NONEQUILIBRIUM QUANTUM LATTICE SYSTEMS

(+ EMERGENCE OF KPZ SCALING IN INTEGRABLE SPIN CHAINS)

Les Houches, April 2024

Lecture I: Integrable spin chains and Quantum circuits

Lecture II: KPZ scaling in XXX circuits and Heisenberg spin chain

Lecture III: The classical limit and superuniversality of superdiffusion

Lecture IV: Full counting statistics (in integrable dynamics)

Lecture I: Integrable spin chains and Quantum circuits

Dual methodological aspects:

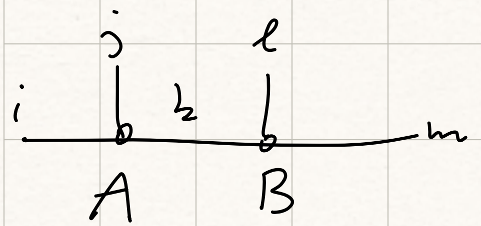
i) Quantum integrability in discrete time

ii) Boundary driven integrable chains and nonequilibrium steady states: noncompact transfer matrices, quasilocal conservation laws, ideal transport

I1) Preliminaries, notation:

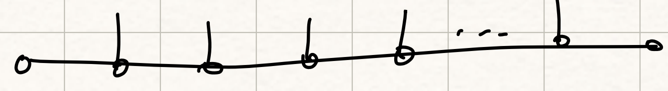
tensor networks: tensor $A^{\,n}_{ij\ldots k}$, $i,j\ldots \in \{1,\ldots,d\}$

$$A \in (\mathbb{C}^d)^{\otimes n}$$



$$= \sum_k A_{ijk} B_{klm}$$

MPS:



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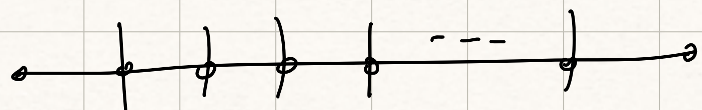
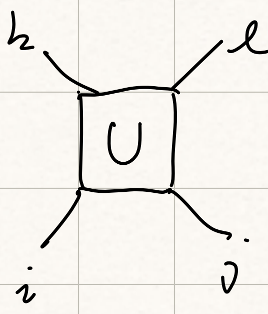


Quantum circuits as local, discrete time many-body dynamical systems

qudit: $\mathcal{H}_1 = \mathbb{C}^d$

local gate: $U \in \mathrm{End}(\mathcal{H}_1 \otimes \mathcal{H}_1)$

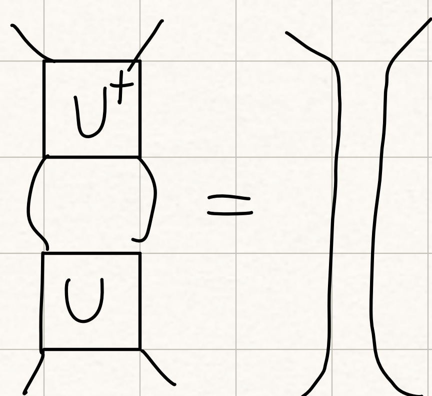


$\equiv U^{kl}_{ij}$ $\qquad U = \sum_{ijkl} U^{kl}_{ij} \, |kl\rangle\langle ij|$

$U|ij\rangle = \sum_{k,l} U^{kl}_{ij} \, |kl\rangle$

U is unitary:

$UU^\dagger = \mathbb{1}$



Brickwork circuit

string of qudits / qudit chain $\mathcal{H}_L = \mathcal{H}_1^{\otimes L}$

L <u>even</u> (for convenience)

$\mathcal{U}_e = U^{\otimes \frac{L}{2}} \in \mathrm{End}(\mathcal{H}_L)$ 4

Shift: $\Pi \in \mathrm{End}(\mathcal{H}_L)$

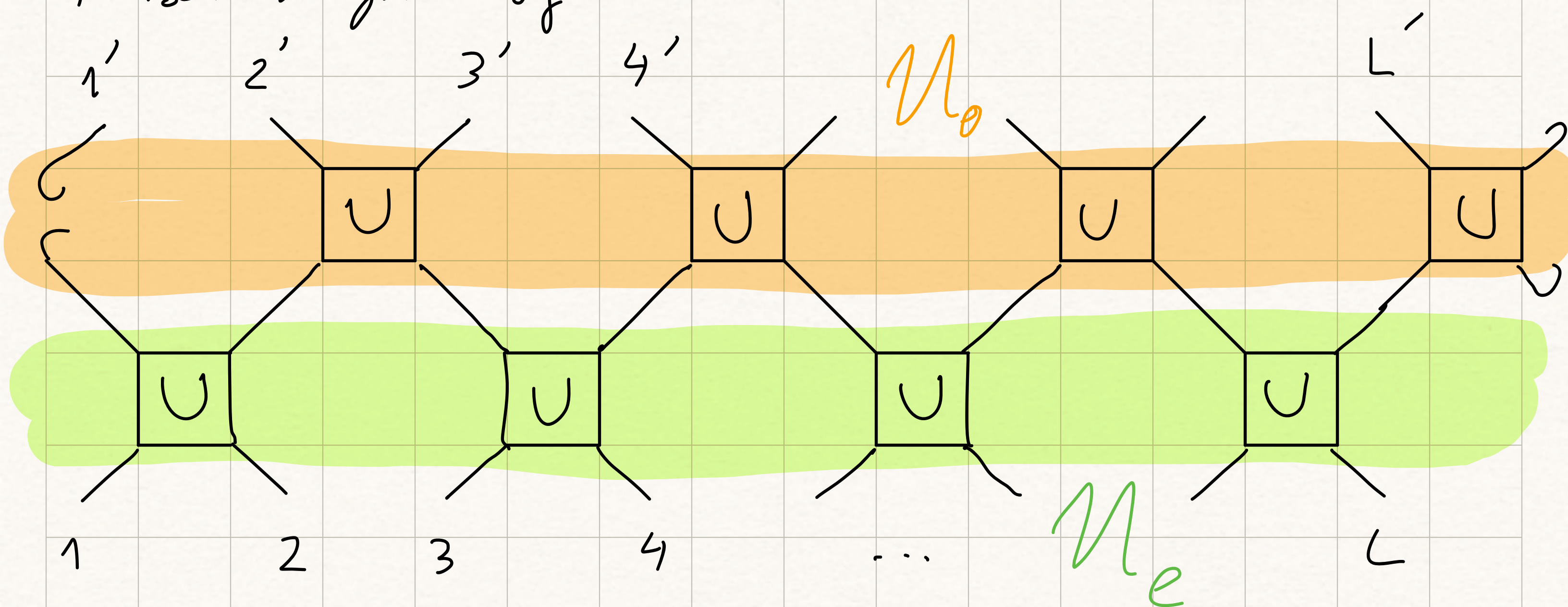
$\Pi\, |\varphi_1\rangle \otimes |\varphi_2\rangle \otimes \dots \otimes |\varphi_L\rangle = |\varphi_2\rangle \otimes |\varphi_3\rangle \otimes \dots \otimes |\varphi_1\rangle$

$\mathcal{U}_o = \Pi\, \mathcal{U}_e\, \Pi^{-1}$ pbc: $L+1 \equiv 1$

One step of time evolution:

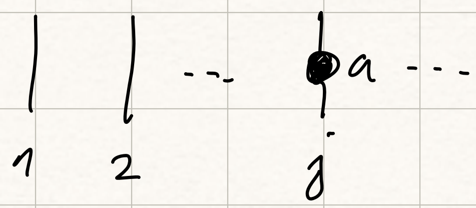
$$\mathcal{U} = \mathcal{U}_o \mathcal{U}_e \in \mathrm{End}(\mathcal{H}_L)$$
$$\in U(d^L)$$

Tensor diagram of $\mathcal{U}$:



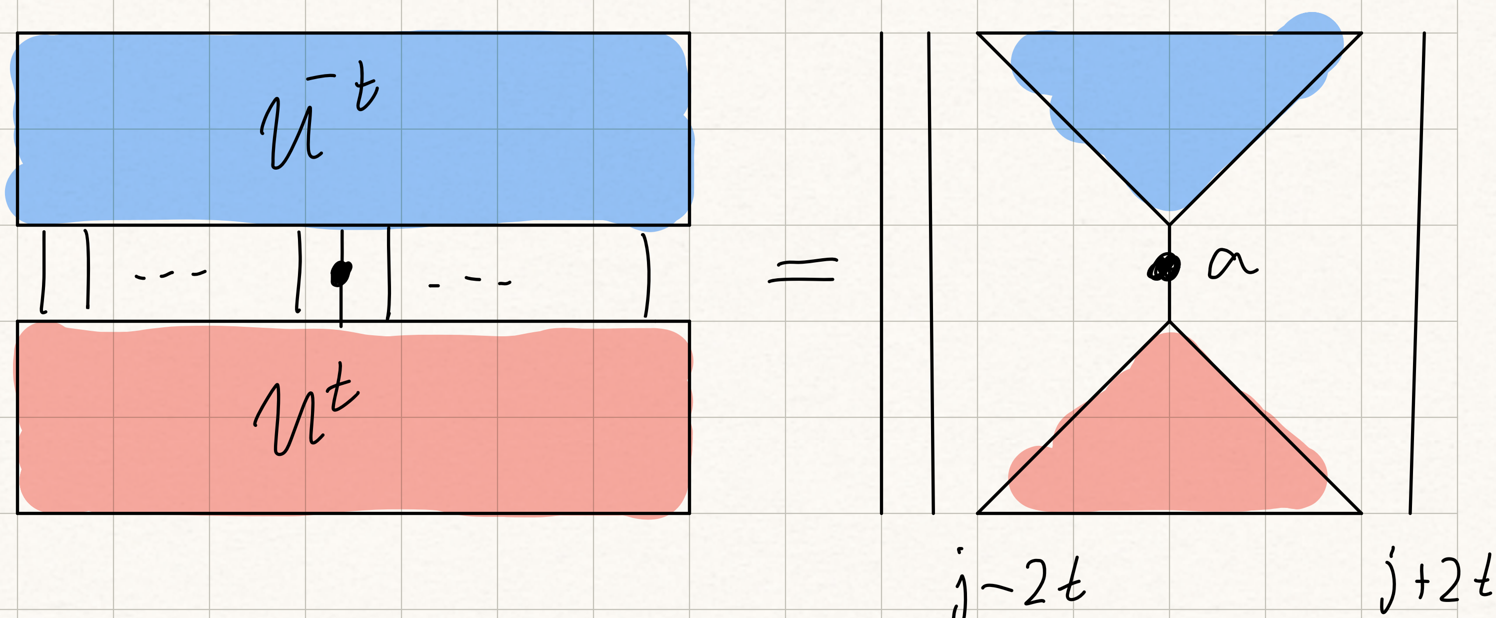
Manifest locality of dynamics:

example: Tensor diagram for Heisenberg dynamics of local operator

$a_j =$ 

$= \mathbb{1}^{\otimes j-1} \otimes a \otimes \mathbb{1}^{\otimes (L-j)}$

$$a_j(t) = \mathcal{U}^{-t}\, a\, \mathcal{U}^{t}$$



Example: Trotterization of local Hamiltonian

$$H = \sum_{j=1}^{L} h_{j,j+1} \qquad U = e^{-i\tau h_{1,2}}$$

$$e^{-i\tau H} \simeq \mathcal{U}_o \, \mathcal{U}_e$$

$$\mathcal{U}_e = \prod_{j=1}^{L/2} U_{2j-1,2j} = U^{\otimes L/2}$$

$$\mathcal{U}_o = \prod_{j=1}^{L/2} U_{2j,2j+1} = \Pi \, U^{\otimes L/2} \, \Pi^{-1}$$

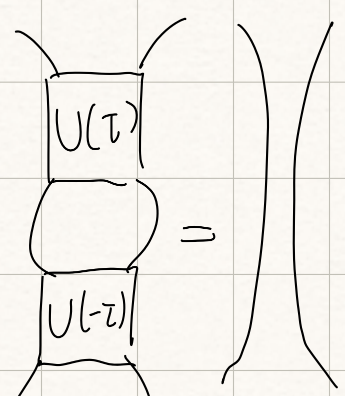
II.2 Yang-Baxter integrable circuits

Start with an example:

$$U(\tau) = \frac{\mathbb{1} + i\tau P}{1 + i\tau} \qquad P|ij\rangle \equiv |ji\rangle \quad \text{(SWAP)}$$

eg for $d=2$:

$$U = \begin{pmatrix} 1 & & & \\ & a & b & \\ & b & a & \\ & & & 1 \end{pmatrix} \qquad a = \frac{1}{1+i\tau} \qquad b = \frac{i\tau}{1+i\tau}$$



$a + b = 1$

$|a|^2 + |b|^2 = 1$

$\rightarrow$ $\tau \in \mathbb{R}$ unitary

$\rightarrow$ $i\tau \in \mathbb{R}$ stochastic

In the Trotter limit, $\tau \to 0$, $\mathcal{U}$ corresponds to trotterization of $SU(d)$ exchange Hamiltonian

$$H = \sum_{j=1}^{L} P_{j,j+1}$$

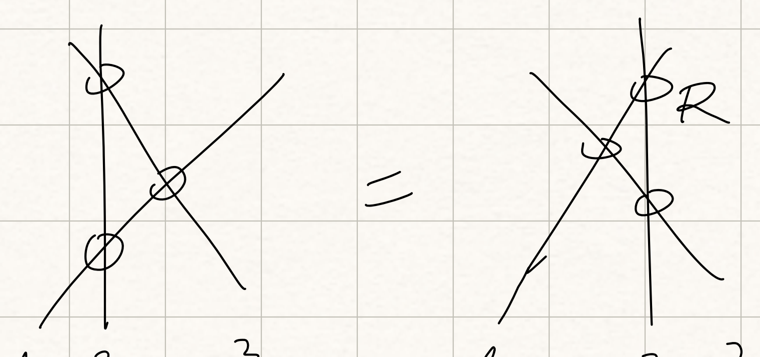
(XXX model for $d=2$)

$U_{ij}(\tau)$ satisfies so-called <u>braid relation</u>:

$$U_{12}(\mu)\, U_{23}(\mu+\tau)\, U_{12}(\tau) =$$
$$= U_{23}(\tau)\, U_{12}(\tau+\mu)\, U_{23}(\mu)$$

In fact: $U_{12}(\tau) = P_{12}\, R_{12}(\tau)$

where $R_{12}(\tau)$ is an R-matrix obeying Yang-Baxter equation

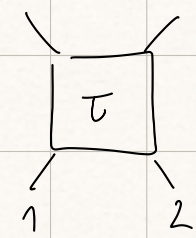


$$\begin{cases} U_{12}(0) = \mathbb{1} \\ R_{12}(0) = P \end{cases}$$

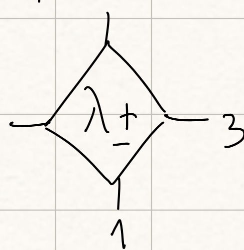
$$R_{23}(\mu) R_{13}(\tau+\mu) R_{12}(\tau) = R_{12}(\tau) R_{13}(\tau+\mu) R_{23}(\mu)$$

set $\mu = \lambda - \frac{\tau}{2}$ and left-multiply by P_{12}

$$(*) \quad R_{13}(\lambda - \tfrac{\tau}{2}) R_{23}(\lambda + \tfrac{\tau}{2}) U_{12}(\tau) = U_{12}(\tau) R_{13}(\lambda + \tfrac{\tau}{2}) R_{23}(\lambda - \tfrac{\tau}{2})$$

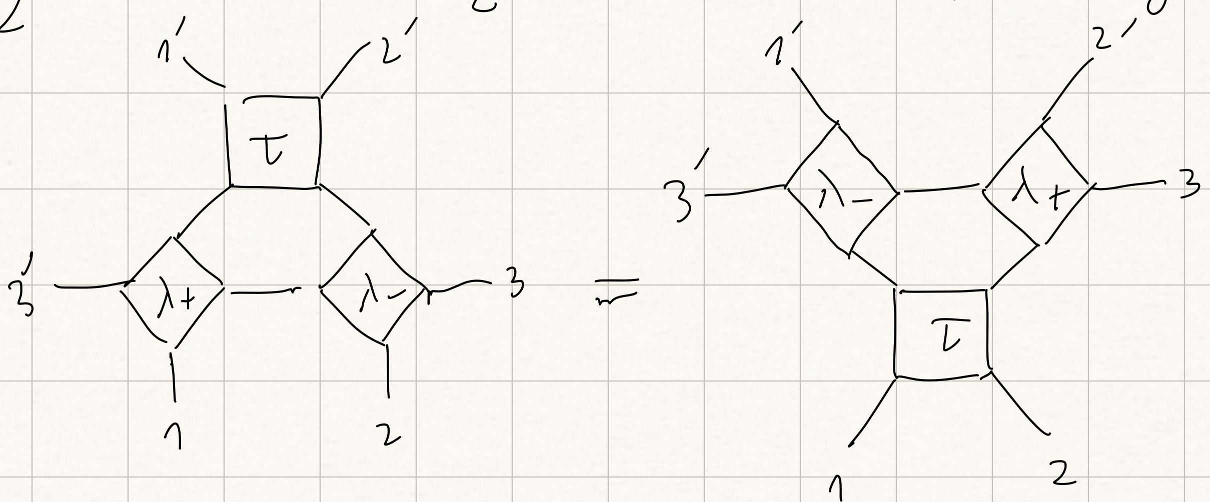


$\equiv U_{12}(\tau)$



$\equiv R_{13}(\lambda \pm \frac{\tau}{2})$

Eq (*) is thus equivalent to the following diagram:



Claim: The following transfer matrix $T(\lambda) \in \mathrm{End}(\mathcal{H}_L)$

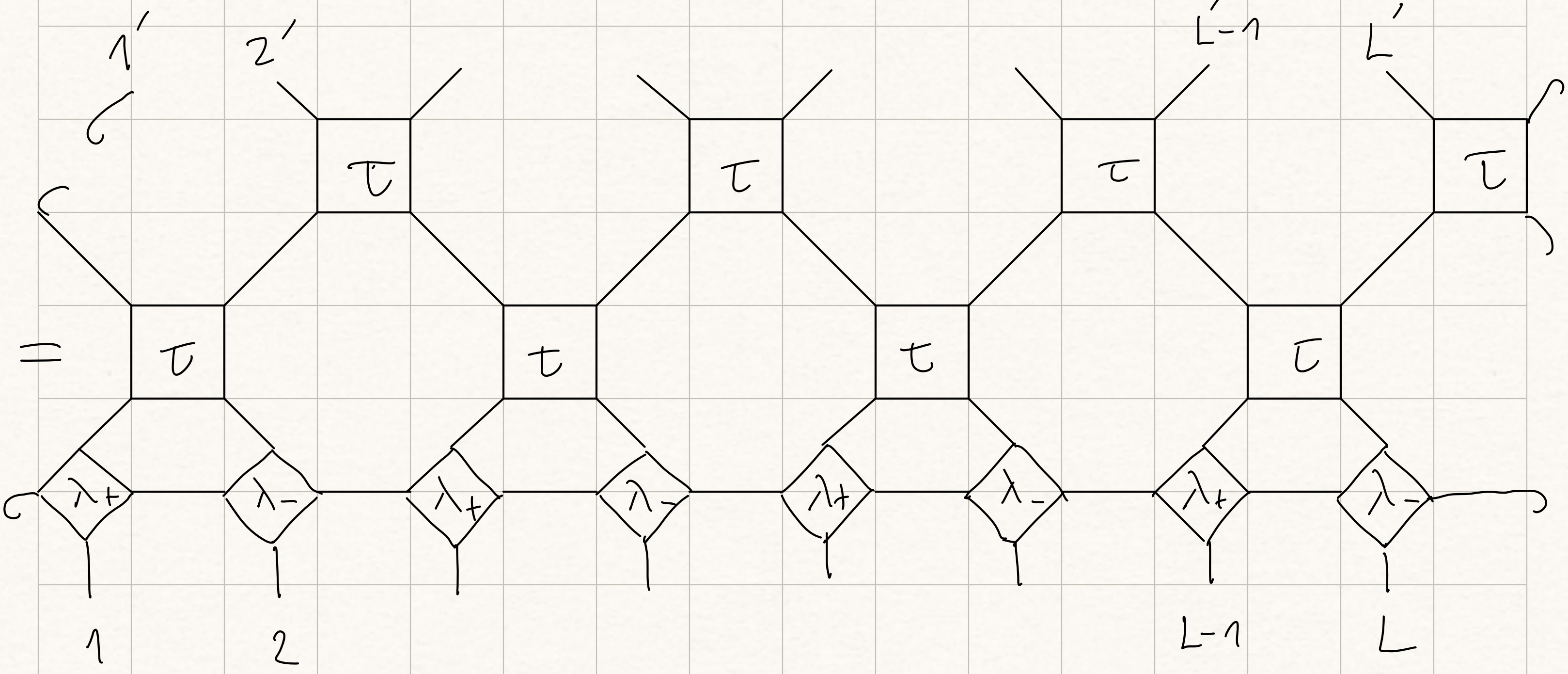
$$T(\lambda) = \mathrm{tr}_a \left(\overrightarrow{\prod_{1 \leq j \leq L/2}} R_{2j-1,a}(\lambda_+) R_{2j,a}(\lambda_-) \right)$$

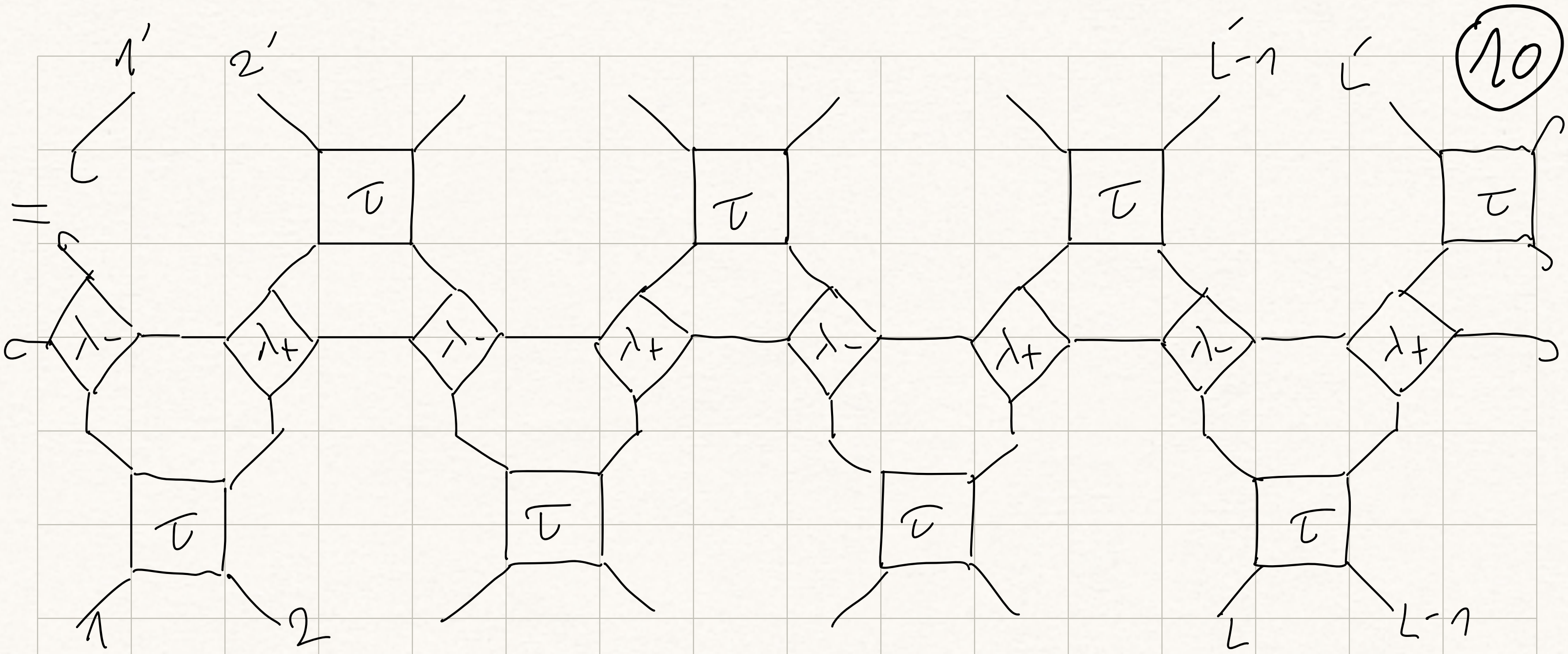
commutes with the time evolution of the circuit

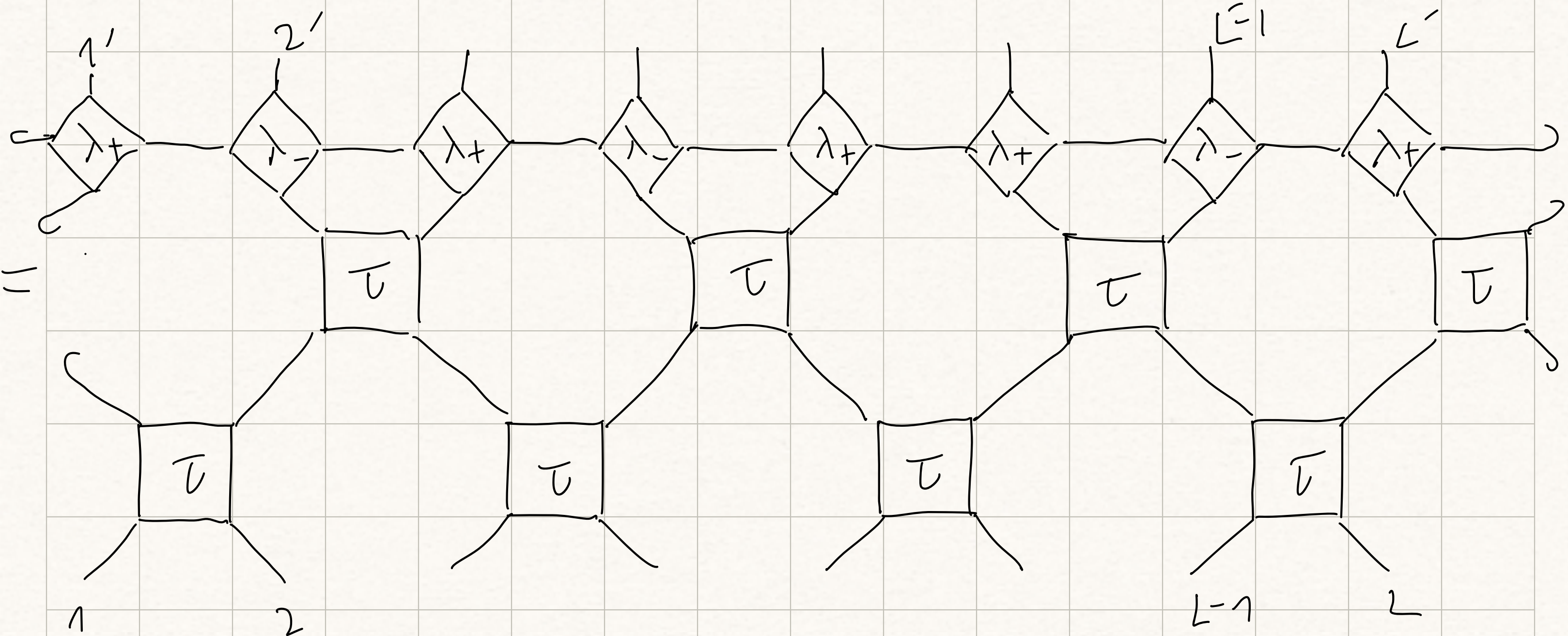
$$[\mathcal{U}, T(\lambda)] = 0$$

Proof: pure diagrammatics

$$\mathcal{U}\, T(\lambda) =$$

$=$ 



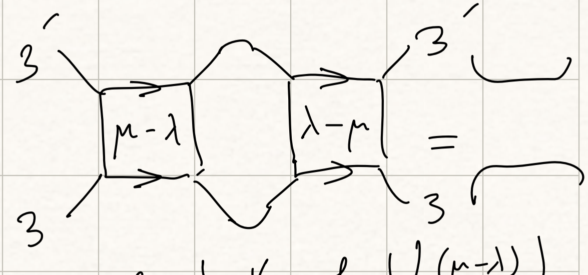


$= T(\lambda)\, \mathcal{U}$ q.e.d.

Moreover, $T(\lambda)$ forms a commuting family of (nearly unitary) operators:

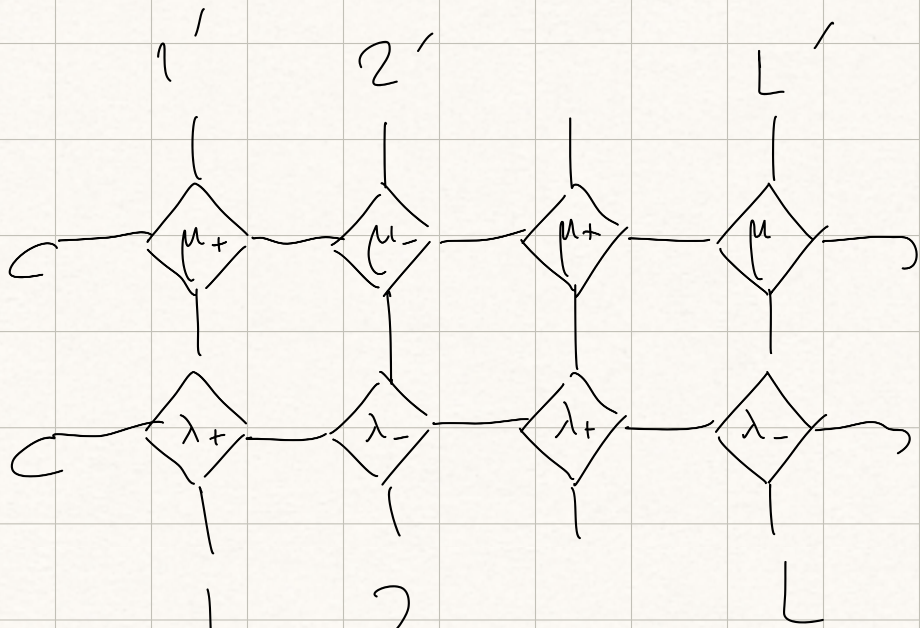
$$T(\mu)\,T(\lambda) = T(\lambda)\,T(\mu)$$

Proof (diagrammatics): &



(unitarity of $U(\mu-\lambda)$)

$T(\mu)\,T(\lambda) =$



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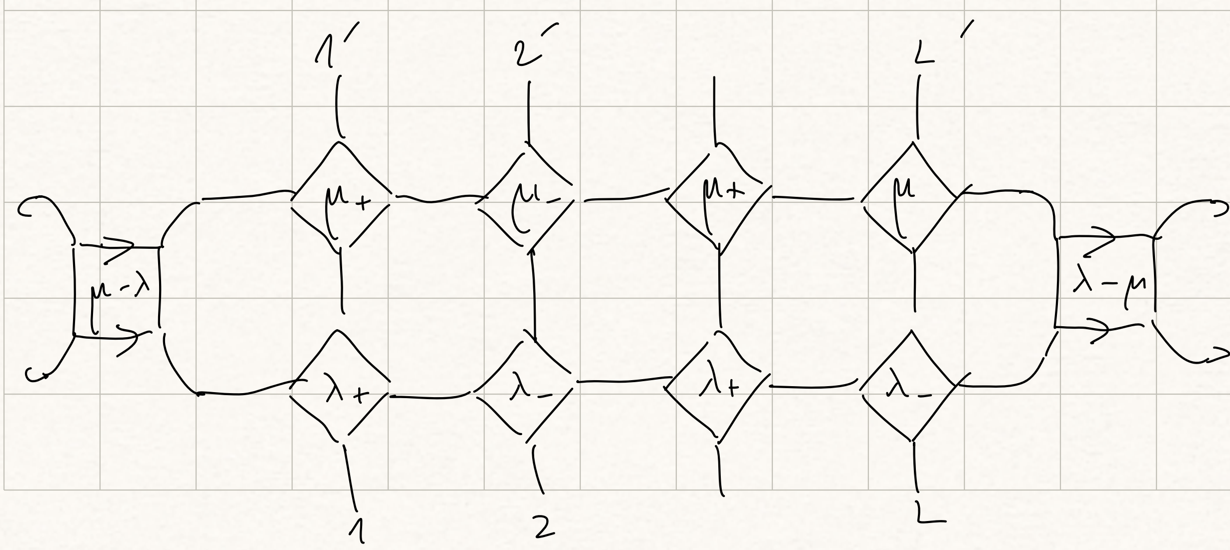

(12)

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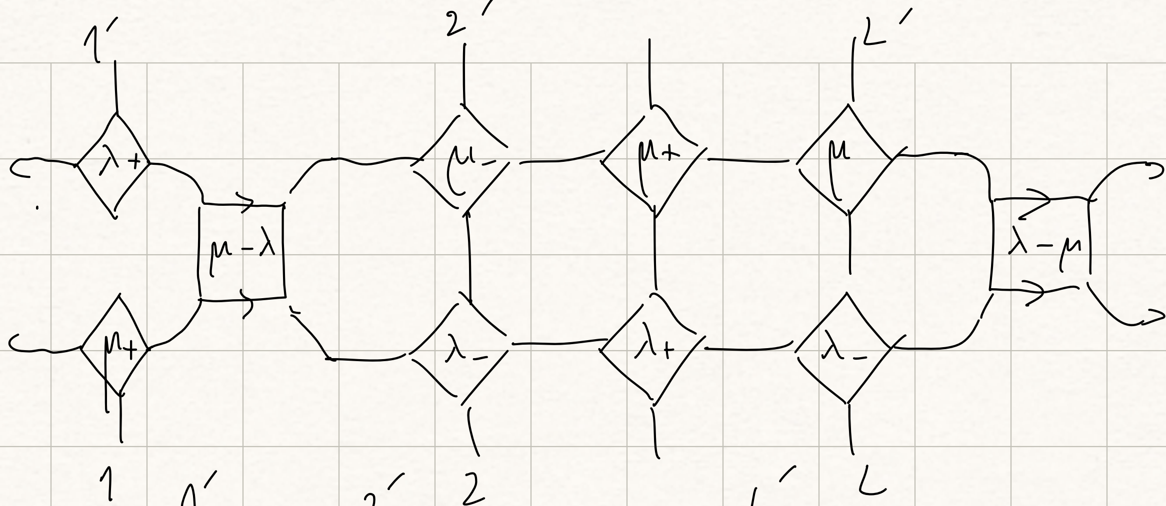

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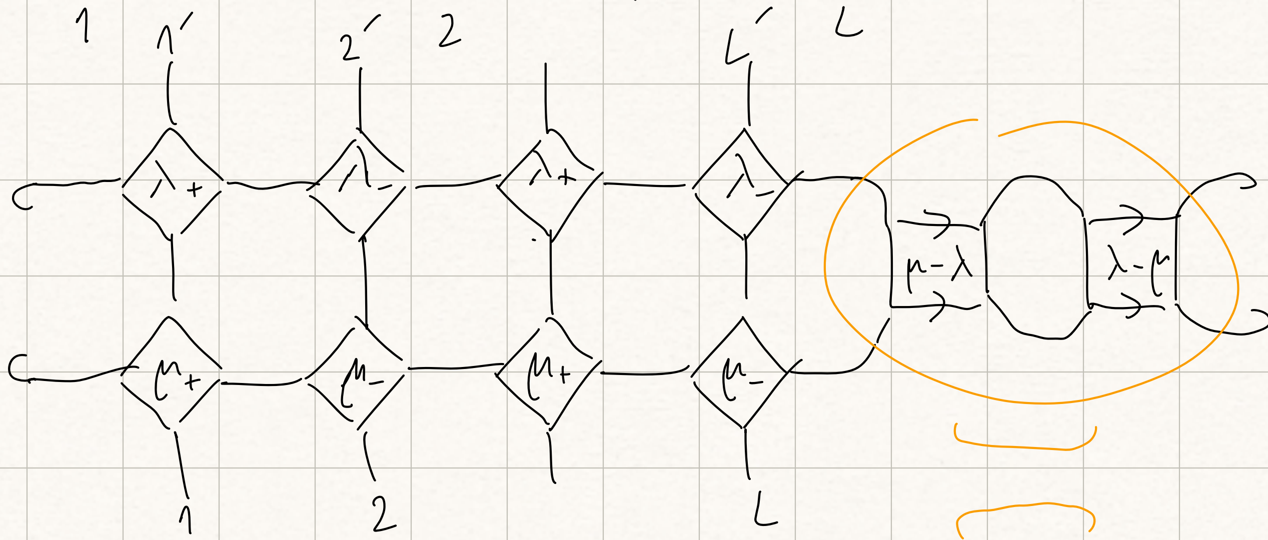


$= \quad T(\lambda)\, T(\mu)$ q.e.d.

Local integrals of motion

$$\boxed{Q_n^{\pm} = \left(\frac{d}{d\lambda}\right)^n \log T(\lambda)\Big|_{\lambda = \pm\frac{\tau}{2}}}$$

$Q_n^{\pm}$ local with support-size $n+1$

Alternative representation of time evolution

$$\boxed{\mathcal{U} = T(-\tfrac{\tau}{2})^{-1} T(\tfrac{\tau}{2})}$$

<u>Excercise</u>: Prove it using tensor network diagrammatics and YBE!

All algebraic constructions generalize to XXZ case

$$U = \exp\left(-iJ(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y) - iJ'(\sigma^z \otimes \sigma^z - \mathbb{1})\right)$$

$$= \begin{pmatrix} 1 & & & \\ & \frac{\sin\eta}{\sin(\lambda+\eta)} & \frac{\sin\lambda}{\sin(\lambda+\eta)} & \\ & \frac{\sin\lambda}{\sin(\lambda+\eta)} & \frac{\sin\eta}{\sin(\lambda+\eta)} & \\ & & & 1 \end{pmatrix}$$

$$e^{2i(J \pm J')} = \frac{\sin\eta - \sin\lambda}{\sin(\eta \pm \lambda)}$$

Two unitary solutions:

- imaginary λ, real η : easy plane XXZ $(\tau = i\lambda)$
- real λ, imaginary η : easy axis XXZ $(\tau = \lambda)$

III 3 Dissipative (open) integrable circuits

boundary driving

$$\rho_{t+1} = \mathcal{M}\, \rho_t$$

$$\mathcal{M} = \mathcal{M}_o \mathcal{M}_e$$

$$\mathcal{M}_e = \sum_{l=1}^{2} K_l^R \, \mathcal{U}_e \, \rho \, \mathcal{U}_e^\dagger \, K_l^{R\dagger}$$

$$\mathcal{M}_o = \sum_{l=1}^{2} K_l^L \, \mathcal{U}_o \, \rho \, \mathcal{U}_o^\dagger \, K_l^{R\dagger}$$

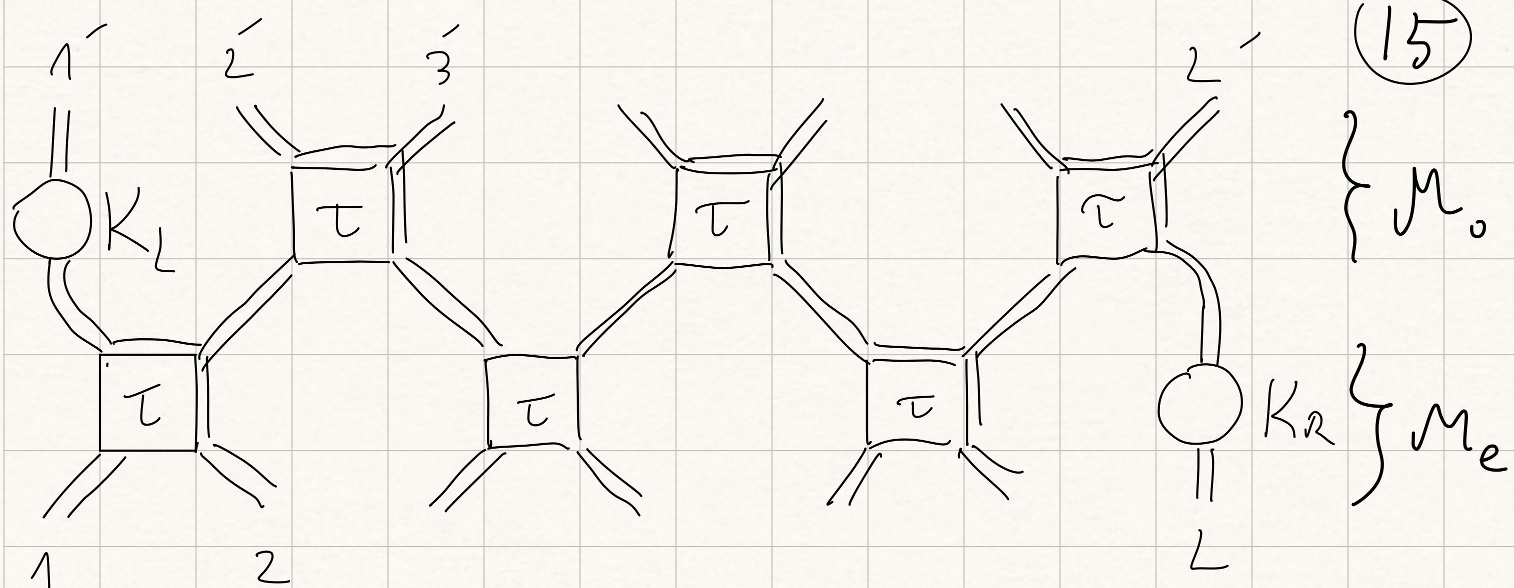
$$\mathcal{U}_e = U_{12} U_{34} \cdots U_{L-2,L-1} e^{i b_R \sigma_2^z} \qquad L \text{ odd}$$

$$\mathcal{U}_o = e^{i b_L \sigma_1^z} U_{23} U_{45} \cdots U_{L-1,L}$$

$$K_1^L = \frac{\mathbb{1} + \sigma_1^z}{2} + \sqrt{1-\gamma_L}\, \frac{\mathbb{1} - \sigma_1^z}{2} \qquad K_2^L = \sqrt{\gamma_L}\, \sigma_1^+$$

$$K_1^R = \frac{\mathbb{1} - \sigma_1^z}{2} + \sqrt{1-\gamma_R}\, \frac{\mathbb{1} + \sigma_L^z}{2} \qquad K_2^R = \sqrt{\gamma_R}\, \sigma_L^-$$

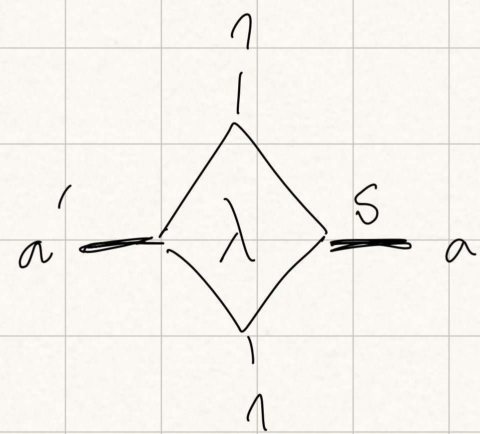
15



Matrix product ansatz for NESS

$$\rho_\infty = \mathcal{M}\,\rho_\infty$$

$$\rho_\infty = \frac{\Omega\Omega^+}{\operatorname{tr}\Omega\Omega^+}$$

L : 

$$\Omega = \langle 0|\, L_{a_1}(\lambda,s)\, L_{a_2}(\lambda-\tau,s) \cdots L_{a_{L-1}}(\lambda-\tau,s)\, L_{a_L}(\lambda,s)\,|0\rangle D^{\otimes L}$$

$$D = \begin{pmatrix} \alpha^{1/4} & 0 \\ 0 & \alpha^{-1/4} \end{pmatrix} \qquad L(\lambda,s) = \begin{pmatrix} i\lambda + S^z & S^- \\ S^+ & i\lambda - S^z \end{pmatrix}$$

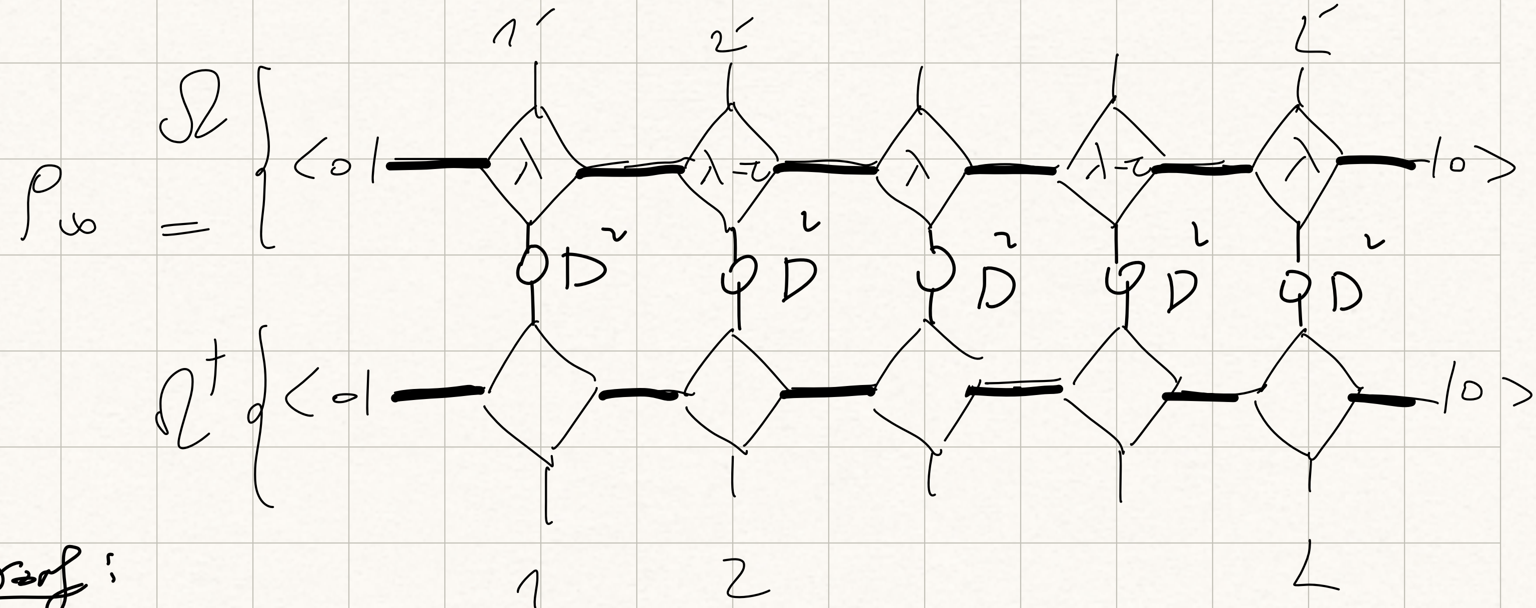
$$S^z = \sum_{k=0}^{\infty} (s-k)\,|k\rangle\langle k| \qquad S^+ = \sum_{k=0}^{\infty} (k+1)\,|k\rangle\langle k+1|$$

$$S^- = \sum_{k=0}^{\infty} (2s-k)\,|k+1\rangle\langle k|$$

$$\lambda = \frac{\tau}{2}\left(\frac{1}{1-e^{-2ib_R}\sqrt{1-\gamma_R}} - \frac{e^{2ib_L}\sqrt{1-\gamma_L}}{1-e^{2ib_L}\sqrt{1-\gamma_L}}\right)$$

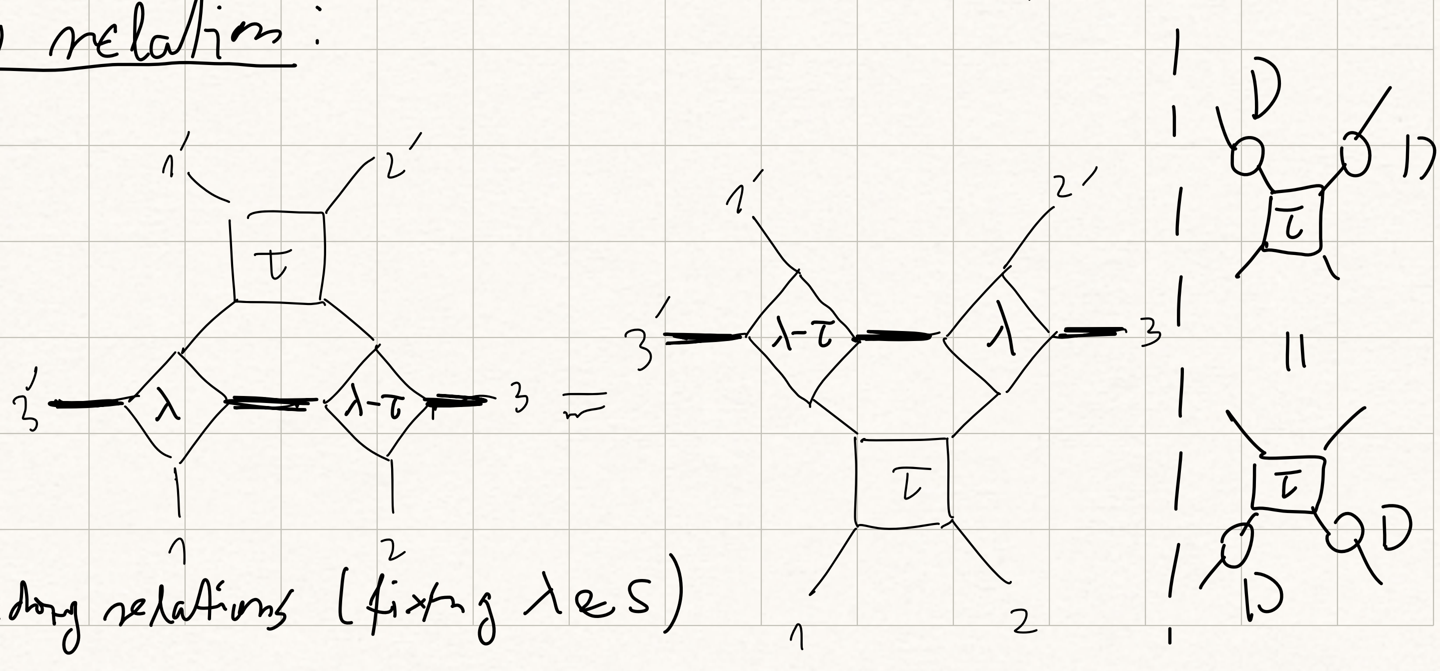
$$\delta = \frac{i\tau}{2}\left(\frac{1}{1-e^{-2ib_R}\sqrt{1-\gamma_R}} + \frac{e^{2ib_L}\sqrt{1-\gamma_L}}{1-e^{2ib_L}\sqrt{1-\gamma_L}}\right)$$

$$x = \frac{\gamma_L}{\gamma_R}\,\frac{2(1-\cos(2b_R)\sqrt{1-\gamma_R})-\gamma_R}{2(1-\cos(2b_L)\sqrt{1-\gamma_L})-\gamma_L}$$



<u>Proof:</u>

<u>Key relation:</u>



+ bonding relations (fixing λ & δ)

byproduct: Quasi-local charge

$$\Omega(\lambda,s) = \langle 0 | L_{a_1}(\lambda,s) L_{a_2}(\lambda-\tau,s) \cdots L_{a_{L-1}}(\lambda-\tau,s) L_{a_L}(\lambda,s) | 0 \rangle$$

$$\boxed{Z(\lambda) = \frac{\partial}{\partial s} \Omega(\lambda,s) \Big|_{s=0}}$$

$$U^{\dagger} Z U - Z = \text{boundary terms}$$

$$\langle Z Z^{\dagger} \rangle := \frac{\mathrm{tr}\, Z^{\dagger} Z}{\mathrm{tr}\, \mathbb{1}} = O(L) \quad \text{extensivity}$$

(for $\eta = \cos \frac{\pi l}{m}$, easy plane XXZ)

Z breaks spin reversal symmetry $\left(F = (\sigma^x)^{\otimes L} \right)$

$$F Z(\lambda) F^{\dagger} \neq Z(\lambda)$$

unlike transfer matrices w.r.t. unitary irreps

$$F T(\lambda) F^{\dagger} = T(\lambda)$$

$Q(\lambda) = i\,(Z(\lambda) - Z^{+}(\lambda))$ 18

$F\,Q(\lambda)\,F^{+} = -Q(\lambda)$

similar to spin current

$J = \sum_j i(\sigma_j^{+}\sigma_{j+1}^{-} - \sigma_j^{-}\sigma_{j+1}^{+})$

$F J F^{+} = -J$

Hence can be used to bound spin Drude weight

$$D_s = \lim_{t\to\infty}\lim_{L\to\infty}\frac{1}{2LT}\sum_{t=1}^{T}\langle J(t)J\rangle$$

Mazur bound $D_s \geq \dfrac{|\langle JQ\rangle|^2}{2L\langle Q^2\rangle}$

$\langle\left(\int_0^T J(t)\,dt + \alpha Q\right)^2\rangle \geq 0$

show slides

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same technique works for other integrable models:

e.g. Hubbard, SU(N) (Lai-Sutherland) chains etc

Lecture II: KPZ scaling in XXX circuits and Heisenberg spin chain

II 1) Weak inhomogeneous quench

Continuity equation

$$\vec{S}_x = \frac{1}{2}\vec{\sigma}_x$$

continuous time: $H = \sum_{r=1}^{L} \vec{S}_r \cdot \vec{S}_{r+1}$

$$j_x = S^x_r S^y_{r+1} - S^y_r S^y_{r+1}$$

$$\frac{dS_r}{dt} = i[H, S_r] = j_{r-1} - j_r$$

discrete time: $U_{12} = e^{-i\tau \vec{S}_1 \cdot \vec{S}_2}$

$$m_r = S^z_r + S^z_{r+1}$$

$$U^\dagger m_{2r} U - m_{2r} = j^o_{2r-1} - j^o_{2r+1}$$

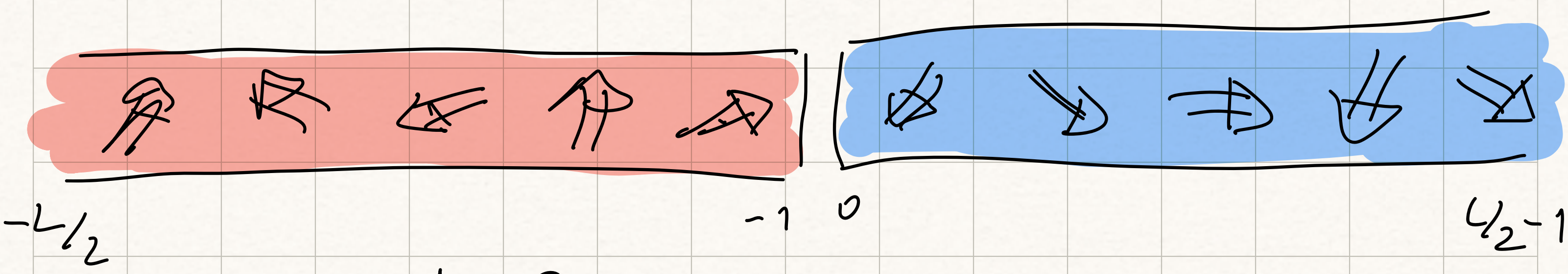
$$U^\dagger m_{2r-1} U - m_{2r-1} = j^e_{2r-2} - j^e_{2r}$$

$$j^o_{2r-1} = (2\sin\tau) j_{2r-1} - \left(\frac{1}{2}\sin^2\frac{\tau}{2}\right)(S^z_{2r} - S^z_{2r-1})$$

$$j^e_{2r} = \mathcal{U}_e^\dagger j^o_{2r} \mathcal{U}_e \quad \text{(acts on 4 adjacent sites!)}$$

We start from initial state density matrix (20)

$$\boxed{\rho(t=1) \propto \rho_\mu = \left(e^{\mu S^z}\right)^{\otimes \frac{L}{2}} \otimes \left(e^{-\mu S^z}\right)^{\otimes \frac{L}{2}}}$$



$$\langle a \rangle_\mu = \frac{\mathrm{tr}\,\rho_\mu a}{\mathrm{tr}\,\rho_\mu} \qquad \left(\bar{r} = \frac{L}{2} + \frac{1}{2}\right)$$

Linear response

$$\langle S^z_r(t)\rangle_\mu = -\mu \sum_{r'} \mathrm{sgn}\left(r'+\tfrac{1}{2}\right)\langle S^z_r(t)\, S^z_{r'}\rangle$$

$$+\, O(\mu^2) \quad \text{(expansion to 2nd order)}$$

$$\langle a \rangle \equiv \langle a \rangle_0 \quad \text{(equilibrium)}$$

transl. invariance

$$\langle S^z_{r-1}(t)\rangle_\mu - \langle S^z_r(t)\rangle_\mu = \qquad \langle S^z_{r-1}(t)\, S^z_{r'}\rangle = \langle S^z_r(t)\, S^z_{r'+1}\rangle$$

$$\mu \left\langle S^z_r(t) \sum_{r'} \mathrm{sgn}\left(r'+\tfrac{1}{2}\right)\left(S^z_{r'} - S^z_{r'+1}\right)\right\rangle =$$

$$= 2\mu \langle S^z_r(t) S_0 \rangle - 2\mu \langle S^z_r(t) \underbrace{S_{-\frac{L}{2}}}_{\downarrow\, 0 \text{ if } t \ll L} \rangle + O(\mu^2)$$

21

$$\Rightarrow \quad \langle S^z_r(t) \rangle_\mu = 2\mu \langle S^z_r(t) S_0 \rangle + O(\mu^2)$$

or:

$$\boxed{\langle S^z_0(0)\, S^z_r(t) \rangle = \lim_{\mu \to 0} \frac{\langle S^z_{r-1}(t) \rangle_\mu - \langle S^z_r(t) \rangle_\mu}{\mu}}$$

weak quantum quench is a good probe to measure equilibrium dynamical correlation functions

Protocol: i) initialize $\rho(t) = \rho_\mu$

ii) Write MPO ansatz

$$\rho(t) = \sum A^{[1]}_{s_1} A^{[2]}_{s_2} \cdots A^{[L]}_{s_L}\, \sigma^{s_1} \otimes \cdots \otimes \sigma^{s_L}$$

and compute $A^{[1]}_s(t)$ using TEBD algorithm

iii) compute

$$\langle S_r^z(t)\rangle_\rho = \mathrm{tr}(S_r^z \rho(t))$$

and 2-point function

$$C(r,t) = \langle S_0^z(0)\, S_r^z(t)\rangle$$

KPZ scaling conjecture - based on numerical (& experimental) data:

$$\partial_t h = \frac{1}{2}\lambda(\partial_r h)^2 + \nu\, \partial_r^2 h + \sqrt{\Gamma}\,\xi$$ ← uncorrelated noise

$$\langle \xi(r,t), \xi(r',t')\rangle = \delta(r-r')\delta(t-t')$$

$$\tilde{C}(r,t) = \langle [h(r,t) - h(0,0) - t\langle \partial_t h\rangle]^2\rangle$$

$$\frac{1}{2}\partial_r^2 \tilde{C}(r,t) = \langle \partial_r h(r,t)\, \partial_r h(0,0)\rangle$$

scaling functions $$g(\varphi) = \lim_{t\to\infty} \frac{\tilde{C}((2\lambda^2 t^2 \Gamma \nu^{-1})^{1/3}\varphi, t)}{(\frac{1}{2}\lambda t \Gamma^2 \nu^{-2})^{2/3}}$$

$$f(\varphi) = \frac{1}{4} g''(\varphi) \propto \partial_r^2 C(r,t)$$

We find that spin-spin c.f. is given in terms of $f(y)$

$$\boxed{\partial_r \langle S^z(r,t)\rangle_\mu = \frac{a\mu}{t^{2/3}}\, f\left(\frac{br}{t^{2/3}}\right)}$$

$\mu\, C(r,t)$

Similarly, for the spin current we find, using continuity eq.

$$-\partial_r \langle j(r,t)\rangle_\mu = \partial_t \langle S^z_r(r,t)\rangle_\mu$$

$$\boxed{\langle j(r,t)\rangle_\mu = \frac{2a\mu}{3b^2 t^{1/3}}\, h\left(\frac{br}{t^{2/3}}\right)}$$

$$h(y) = \frac{g(y) - y\, g'(y)}{y}$$

Naive identification:

$$\partial_r h(r,t) \Longleftrightarrow S^z(r,t)$$

Transferred magnetization

$$Q(t) = \int_0^T dt\, j(0,t) = -\int_{-\infty}^{0} dr \left(S^z(r,T) - S^z(r,0) \right)$$

$$\text{"} \approx -\left(h(0,T) - h(0,0) \right) \text{"}$$

$h \Longleftrightarrow -h$ no symmetry

$\rightarrow\leftarrow$?

Can't really work:

Fluctuations of $\Delta h(T)$ are <u>skewed</u>

— " — $Q_{mag}(t)$ are symmetric (equilibrium!)

Key features:

- integrability
- non-abelian symmetry ($SU(2)$ here)
- We consider spin reversed odd sector (spin current, magnetization are all odd under F) while integrability transfer matrix is even

Microscopic picture

Explanation of $z = 3/2$ in terms of giant bound-state quasiparticles

Starting point: Spin conductivity bounded by the curvature of Drude weight:

i) $\sigma_s(h=0) \geq \beta \frac{\partial^2}{\partial h^2} \mathcal{D}(h) \Big|_{h=0}$

ii) Decomposing Drude weight into quasiparticle contributions

$$D(h) = \sum_s \int d\lambda \, D(h,s,\lambda)$$

$\Rightarrow$ TBA for a time-dependent diffusion constant diverges as $\boxed{D(t) = t^{1/3}}$

or $D(s^*) \sim s^{\alpha}$, until $t = s^{*3}$

s^* longest string included

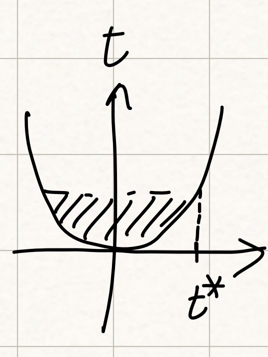
$$\Rightarrow \; x(t) \sim \sqrt{D(t)\, t} = t^{2/3} \;\Rightarrow\; z = 3/2$$

Nearly isotropic case: $\Delta = \cosh \eta$, $|\eta| \ll 1$

$\boxed{D \propto \frac{1}{\sqrt{\Delta - 1}}}$ crossover time to diffusion $\boxed{t^* \sim (\Delta-1)^{-3/2}}$

Consistency with <u>Lindblad picture</u>

$$j(L) \sim L^{-\gamma} = \frac{D(L)}{L}$$



$t^{*\,1/z} \sim L$

$$\langle J(t) J \rangle \sim t^{-1/z} - t^{-2/3}$$

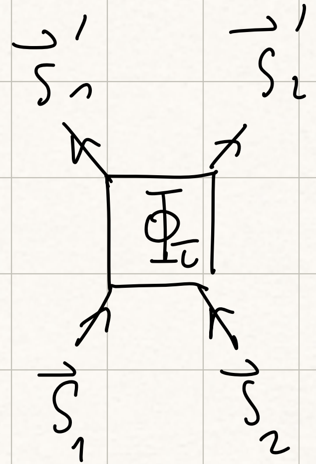
$t^* \sim L^z$

$$D(L) \simeq t^{*\,1-1/z} = L^{z-1} = L^{1-\gamma}$$

$$\boxed{\gamma = 2 - z = 1/2}$$

Lecture III: Classical limit and superuniversality of superdiffusion

Start with a simple dynamical system, formed from a 2-spin black box:

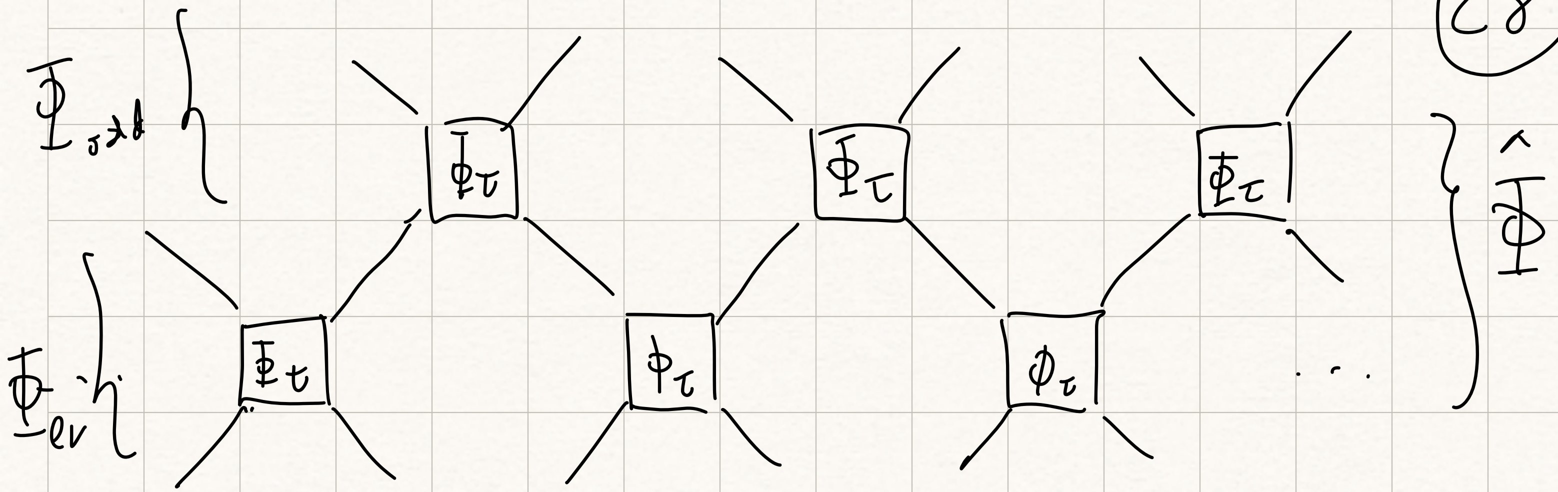


$$\Phi_\tau(\vec{S}_1,\vec{S}_2) = \frac{1}{\sigma^2+\tau^2}\left(\sigma^2\vec{S}_1 + \tau^2\vec{S}_2 + \tau\,\vec{S}_1\times\vec{S}_2,\ \sigma^2\vec{S}_2 + \tau^2\vec{S}_1 + \tau\,\vec{S}_2\times\vec{S}_1\right)$$

$$\sigma^2 = \frac{1}{2}\left(1+\vec{S}_1\cdot\vec{S}_2\right)$$

Nice properties: i) $\Phi_{-\tau} = \Phi_\tau^{-1}$

ii) Φ_τ <u>symplectic</u> w.r.t. $\{S_i^\alpha, S_j^\beta\} = \varepsilon_{\alpha\beta\gamma} S_i^\gamma \delta_{ij}$



Deterministic spin lattice gas:

$$\Phi_{ev}: \quad (\vec{S}^{2t+1}_{2x}, \vec{S}^{2t+1}_{2x+1}) = \Phi_\tau(\vec{S}^{2t}_{2x}, \vec{S}^{2t}_{2x+1})$$

$$\Phi_{odd}: \quad (\vec{S}^{2t+2}_{2x-1}, \vec{S}^{2t+2}_{2x}) = \Phi_\tau(\vec{S}^{2t+1}_{2x-1}, \vec{S}^{2t+1}_{2x})$$

Local map generates symplectic 2-spin Landau-Lifshitz type dynamics with Hamiltonian

$$H(\vec{S}_1, \vec{S}_2) = 2h(\sigma), \quad \sigma^2 = \frac{1}{2}(1 + \vec{S}_1 \cdot \vec{S}_2)$$

$$h(\sigma) = \sigma \arctan\left(\frac{2\sigma\tau}{\tau^2 - \sigma^2}\right) - \tau \log(\tau^2 + \sigma^2)$$

Nice limits:

i) $\tau \to 0$:

$$\frac{d\vec{S}_x}{dt} = \{\vec{S}_x, H\} = 2\vec{S}_x \times \left(\frac{\vec{S}_{x-1}}{1+\vec{S}_{x-1}\cdot\vec{S}_x} + \frac{\vec{S}_{x+1}}{1+\vec{S}_x\cdot\vec{S}_{x+1}} \right)$$

where $$H = -\sum_x 2\log\tfrac{1}{2}(1+\vec{S}_x\cdot\vec{S}_{x+1})$$

(Ishimori chain)

ii)

$$\vec{S}_x(t) \equiv \vec{S}(x\cdot\delta, t)$$

$\delta \to 0$: $\quad \partial_t \vec{S} = \vec{S} \times \partial_x^2 \vec{S}$

Landau Lifshitz field theory

The model is integrable (will be shown later)!

Let us study spin transport

$$\sum_x \vec{S}_x = \text{konst} \qquad C(x,t) = \langle S_0^z S_x^z(t) \rangle$$

30

Lecture IV:

Full counting statistics in deterministic integrable many-body dynamics

(slides)