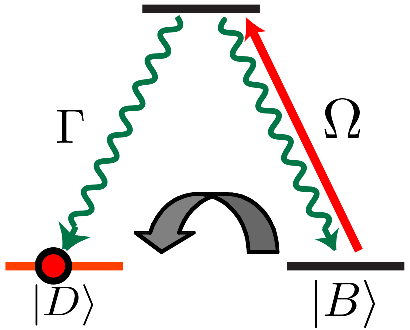
$$(a_1^\dagger + a_2^\dagger)\,|\mathrm{vac}\rangle$$

symmetric
superposition

Dark states: From quantum optics to many particles

- optical pumping: three internal (electronic) levels Aspect et al., PRL (1998); Kasevich, Chu, PRL (1992)

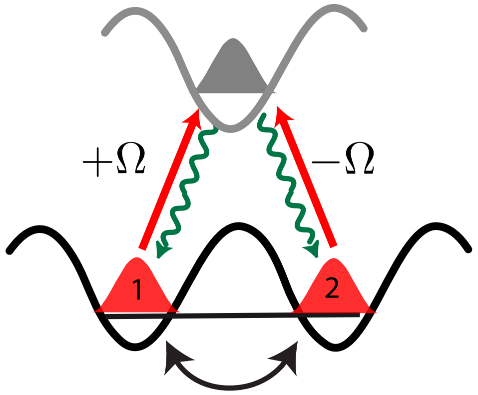


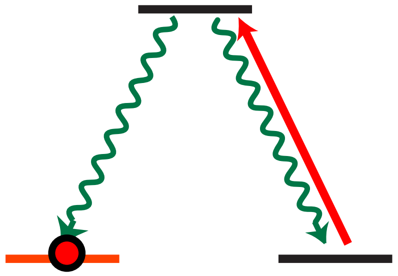


dark state

$$|D\rangle \sim |g_{+1}\rangle + |g_{-1}\rangle$$

- 1 atom on 2 sites: external (spatial) degrees of freedom (atoms on optical lattice)

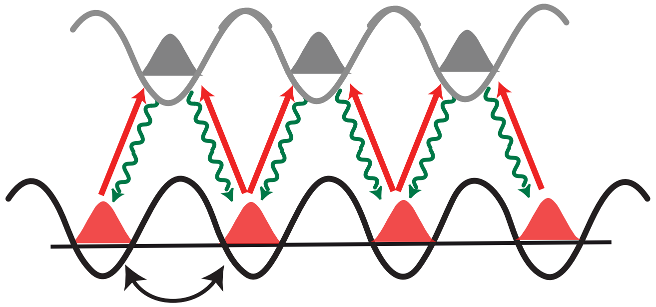




$$(a_1^\dagger + a_2^\dagger)\,|\mathrm{vac}\rangle$$

symmetric superposition

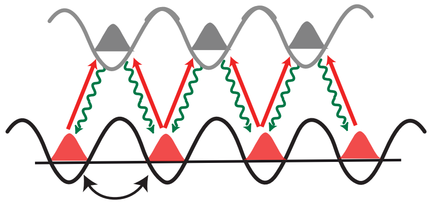
- N atoms on M sites (optical lattice)



$$|BEC\rangle = \frac{1}{N!}\Big(\sum_\ell a_\ell^\dagger\Big)^N |vac\rangle$$

➡ combination of drive and dissipation enables purification

Dissipative Many-Body State Preparation

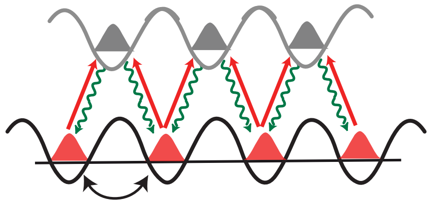


- Lindblad operators for BEC dark state:

$$L_i = (a_i^\dagger + a_{i+1}^\dagger)(a_i - a_{i+1}) \qquad\qquad L_i|\mathrm{BEC}\rangle = 0 \;\forall\, i$$

➡ Long range phase coherence/ boson condensation from quasilocal dissipative operations

Dissipative Many-Body State Preparation



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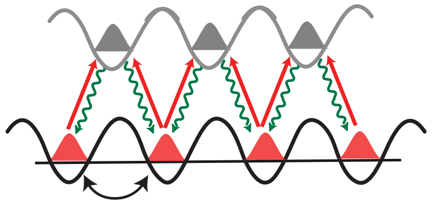
- Uniqueness of stationary solution can be shown
  - ➡ Ordered phase reached from arbitrary initial state

$$\Longrightarrow \rho(t) \longrightarrow |BEC\rangle\langle BEC| \text{ for } t \to \infty$$

Dissipative Many-Body State Preparation



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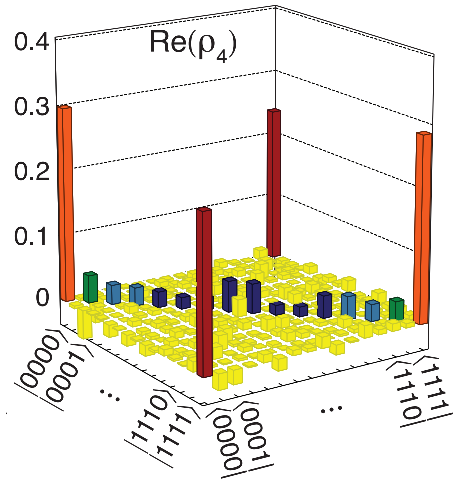
- Experimental realizations:

Entanglement generation



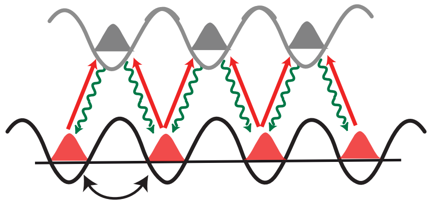
$$\rho = |D\rangle\langle D|$$

$$|D\rangle = \frac{1}{2}(|0000\rangle + |1111\rangle)$$

GHZ state of
four ions

Universal open-system quantum
simulator,
Blatt group, Nature (2011)

Dissipative Many-Body State Preparation



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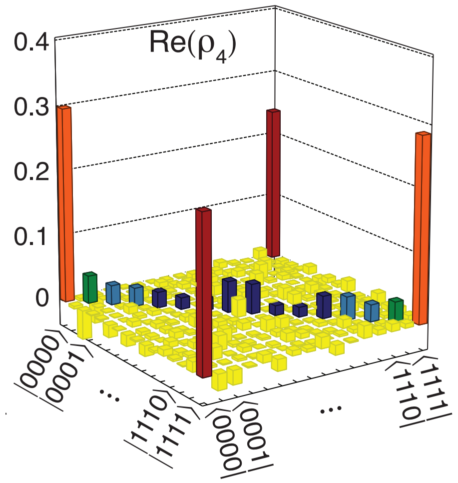

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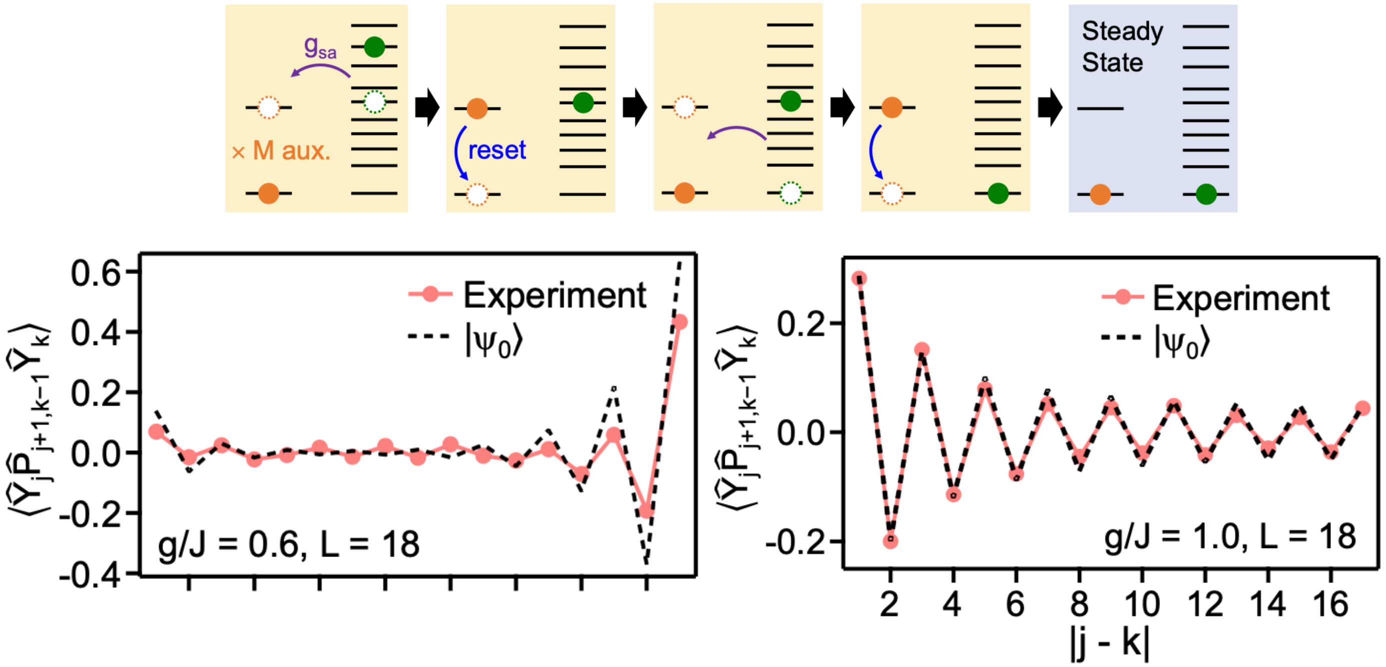
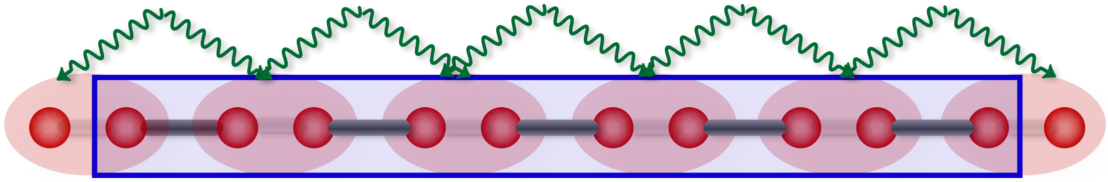
Long range quantum correlations



cooling the quantum Ising model by engineered dissipation
Google Quantum AI collaboration, arxiv (2023)

Fermions:
Topological order by dissipation in a dissipative Kitaev chain



SD, E. Rico, M. A. Baranov, P. Zoller, Nat. Phys. (2011)
Review: C.-E. Bardyn, C. Kraus, E. Rico, M. Baranov, A. Imamoglu, SD, NJP (2013)

Microscopic
Quantum Optics

“Thermodynamic”
Many-body physics

Long wavelength
Statistical mechanics

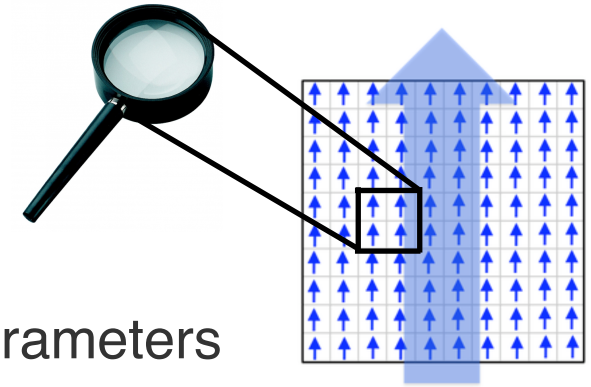
Topological States in Condensed Matter

Nayak et al., RMP (2008)

Hasan and Kane, RMP (2010)

Qi and Zhang, RMP (2011)

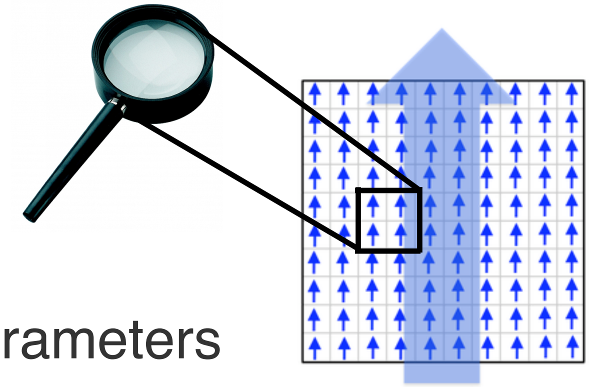
- Characteristics of topological states of matter
  - New paradigm for ordered states of matter: beyond Landau’s local order parameters
    - ➡ Nonlocal “order parameters”: Topological invariants
    - ➡ Topological order not visible by local probes



magnetization
visible locally

classification: Schnyder et al. PRB (2008); Kitaev (2009)
based on Altland and Zirnbauer, PRB (1997)

Topological States in Condensed Matter



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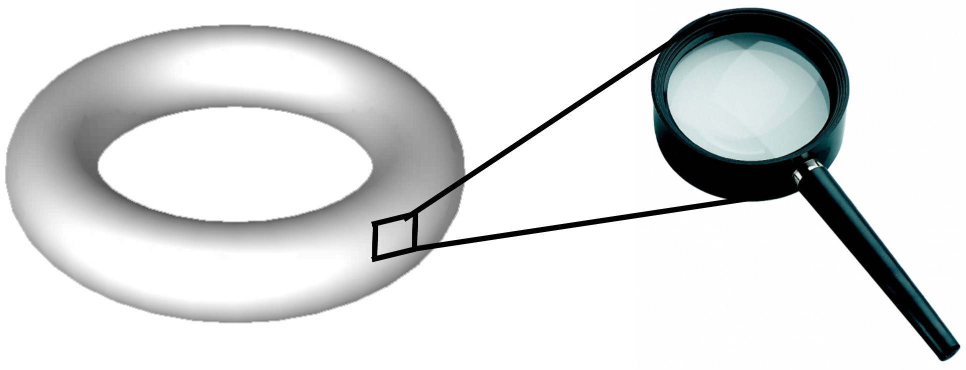

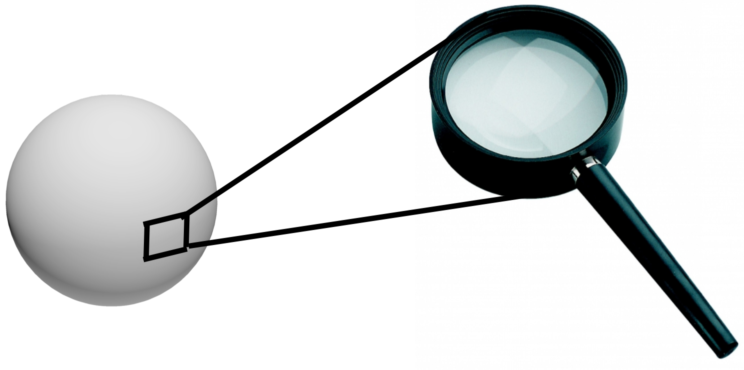

- Observable consequences: bulk-boundary correspondence
  - ➡ robust edge states (e.g. Majorana fermions)
  - ➡ may carry non-abelian exchange statistics

- Applications: topological quantum memories and computing
- Systems:
  - Quantum Hall systems
  - topological insulators/superconductors

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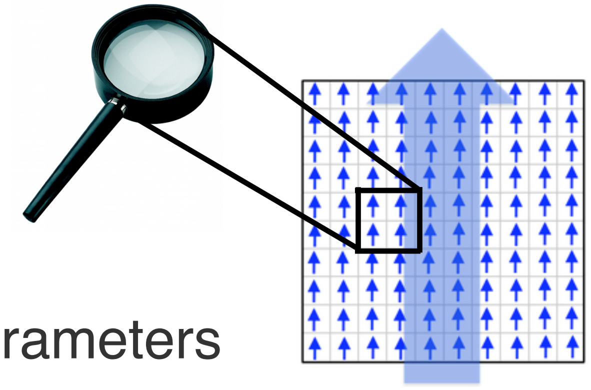

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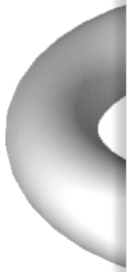

Can we prepare such states efficiently out-of-equilibrium?

Is topology uniquely tied to ground or equilibrium states?

- Observable consequences: bulk-boundary correspondence
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Majorana edge
mode

- Applications: topological quantum memories and computing

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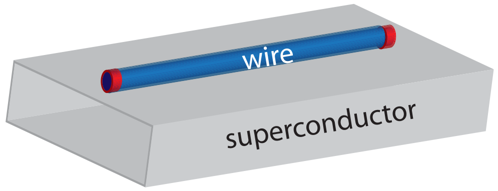
Kitaev‘s quantum wire (Hamiltonian scenario)

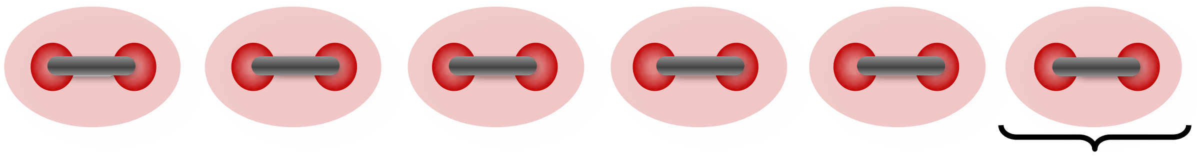
Kitaev, Physics Uspekhi (2001)

- spinless superconducting fermions on a lattice $H=\sum_i\left[-Ja_i^\dagger a_{i+1}+\Delta a_i a_{i+1}+\text{h.c.}-\mu\left(a_i^\dagger a_i-\frac{1}{2}\right)\right]$



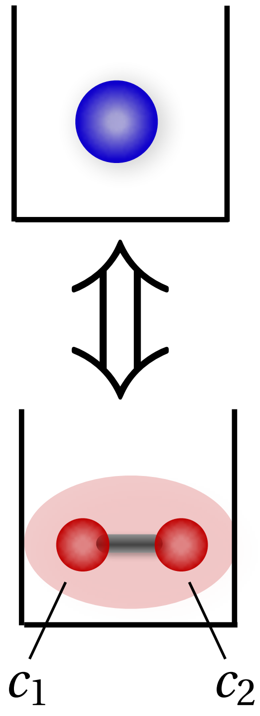
- Hamiltonian in Bogoliubov basis $H=\sum\epsilon_i\tilde{a}_i^\dagger\tilde{a}_i$ $\tilde{a}_i|G\rangle=0\,\forall\,i$
- two inequivalent representatives

$$\tilde{a}_i=a_i$$



physical site

trivial phase



complex basis

$$H=\epsilon\,a_1^\dagger a_1$$

$$a_1\equiv\frac{1}{2}(c_1+\mathrm{i}c_2)$$

Majorana (real) basis

$$H=\tfrac{\mathrm{i}}{2}\epsilon\,c_1c_2$$

fermion as onsite pairing of two Majoranas

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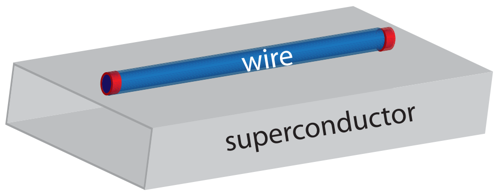
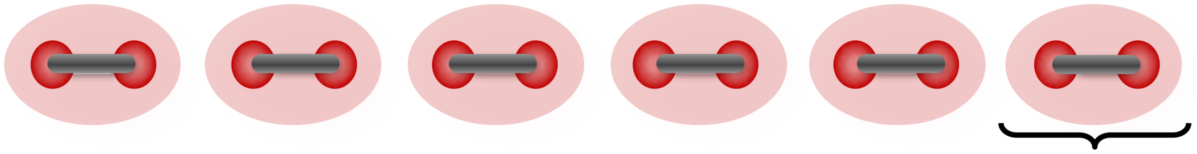


- spinless superconducting fermions on a lattice
- Hamiltonian in Bogoliubov basis $H = \sum \epsilon_i \tilde{a}_i^\dagger \tilde{a}_i$ $\qquad \tilde{a}_i|G\rangle = 0 \, \forall \, i$
- two inequivalent representatives

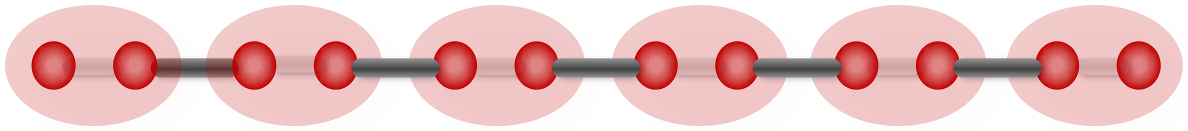
quasilocal!

$$\tilde{a}_i = a_i$$

$$\tilde{a}_i = \tfrac{1}{2}(a_{i+1} + a_{i+1}^\dagger - a_i + a_i^\dagger)$$



trivial phase



nontrivial phase

bulk

- BCS p-wave superfluid in ground state
- gapped spectrum

edge

- unpaired zero energy Majorana edge modes, or
- non-local Bogoliubov fermion

Dissipative Majorana quantum wire

- reconsider simplemost Lindblad operators and dark state condition:

$$L_i = (a_i^\dagger + a_{i+1}^\dagger)(a_i - a_{i+1}) \equiv C_i^\dagger A_i \qquad L_i|D\rangle = 0 \; \forall \, i$$

- main insight:
  - $a_i^\dagger$ boson creation => $|D\rangle = |\mathrm{BEC}, N\rangle$ fixed number BEC dark state
  - $a_i^\dagger$ fermion creation => $|D\rangle = |\mathrm{BCS}, N\rangle$ fixed number BCS pair dark state

- this example: $|\mathrm{BCS}, N\rangle = |\mathrm{Kitaev}, N\rangle$

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- connection to Kitaev model: emergent eigenoperators in thermodynamic limit

SD, E. Rico, M. A. Baranov, P. Zoller, Nat. Phys. (2011)

fixed number

fixed phase

$$L_i = \underbrace{(a_i^\dagger + a_{i+1}^\dagger)}_{}\underbrace{(a_i - a_{i+1})}_{} \xrightarrow[\hat{=}\text{“low energies”}]{\text{long times}} \ell_i = \underbrace{(a_i^\dagger + a_{i+1}^\dagger}_{} + \underbrace{a_i - a_{i+1})}_{}$$

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$$\propto \tilde{a}_i$$

Kitaev’s Majorana operators

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$$|D\rangle = |\mathrm{Kitaev}, \theta\rangle \qquad\qquad \propto \tilde{a}_i$$

standard Kitaev p-wave superfluid state

Kitaev's Majorana operators

Mechanism: Fixed number vs. fixed phase Lindblad operators

- spinless fermions for simplicity

SD, W. Yi. A. Daley, P. Zoller, PRL (2010)

- fixed number Lindblad operators

$$L_i \quad = \quad C_i^\dagger A_i$$

- fixed phase Lindblad operators

$$\ell_i = C_i^\dagger + r e^{\mathrm{i}\theta} A_i$$

$$\Longleftrightarrow$$

- resulting dark state

$$|BCS, N\rangle \quad = \quad G^{\dagger\,N} |\mathrm{vac}\rangle$$

- resulting dark state (with $\Delta N \sim 1/\sqrt{N}$)

$$|BCS, \theta\rangle \quad = \quad \exp(r e^{\mathrm{i}\theta} G^\dagger) |\mathrm{vac}\rangle$$

- requirements

translation invariant creation and annihilation part

$$C_i^\dagger = \sum_j v_{i-j} a_j^\dagger \qquad C_k^\dagger = v_k a_k^\dagger$$

$$A_i = \sum_j u_{i-j} a_j \qquad A_k = u_k a_k$$

antisymmetry

$$\varphi_k = \frac{v_k}{u_k} = -\varphi_{-k}$$

$$G^\dagger \quad = \quad \sum_k \varphi_k c_{-k}^\dagger c_k^\dagger$$

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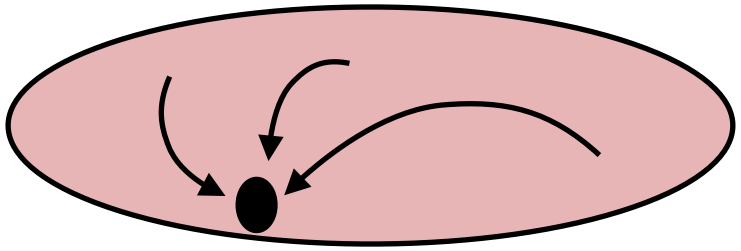
Topological field theory far from equilibrium



$$\frac{\theta}{4\pi}\int A_+\,dA_+ - A_-\,dA_-$$

microphysics

macrophysics

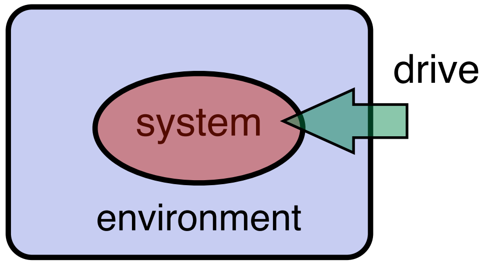
F. Tonielli, J. Budich, A. Altland, SD
PRL (2020)

Motivation: Quantum states vs. quantum dynamics

- quite general quantum evolution:

Lindblad operators

$$\partial_t \hat{\rho} = \underbrace{-i[\hat{H}, \hat{\rho}]}_{\text{coherent evolution}} + \kappa \underbrace{\sum_\alpha (\hat{L}_\alpha \hat{\rho} \hat{L}_\alpha^\dagger - \frac{1}{2}\{\hat{L}_\alpha^\dagger \hat{L}_\alpha, \hat{\rho}\})}_{\text{driven-dissipative evolution}}$$



- simple example: damped harmonic oscillator (quantum cavity)

$$H = \omega_0 \, a^\dagger a \qquad\qquad L = a$$

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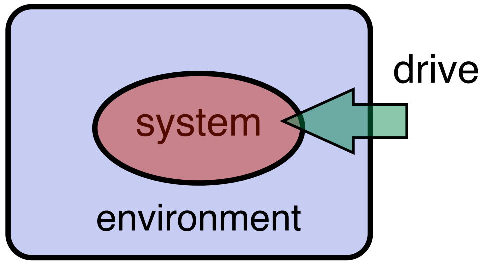

- simple example: damped harmonic oscillator (quantum cavity)

$$\begin{tikzcd}
 & H = \omega_0 \, a^\dagger a \arrow[dl, "\kappa = 0"'] & L = a \arrow[dr, "\omega_0 = 0"] & \\
\partial_t \rho = -i[H, \rho] & & & \partial_t \rho = \kappa (a \rho a^\dagger - \tfrac{1}{2}\{a^\dagger a, \rho\})
\end{tikzcd}$$

evolution

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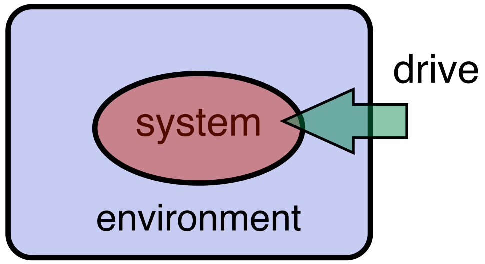

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$$\kappa = 0 \qquad\qquad \omega_0 = 0$$

evolution

$$\partial_t \rho = -i[H, \rho] \qquad\qquad \partial_t \rho = \kappa(a\rho a^\dagger - \tfrac{1}{2}\{a^\dagger a, \rho\})$$

stationary state

$$\rho_{\mathrm{eq}} = e^{-\beta H} \qquad\qquad \rho_{\mathrm{neq}}$$

$$\lim_{\beta\to\infty} e^{-\beta H} = |0\rangle\langle 0| \quad \xleftrightarrow{\text{coincide!}} \quad \rho_{\mathrm{neq}} = |0\rangle\langle 0|$$

ground state

dark state $L_i|D\rangle = 0 \,\forall\, i$

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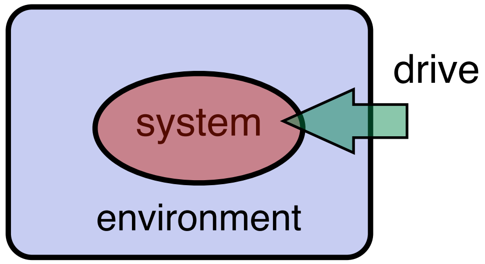
$$\partial_t\hat{\rho} = \underbrace{-i[\hat{H},\hat{\rho}]}_{\text{coherent evolution}} + \underbrace{\kappa\sum_\alpha\left(\hat{L}_\alpha\hat{\rho}\hat{L}_\alpha^\dagger - \tfrac{1}{2}\{\hat{L}_\alpha^\dagger\hat{L}_\alpha,\hat{\rho}\}\right)}_{\text{driven-dissipative evolution}}$$



- simple example: damped harmonic oscillator (quantum cavity)

evolution

$$\begin{tikzcd} H = \omega_0\, a^\dagger a \arrow[d, "\kappa = 0"'] & L = a \arrow[d, "\omega_0 = 0"] \\ \partial_t\rho = -i[H,\rho] & \partial_t\rho = \kappa(a\rho a^\dagger - \tfrac{1}{2}\{a^\dagger a,\rho\}) \end{tikzcd}$$

stationary state

$$\rho_{\mathrm{eq}} = e^{-\beta H} \qquad\qquad \rho_{\mathrm{neq}}$$

$$\lim_{\beta\to\infty} e^{-\beta H} = |0\rangle\langle 0| \xleftrightarrow{\text{coincide!}} \rho_{\mathrm{neq}} = |0\rangle\langle 0|$$

ground state

dark state $L_i|D\rangle = 0\,\forall\, i$

dynamical response to $j^\dagger a + a^\dagger j$

$$\theta(t)e^{-i\omega_0 t} \xleftrightarrow{\text{qualitatively different!}} \theta(t)e^{-\gamma t}$$

reversible, equilibrium

irreversible, non-equilibrium

Motivation: Quantum states vs. quantum dynamics

- quite general quantum evolution: eliminate environment

Lindblad operators

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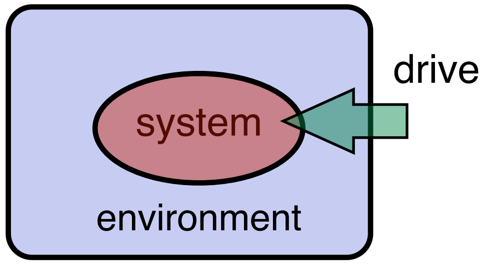
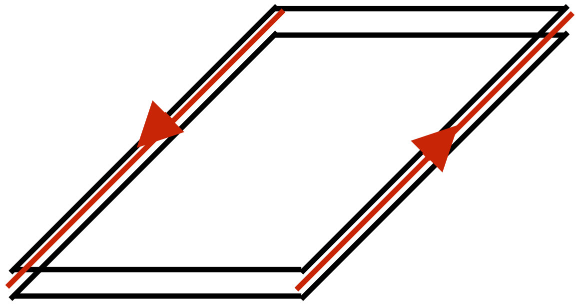


- enter topology:
  - topology is encoded in the state (wave function)
  - observables are oftentimes dynamical responses (to gauge fields)



- example: Chern insulator (e.g. quantum Hall)

cf., for single particle problems: Albert, Bradlyn, Fraas, Jiang, PRX (2016)
following Avron, Fraas, Graf, J. Stat. Phys. (2012); Avron, Fraas, Graf, Kenneth, New J. Phys. (2010)

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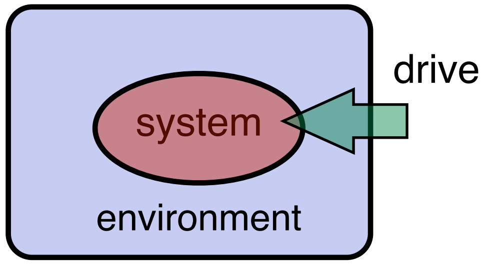

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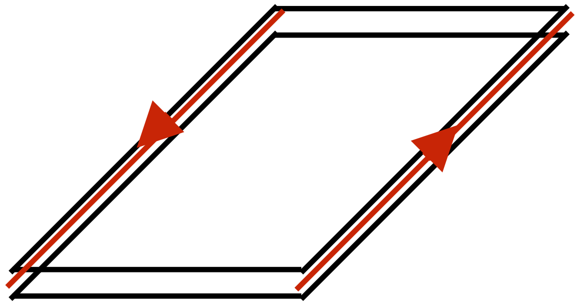

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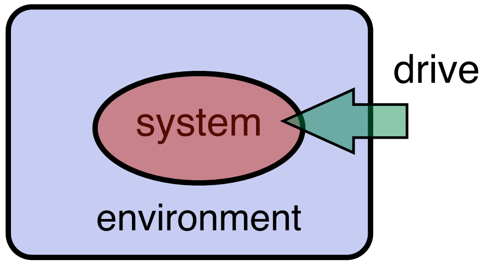

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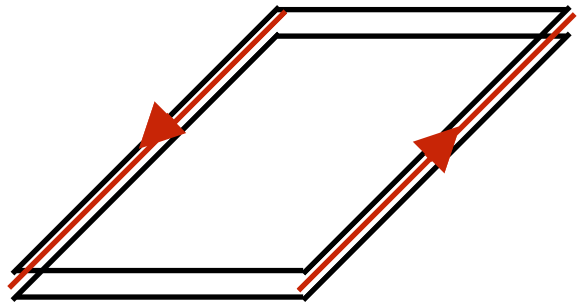

Questions:
Is there a topological response in irreversible out of equilibrium dynamics?
Are there chiral edge modes (reversible) on top of a dissipative bulk (irreversible)?

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dissipative topological superconductors: SD, Rico, Baranov, Zoller, Nat. Phys. (2011),
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Different dynamics

Hamiltonian scenario

State

- unique ground state

$$H|D\rangle = 0$$

Lindblad scenario

- unique dark state

$$L_\alpha|D\rangle = 0 \quad \forall\alpha$$

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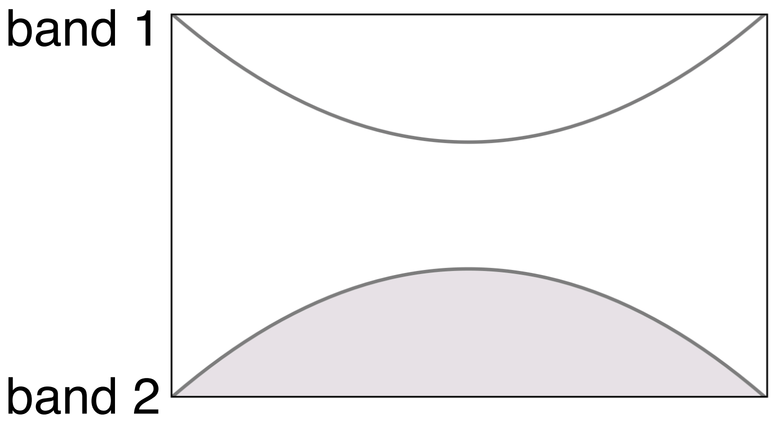

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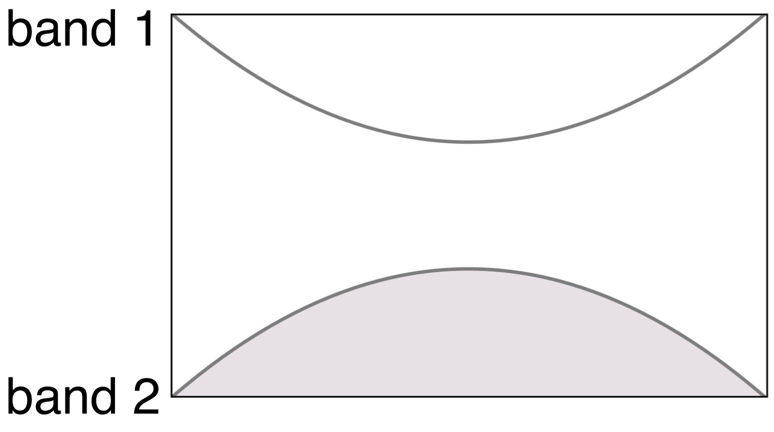

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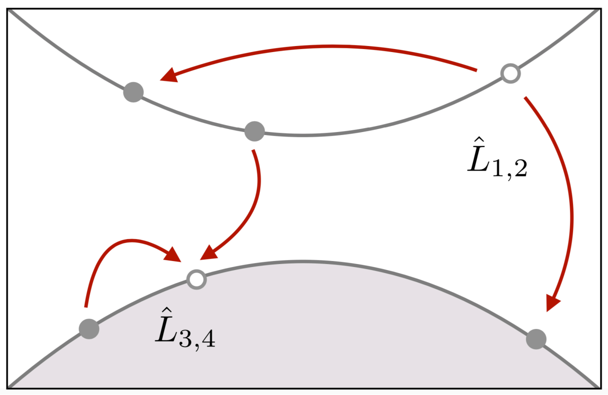

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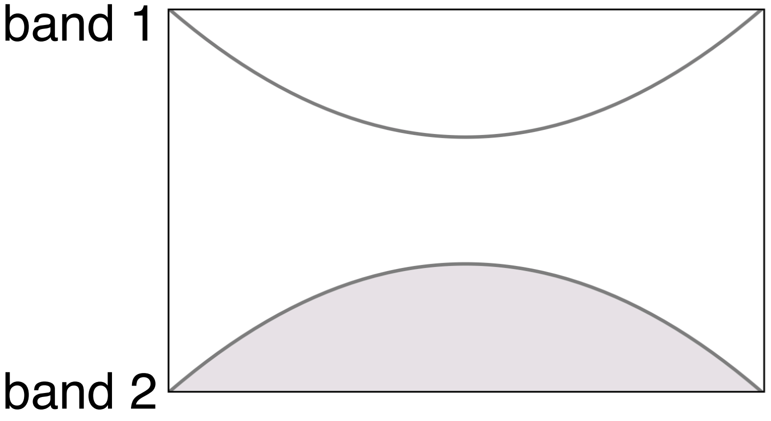

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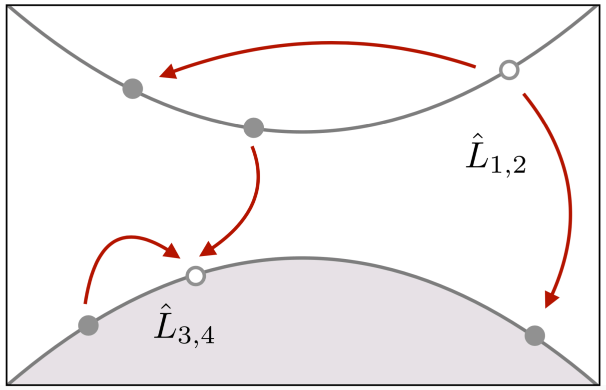

Dynamics

- (micro) reversible / unitary
- thermal equilibrium (detailed balance)

- irreversible / dissipative
- far from equilibrium: detailed balance violated

Sieberer, Chiocchetta, Taeuber, Gambassi, SD, PRB (2015)

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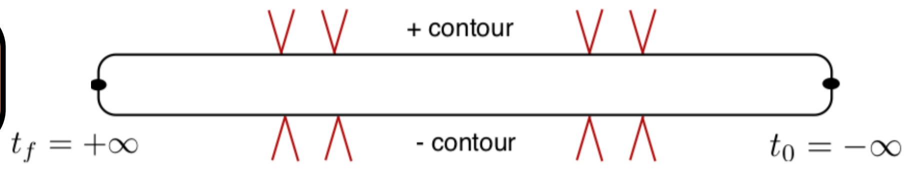
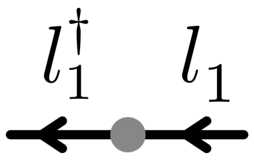
Analysis of dark state Lindbladian

Many-Body Master Equation → Keldysh functional integral



- strongly interacting problem, but dark state exactly known: self-consistent Born mean field approximation

$$(L^\dagger_\alpha L_\alpha)_{\mathrm{mf}} =$$



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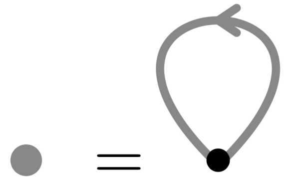

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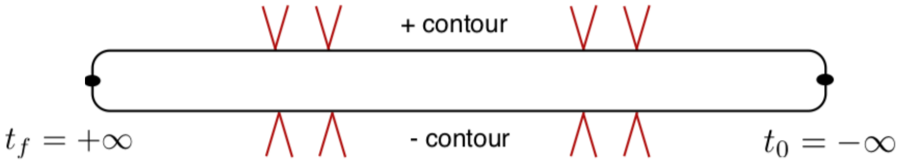

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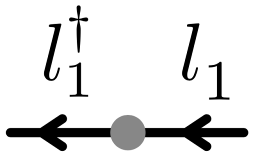

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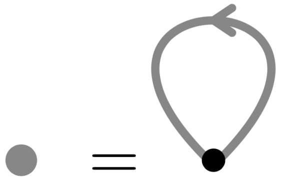

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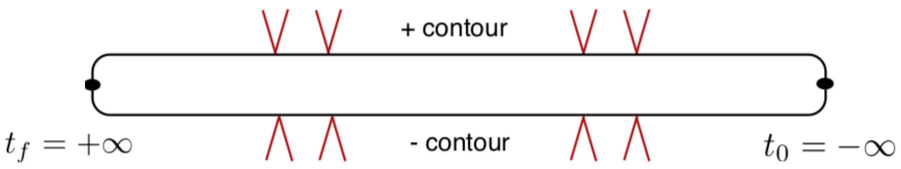

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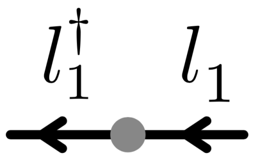

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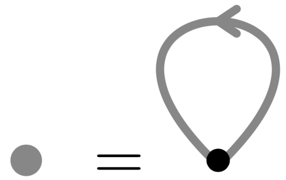

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- robustness: emergent dissipative single-particle dissipative gap: $\boxed{\gamma_{\mathbf{q}=0} = \bar{\gamma} m^2}$
- after mean field approximation: particle number conservation masked / quantum symmetry spont. broken (cf. Halperin-Hohenberg models with conserved charge: no quantum symmetry)

➡ Interpretation: system acts as its own bath

➡ Quantization of the response? How does the system remember its number conserving origin?

Microscopic gauge-matter action: minimal coupling and response

- resolution: full functional integral knows about number conservation, lost only by approximation
  - ➡ first couple to the gauge field, then do the approximations

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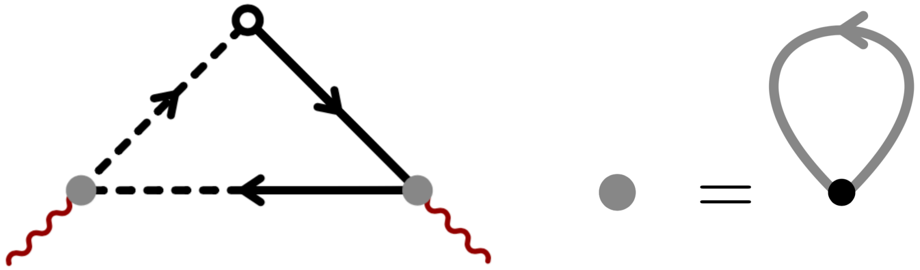

- resulting theory:

$$Z = \int D[\psi^*,\psi] e^{iS_\mathrm{f}[\psi^*,\psi,A]} \underset{\substack{\uparrow \\ \text{self-consistent Born}}}{\approx} \int D[\psi^*,\psi] e^{iS_\mathrm{SCB}[\psi^*,\psi,A]} \underset{\substack{\uparrow \\ \text{Gaussian integration}}}{=} e^{\frac{i}{2}\mathrm{tr}\log[G^{-1}[A]]} \equiv e^{iS[A]}$$

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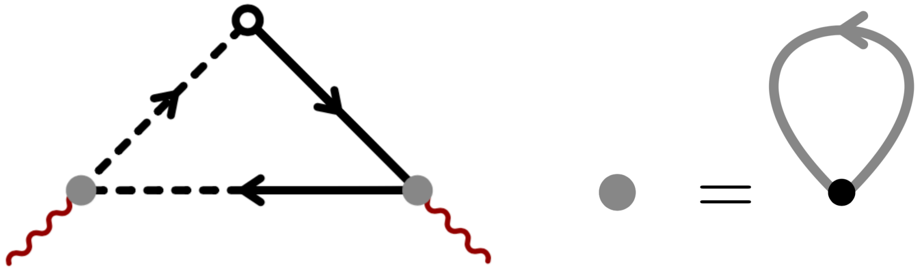

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- Electromagnetic response: all possible contributions in 1-loop approximation

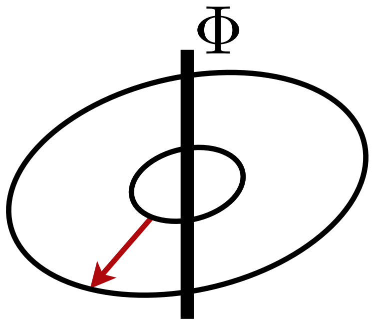
$$S[A_c,A_q] = \int \left[\frac{\theta}{4\pi}\epsilon^{\mu\nu\rho}\left(A_{\mu,c}\partial_\nu A_{\rho,q} \quad + A_{\mu,q}\partial_\nu A_{\rho,c}\right) + \lambda_1 B_c B_q + i\lambda_2 B_q^2\right] \qquad B_I = \epsilon_{ij}\partial_i A_{j,I}$$

  - ➡ $\theta = -1$ quantised
  - ➡ subleading Maxwell-like terms

Complementary symmetry analysis

Laughlin, PRB (1981)

Key ingredients: Laughlin argument



physically:

- adiabaticity
- no particles lost on the way
- pure states (but see outlook)

technically:

- many-body damping gap (replaces Hamiltonian gap)
- $U_c(1) \times U_q(1)$ ‘strong’ invariance (number conservation)
- large gauge invariance (number quantization)

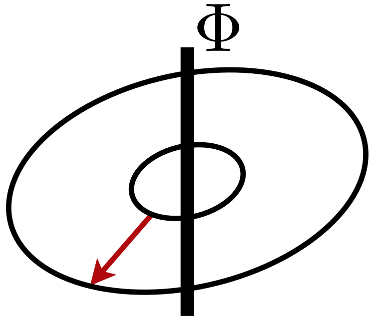
➡ universally in- and out of equilibrium

Complementary symmetry analysis


microphysics

macrophysics

Laughlin, PRB (1981)



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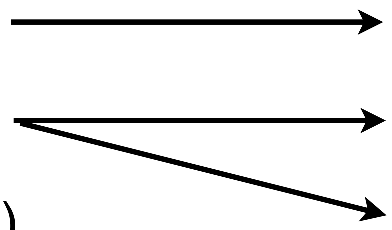

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Implications:

- standard Chern-Simons action emerges (dissipative corrections subleading in derivative expansion)

$$\boxed{S_{\mathrm{CS}}[A] = \frac{\theta}{4\pi} \int A_+\, dA_+ - A_-\, dA_-}$$

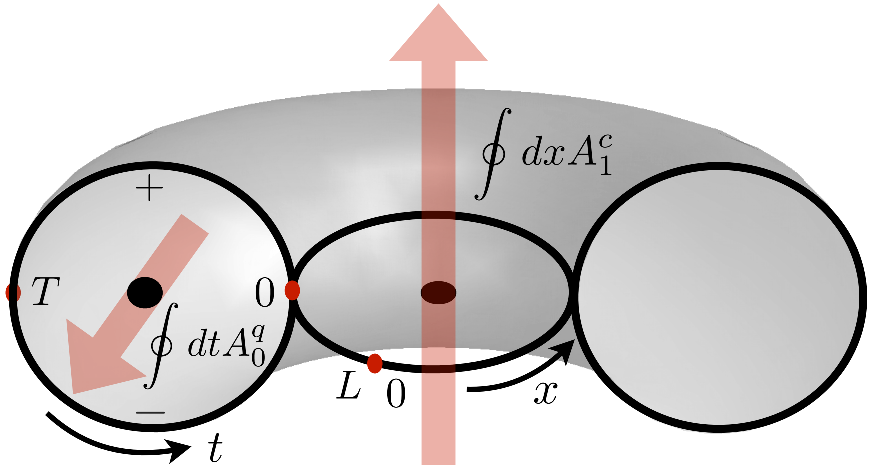
➡ contour decoupled Hamiltonian structure w/ quantized coefficient on top of purely dissipative bulk

- bulk-boundary correspondence: sharp chiral edge models (subleading width)

$$\omega = vq + iDq^2$$

v determined by Lindblad parameters

Topological field theory for mixed quantum states



Z. Huang, X. Sun, SD, PRB (2022)

Prerequisite: spectral and purity gaps

- setting: quadratic (lattice) fermion action (e.g. after mean field decoupling)

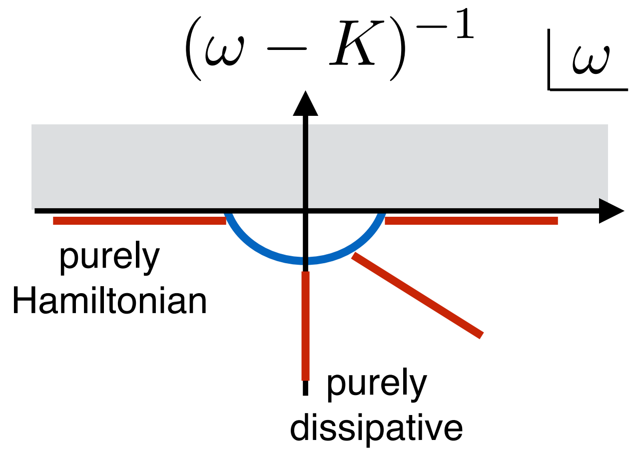
$$S = \int_t (\bar{\psi}_c, \bar{\psi}_q) \begin{pmatrix} 0 & \omega - K^\dagger \\ \omega - K & 2P \end{pmatrix} \begin{pmatrix} \psi_c \\ \psi_q \end{pmatrix}$$

- robustness of classification scheme: 2 gaps
  - spectral gap: no zero modes of spectral matrix $K = H - iD$
  - purity gap (fermions): no zero modes of Hermitian covariance matrix

$$\Gamma_{ab} = \langle [\hat{\psi}_a, \hat{\psi}_b^\dagger] \rangle = -2i \int \frac{d\omega}{2\pi} \Big(\frac{1}{\omega - K} P \frac{1}{\omega - K^\dagger} \Big)_{ab}$$

static single-particle correlations

poles of retarded Green's function

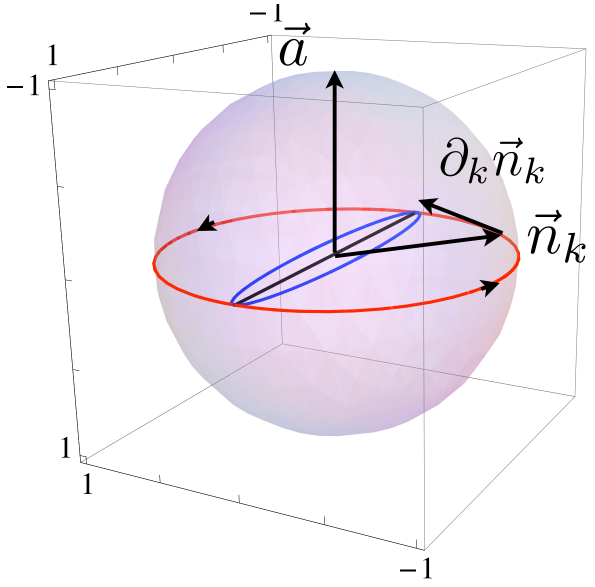


- intuition of purity gap: two-band chiral model in 1D (SSH)

$|\vec{n}_k| = 1$ pure state

$$\Gamma_k = \vec{n}_k \cdot \vec{\sigma} \qquad |\vec{n}_k| \le 1$$

$|\vec{n}_{k_*}| = 0$ purity gap closing



➡ chiral winding ceases to be well defined once there is a purity gap closing

➡ concept well defined in- and out of equilibrium

Prerequisite: spectral and purity gaps

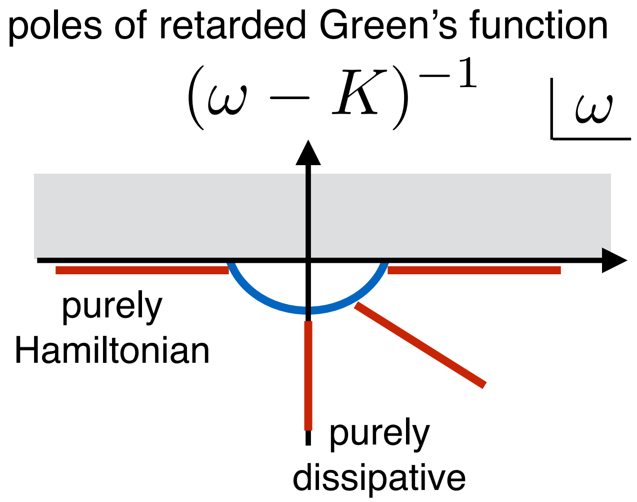
- setting: quadratic (lattice) fermion action (e.g. after mean field decoupling)

$$S = \int_t (\bar{\psi}_c, \bar{\psi}_q) \begin{pmatrix} 0 & \omega - K^\dagger \\ \omega - K & 2P \end{pmatrix} \begin{pmatrix} \psi_c \\ \psi_q \end{pmatrix}$$

- robustness of classification scheme: 2 gaps
  - spectral gap: no zero modes of spectral matrix $K = H - iD$
  - purity gap (fermions): no zero modes of Hermitian covariance matrix

$$\Gamma_{ab} = \langle [\hat{\psi}_a, \hat{\psi}_b^\dagger] \rangle = -2i \int \frac{d\omega}{2\pi} \left(\frac{1}{\omega - K} P \frac{1}{\omega - K^\dagger} \right)_{ab}$$

static single-particle correlations

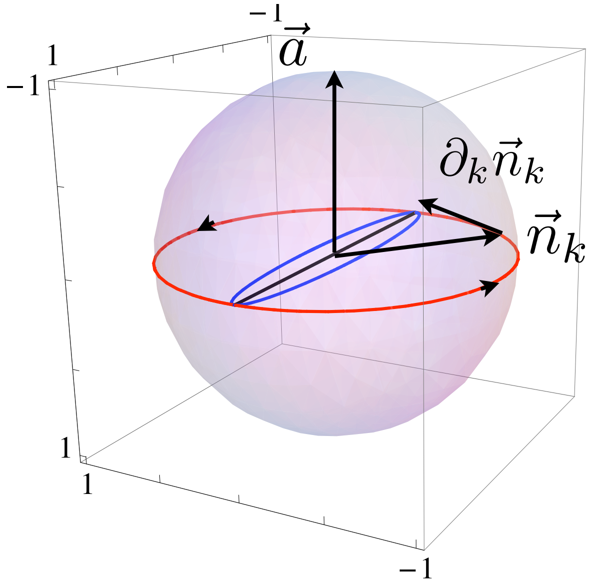


- intuition of purity gap: two-band chiral model in 1D (SSH)

$$\Gamma_k = \vec{n}_k \cdot \vec{\sigma} \qquad |\vec{n}_k| \leq 1$$

$|\vec{n}_k| = 1$ pure state

$|\vec{n}_{k_*}| = 0$ purity gap closing



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Z. Huang, X. Sun, SD, PRB (2022)

Topological field theory for mixed states: approach

- assume spectral and dissipative gap => restrict attention to density matrix

- approximate density matrix by Dirac stationary state ('modular Hamiltonian': H not necessarily generator of dynamics; equilibrium: $G = \beta H$)

$$\hat{\rho} = e^{-\hat{G}} \qquad \hat{G} = \int_{\mathbf{x}} \hat{\psi}^{\dagger}(\mathbf{x}) G \hat{\psi}(\mathbf{x}) \qquad G = \sum_{i=1}^{n} \partial_i \alpha^i + m\alpha^{2n+1} \qquad \{\alpha^i, \alpha^j\} = 2\delta_{ij}$$

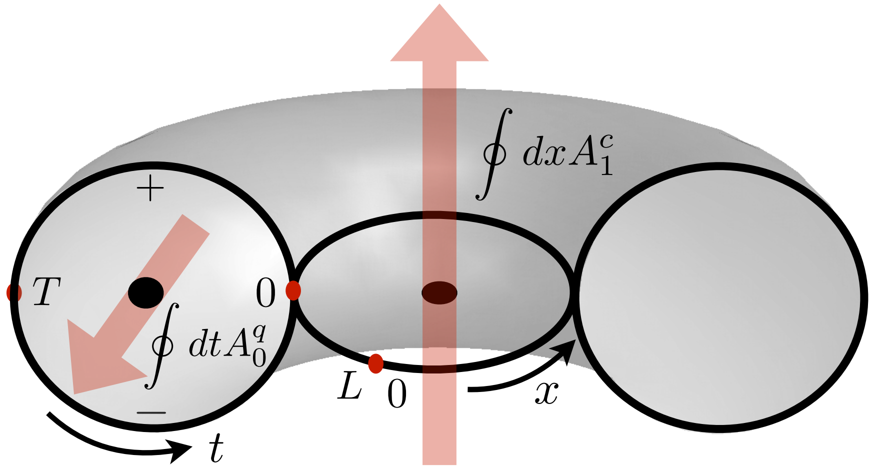
- spatial gauge fields by minimal coupling of spatial derivatives:

$$\partial_i \to \partial_i + iA_i^c(t, \mathbf{x})$$

➡ thread flux in spatial directions

- temporal gauge fields by minimal coupling of time derivative:

$$\partial_t \hat{\rho} = -i(\hat{A}_0^+ \hat{\rho} - \hat{\rho}\hat{A}_0^-) + \hat{\mathcal{L}}[\hat{\rho}] \qquad \hat{A}_0^{\pm} = \int_{\mathbf{x}} \hat{\psi}^{\dagger}(\mathbf{x}) A_0^{\pm}(t, \mathbf{x}) \hat{\psi}(\mathbf{x})$$



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$$\oint dt \hat{A}_0^q \equiv \int_0^T dt(\hat{A}_0^+(t) - \hat{A}_0^-(t))$$

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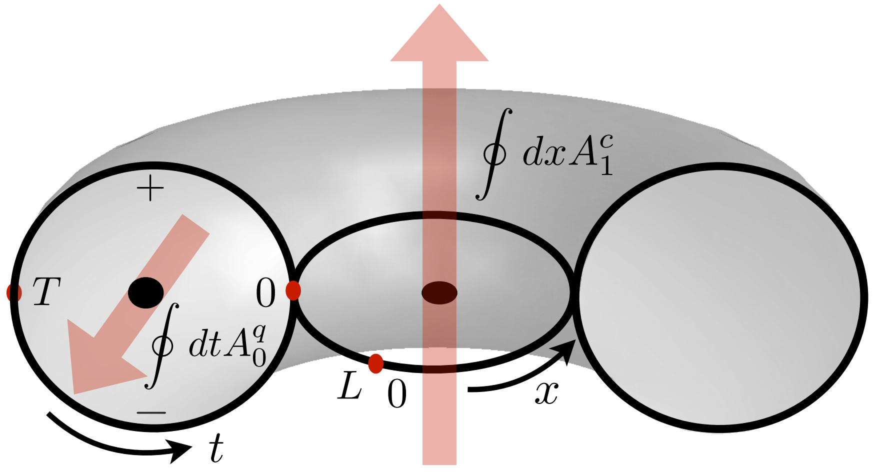

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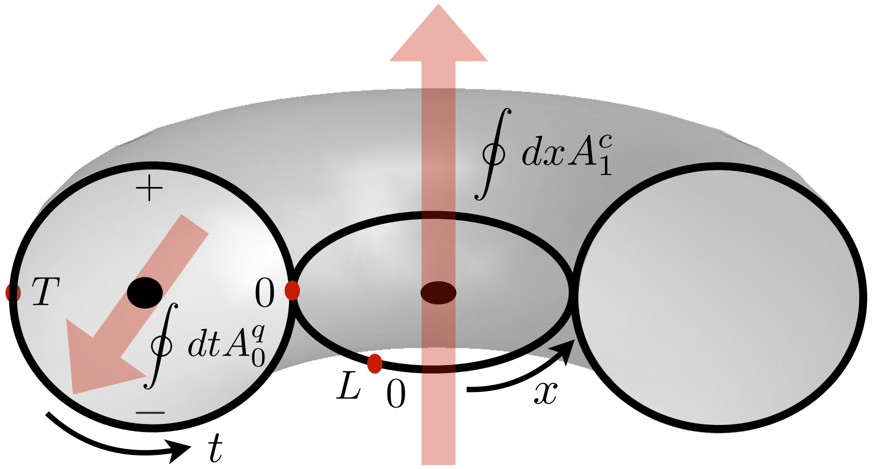

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Topological field theory for mixed states: toy example

- topological field theory in 0+1 dimensions: single fermion mode

$$\hat{H} = m\hat{\psi}^\dagger\hat{\psi} \qquad \text{stationary state} \qquad \hat{\rho} = e^{-\beta\hat{H}} \qquad \beta = 1/T$$

- gauge field: minimal coupling, only 0-component $a_0 = \oint dt A_0^q$

- partition function and effective action can be computed exactly

$$Z[A_0^q] = e^{iS[A_0^q]} \qquad \boxed{S[A_0^q] = -\frac{1}{2}a_0 + \frac{\text{sign}(m)}{2}\text{Re}\{-2i\ln[\cos(\frac{a_0}{2}) + i\tanh(\frac{\beta|m|}{2})\sin(\frac{a_0}{2})]\}}$$

➡ reduces to (0+1) Chern-Simons term at T =0:

$$S[A_0^q] = \frac{1}{2}[-1 + \text{sign}(m)] \oint A_0^q(t).$$

➡ but preserves topology for any T: large U(1) gauge transformation produces quantized coefficient

$$a_0 \to a_0 + 2\pi \implies S|_{a_0}^{a_0+2\pi} = 2\pi\frac{1}{2}[-1 + \text{sign}(m))] \in 2\pi\mathbb{Z}$$

- generalisation to 2n+1 dimensions: Dirac Hamiltonian

$$\hat{H} = \int d^{2n}x\hat{\psi}^\dagger(\mathbf{x})H\hat{\psi}(\mathbf{x}) \qquad H = \sum_{i=1}^{n}\partial_i\alpha^i + m\alpha^{2n+1} \qquad \{\alpha^i, \alpha^j\} = 2\delta_{ij} \qquad \hat{\rho} = e^{-\beta\hat{H}}$$

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Topological field theory for mixed states

- starting point Dirac stationary theory in arbitrary dimension (eq./neq.)
- compute the full topological contribution to the real time electromagnetic response (e.g. odd spacetime dim.)

temporal gauge field dependence

$$a_0 = \oint dt A_0^q$$

only dim.less ratio:
purity gap

spatial gauge field dependence

$$C_{2n}[A_i] = \frac{1}{(2\pi)^n n!}\epsilon^{0\mu_1\mu_2\cdots}\partial_{\mu_1}A_{\mu_2}...\partial_{\mu_{2n-1}}A_{\mu_{2n}}$$

Chern character density

$$S = \underbrace{\mathrm{ch}}_{\substack{\text{Chern number}\\ \text{(quantized)}}}\frac{1}{T}\int dt d^{2n}\boldsymbol{x}\,\underbrace{\mathcal{I}[a_0,\beta|m|]}_{\text{<-> mixedness of state}}\,\underbrace{\mathcal{C}^0_{(2n)c}[A_i]}_{\text{<-> external magnetic fields}}+$$

(terms from current conservation)

<-> topology of state

$$\mathcal{I}(a_0,\beta|m|) = -2i\ln[\cos(\frac{a_0}{2}) + i\tanh(\frac{\beta|m|}{2})\sin(\frac{a_0}{2})]$$

- non-linear in temporal gauge field
  - ➡ linear response observables non-quantized (for mixed states)
- reduction to standard result for pure states $\lim_{\beta\to\infty}\mathcal{I}[a_0,\beta|m|] = a_0$
  - ➡ linear response observables quantized for pure states (ground or not)
- number quantization: large U(1) invariance

$$\mathcal{I}[a_0+2\pi,\beta|m|] = \mathcal{I}[a_0,\beta|m|] + 2\pi$$

➡ non-linear response observables can be quantized

generalizes ‘ensemble geometric phase’
Bardyn, Wawer, Altland, Fleischauer, SD PRX (2017)

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Chern number (quantized)

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Z. Huang, SD, arxiv (2024)

Observable topology in mixed quantum states

- physical interpretation: ensemble geometric phase

$$S[a_0(x) = \tfrac{2\pi}{L}x] = \operatorname{Im}\ln\operatorname{Tr}\left(\hat{\rho}e^{i\frac{2\pi}{L}\hat{X}}\right)$$

position operator

$$\hat{X} = \sum_i x_i \hat{c}_i^\dagger \hat{c}_i$$

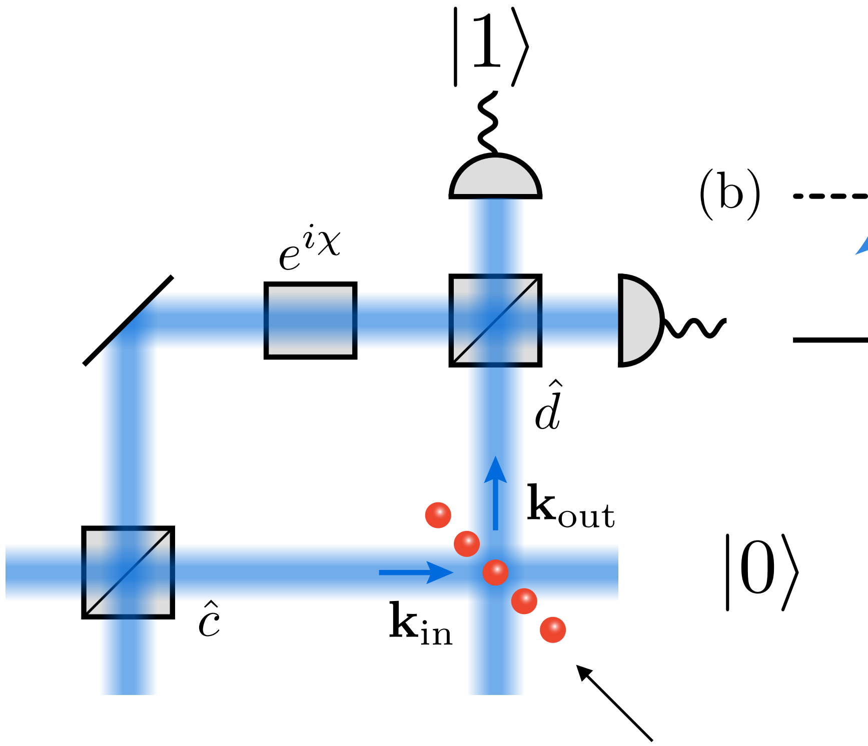
(1+1 dimensions)

- detection: Mach-Zehnder interferometer

Sjoeqvist et al., PRL (2000)

$$\mathcal{U} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes U + \begin{pmatrix} e^{i\chi} & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{1} \qquad U = e^{i\frac{2\pi}{L}\hat{X}}$$

interferometer fermion system



1D fermion system

$$\delta k = \frac{2\pi}{L} = \mathbf{e}_z(\mathbf{k}_{\mathrm{in}} - \mathbf{k}_{\mathrm{out}})$$

  - signal e.g. in interferometer arm $|0\rangle$

$$\operatorname{tr}[e^{-i\chi}\rho U + h.c.] \sim \cos[\chi - \arg\langle e^{i\delta k \hat{X}}\rangle]$$

  - mesoscopic setup for reasonable visibility (N ~ 40)

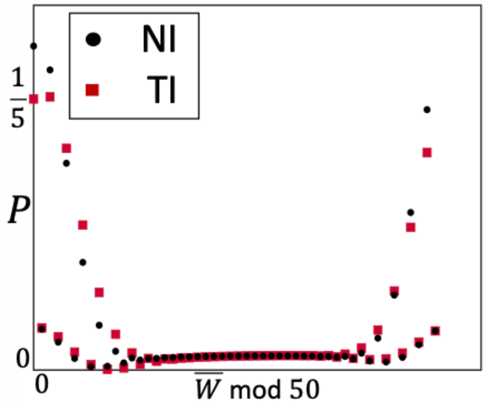
- alternative: full counting statistics of position operator

Z. Huang, SD, arxiv (2024)

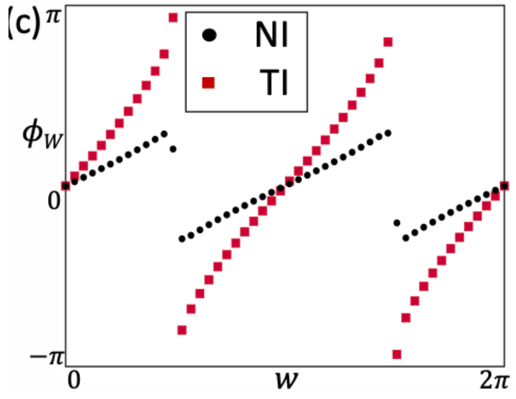
$$P(x) = \operatorname{Tr}[\hat{\rho}\delta(x - \hat{X})] = \int dq e^{-iqx}\operatorname{tr}[\hat{\rho}e^{iq\hat{X}}]$$

—> Fourier component $q = \frac{2\pi}{L}$

real space



Fourier space



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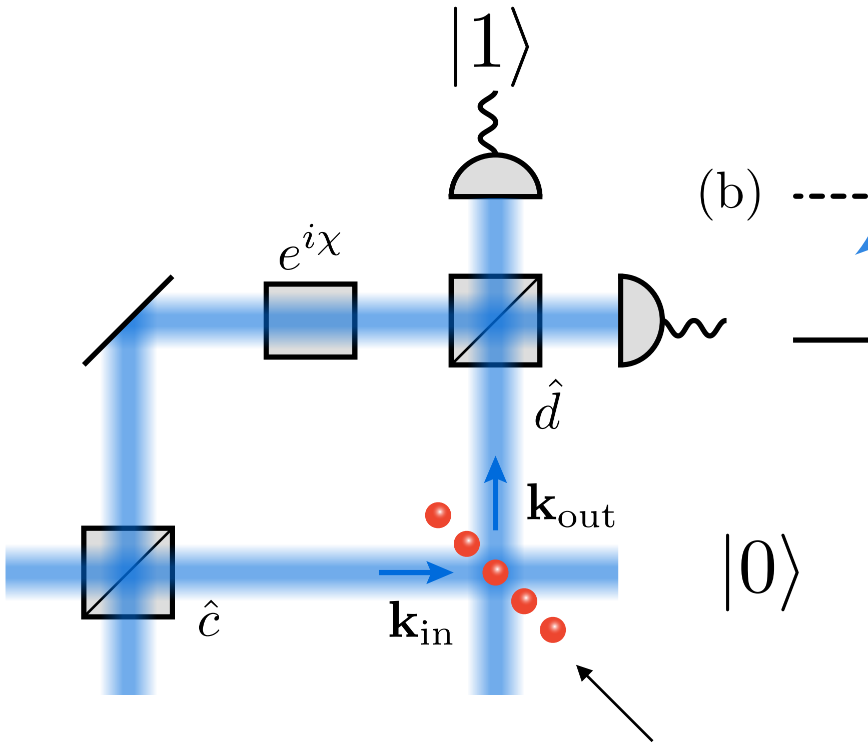

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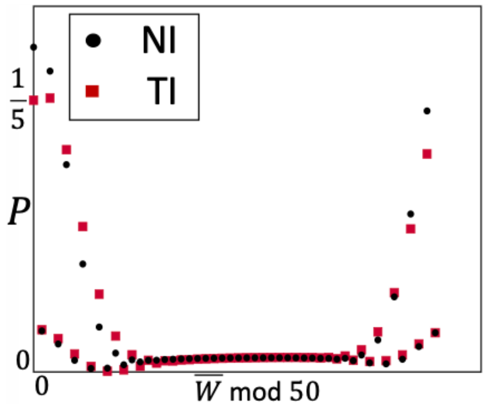

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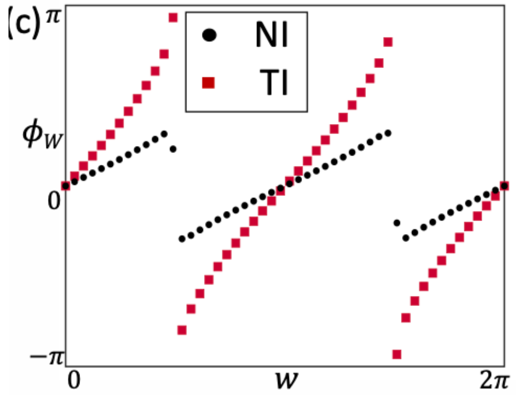
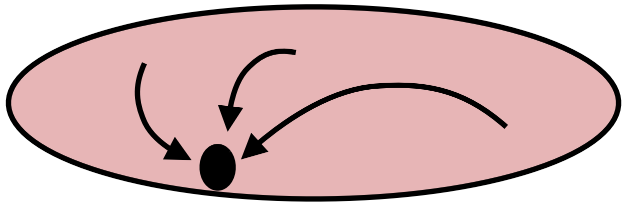
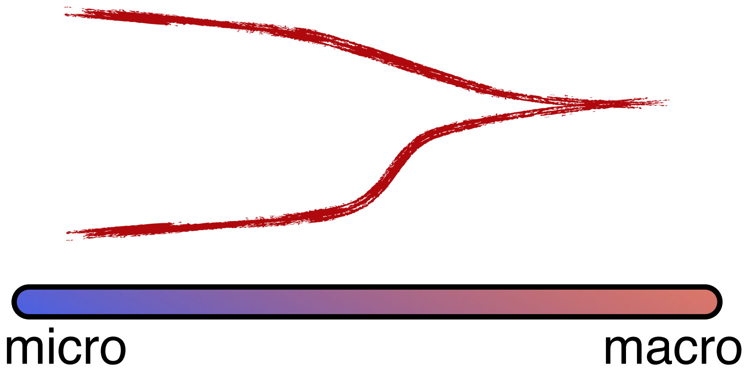


Summary & perspective: Topology out-of-equilibrium

- topological states can be prepared by targeted non-equilibrium cooling

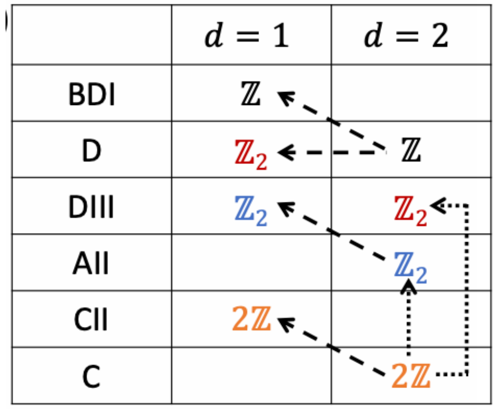


- topology provides robust ordering principle
  - ➡ ‘topology beats dynamics’: pure eq. vs. neq. states feature Chern-Simons response
  - ➡ ‘topology beats mixedness’: non-linear quantised observables exist for stationary state in- and out-of-equilibrium



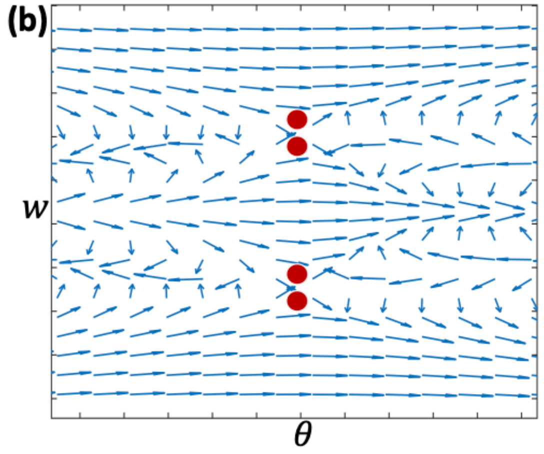
- quantised topological order parameters for entire periodic table

Z. Huang, SD, arxiv (2024)



	$d = 1$	$d = 2$
BDI	$\mathbb{Z}$	
D	$\mathbb{Z}_2$	$\mathbb{Z}$
DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AII		$\mathbb{Z}_2$
CII	$2\mathbb{Z}$	
C		$2\mathbb{Z}$

- novel entropy driven topological phase transitions in mixed quantum states?



- connection to error thresholds in stabilizer codes?